4. *Abstraction and Unification*

1820 CE–1894 CE

Emergence of Algebraic Structures and the Rise of Abstract Algebras

Breakaway from Euclidean Geometry

The Arithmetization of Analysis

Complex Analysis; Differential and Integral Equations

Advent of Electromagnetism: Unification of Optics, Electricity and Magnetism

Quantification of Thermal Phenomena; Thermodynamics and Statistical Physics

The Periodic Table of the Elements

Organic Chemistry and Cell Theory

The Theory of Biological Evolution

Oceanography – The Conquest of Inner Space

Alternating Current Technology; Discovery of Photoconductivity

Emergence of World Communication: Telephone, Telegraph, Facsimile

Personae

List of Essays

List of Tables

List of Figures

Science 'Progress Reports'

Worldviews

Environmental Events that Impacted Civilization

1876–1879	Prolonged *drought* in India and China; ca 18 million people perish
1893	The *Krakatoa* volcanic eruption
1887	*Floods* of the Yellow River (China): 1 million people perish
1889–1890	*Influenza pandemic* in the world; millions die
1892–1900	Drought, famine and plague in India and China; ca 7 million perish

Political and Religious Events that Impacted World Order

1826 The last 'auto-da-fe' of the Spanish Inquisition

1839–1842 The '*Opium War*' between China and England

1850–1871 A series of European wars give birth to unifications of Italy and Germany and an unprecedented growth of science in Europe:

- The *Crimean War* (1854–1856): Russia vs. Western powers
- France and Italy against Austria (1859)
- Garibaldi against the French (1860)
- The battle of *Sadowa* (1866): Prussia against Austria
- The battle of *Sedan* (1870): Germany against France

* * *

"I am the daughter of earth and water,
And the nursling of the sky;
I pass through the pores of the ocean and shores;
I change, but I cannot die.
For after the rain, when with never a stain
The pavilion of heaven is bare,
And the winds and sunbeams, with their convex gleams,
Build up the blue dome of air,
I silently laugh at my own cenotaph,
And out of the caverns of rain,
Like a child from the womb, like a ghost from the tomb,
I arise and unbuild it again."

P.B. Shelley ("The Cloud", 1820)

1820 CE **Hans Christian Oersted** (1777–1851, Denmark). Physicist and chemist. Discovered electromagnetism (the magnetic effects of currents) and concluded that there exists a magnetic field surrounding a current. This phenomenon was soon quantified by **Jean Baptiste Biot** (1774–1862, France) and the physician and physicist **Felix Savart** (1791–1841, France).

Oersted was born at the town of Rudkobing, on the Island of Langeland, Denmark.

The Law of Biot-Savart

In 1819 Oersted observed that wires carrying electric currents produced deflections of permanent magnetic dipoles placed in their neighborhood. Thus the currents were sources of magnetic-flux density. Biot and Savart (1820), and later Ampère (1820–1825), in much more elaborate and thorough experiments, established the basic experimental laws relating magnetic induction $\boldsymbol{B}$ *to the source currents, and established the* law of force *between one current and another. The final analytic results were derived by Ampère (1826). In a form later written by* **Maxwell**, *a time-independent magnetic field due to a static current-density* $\boldsymbol{J}(\boldsymbol{r})$ *is given by*

$$\textit{curl}\ \boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{J}(\boldsymbol{r}); \quad \boldsymbol{B} = \textit{curl}\ \boldsymbol{A}, \quad \textit{div}\ \boldsymbol{A} = 0, \quad \nabla^2 \boldsymbol{A} = -\frac{4\pi\mu}{c}\boldsymbol{J}(\boldsymbol{r}).$$

Integration of the vector Poisson equation gives $\boldsymbol{A} = \frac{\mu}{c}\int \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}'$, *and consequently* $\boldsymbol{B} = \frac{\mu}{c}\int \frac{\boldsymbol{J}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}\boldsymbol{r}'$. *If the conductor in which the current flows is sufficiently thin (thin wire), and if we are interested only in the field in the surrounding space, the thickness of the wire may be neglected. The integration over the volume of the conductor is then replaced by an integration along its length, i.e. we put* $\boldsymbol{J}\,\mathrm{d}\boldsymbol{r}' \to J\,\mathrm{d}\boldsymbol{l}$, *where* J *is the total current in the conductor. Hence*

$$\boldsymbol{A} = \frac{\mu J}{c}\oint_{\text{wire}}\frac{\mathrm{d}\boldsymbol{l}}{R}; \qquad \boldsymbol{H} = \frac{J}{c}\oint_{\text{wire}}\frac{\mathrm{d}\boldsymbol{l}\times\boldsymbol{R}}{R^3}; \qquad \boldsymbol{R} = \boldsymbol{r}-\boldsymbol{r}',$$

which is Biot and Savart's law. Note that the field $\boldsymbol{H}$ *is independent of the magnetic susceptibility of the medium. Above,* $\mathrm{d}\boldsymbol{l}$ *is an element of length*

(pointing in the direction of current flow) of a filamentary wire carrying a current J, and $\boldsymbol{R}$ is the coordinate vector from the element of length to an observation point.

Ampère's experiments did not deal directly with the determination of the relation between currents and magnetic induction, but were concerned rather with the force that one current-carrying wire experiences in the presence of another; the force experienced by a current element $J_1 \mathrm{d}\boldsymbol{l}_1$ in the presence of a magnetic induction $\boldsymbol{H}$ is $\mathrm{d}\boldsymbol{F} = \frac{J_1}{c}(\mathrm{d}\boldsymbol{l}_1 \times \boldsymbol{H})$. If the external field $\boldsymbol{H}$ is due to a closed current loop with current J_2, then the total force which another closed current loop with current J_1 experiences is

$$F_{12} = \frac{J_1 J_2}{c^2} \oint \oint \frac{\mathrm{d}\boldsymbol{l}_1 \times (\mathrm{d}\boldsymbol{l}_2 \times \boldsymbol{R}_{12})}{|\boldsymbol{R}_{12}|^3},$$

*where the line integrals are taken around the two loops and $\boldsymbol{R}_{12}$ is the vector distance from line element $\mathrm{d}\boldsymbol{l}_2$ to $\mathrm{d}\boldsymbol{l}_1$. This is the mathematical statement (in modern notation) of Ampère's observations about forces between current-carrying loops, as obtained by **Grassman** (1809–1877) in 1844. By manipulating the integrand it can be recast in a form which is symmetric in $\mathrm{d}\boldsymbol{l}_1$ and $\mathrm{d}\boldsymbol{l}_2$ and which explicitly satisfies Newton's third law. Thus*

$$\frac{\mathrm{d}\boldsymbol{l}_1 \times (\mathrm{d}\boldsymbol{l}_2 \times \boldsymbol{R})}{|\boldsymbol{R}_{12}|^3} = -(\mathrm{d}\boldsymbol{l}_1 \cdot \mathrm{d}\boldsymbol{l}_2)\frac{\boldsymbol{R}_{12}}{|\boldsymbol{R}_{12}|^3} + \mathrm{d}\boldsymbol{l}_2 \left[\frac{\mathrm{d}\boldsymbol{l}_1 \cdot \boldsymbol{R}_{12}}{|\boldsymbol{R}_{12}|^3}\right].$$

The second form involves a perfect differential in the integral over $\mathrm{d}\boldsymbol{l}_1$. Consequently, it gives no contribution to the integral in $\boldsymbol{F}_{12}$, provided the paths are closed or extend to infinity. Ampère's law of force between current loops then becomes

$$\boldsymbol{F}_{12} = -\frac{J_1 J_2}{c^2} \oint \oint \frac{(\mathrm{d}\boldsymbol{l}_1 \cdot \mathrm{d}\boldsymbol{l}_2)\boldsymbol{R}_{12}}{|\boldsymbol{R}_{12}|^3}.$$

The line integral *for the field $\boldsymbol{H}$ can be transformed into an equivalent integral over a* surface *S bounded by the line, obtaining*

$$\boldsymbol{H} = \mathrm{grad}\ \Phi; \qquad \Phi = \frac{J}{c}\int_S \frac{\mathrm{d}\boldsymbol{s} \cdot \boldsymbol{R}}{|\boldsymbol{R}|^3},$$

where $\mathrm{d}\boldsymbol{s}$ is the vectorial area element, and Φ is a harmonic potential. The surface integral is, geometrically, the solid angle *Ω subtended by the closed contour at the point considered. As this point describes a closed path round the wire, the angle Ω changes suddenly from 2π to -2π, rendering Φ a* multi-valued *potential.*

The law of Biot-Savart is applicable, mutatis mutandis, *in the theories of hydrodynamics and elasticity.*

Clearly, the work done by moving a unit magnetic pole once around the wire or a closed curve C *is* $\oint_C \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \oint_C d\Phi = 4\pi J/c$. *This is known as Ampère's circuital theorem.*

1820 CE **Robert Gibbon Johnson** (USA). Consumed two tomatoes on the steps of the courthouse in Salem MA, thus refuting the then widely circulating belief that the tomato was poisonous[1]. This changed forever the human-tomato relationship.

1820–1827 CE **André Marie Ampère** (1775–1836, France). Physicist. Attempted to render a combined theory of electricity and magnetism. Stimulated by the discovery of the *phenomenon* of electromagnetism by Oersted (1820), he soon followed with his own discovery of the basic *laws of electrodynamics* (Ampère coined this term).

He showed that parallel electric currents attract each other if the currents move in the same direction, and repel each other if their directions are opposite. In 1820 he showed that an electric current flowing through a coiled wire acts like a magnet. This led him in 1824 to the invention of the *galvanometer*, an instrument for detecting and measuring electric currents[2].

From his experiments, **Ampère** derived a number of quantitative empirical laws concerning the interaction of circuits carrying *direct* electric currents. Among the laws stated, is the inverse square law of force between two current elements (analogous to Newton's law of gravitation, 1687; and Coulomb's law

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[1] *Tomato* (*Lycopersicum esculentum*; order — *Solanaceae*). Annual plant, native of South America, probably Peru. The family includes: potato, eggplant, bell peppers, hot chili peppers, pimentos, paprika. Within this family are 4 genera which have been involved in most foul murders, in witchcraft and demonology, in military campaigns, in sly seduction, and in sexual orgies. These are: jimsonweed (*Datura*), deadly nightshades (*Atropa*), henbanes (*Hyoscyamus*), and mandrakes (*Mandragora*). These genera contain three *alkaloids*: atropine, hyoscyamine, and scopolamine which are representative of the tropane series of alkaloids, similar in structure to cocaine.

[2] **Johann Salomo Christoph Schweigger** (1779–1857, Germany) made an independent, similar experiment: he built a galvanometer (1820) consisting of a wire wound around a magnetic needle and measured the angle of deflection of the magnetic needle by the current. He named his instrument in honor of Luigi Galvani.

of the static force between point charges, 1785). Another is the Ampère circuital law relating the current flowing in a closed circuit to the magnetic field produced by the current[3]. Consequently, he treated magnetism by postulating small closed circuits inside the magnetized substance. This approach became fundamental in the 19^{th} century. Later, **Maxwell** modified this law for the case of time-varying electric fields.

Ampère was born at Polémieux, near Lyons. At an early age he learned Latin, which enabled him to read the works of Euler and the Bernoullis, but his reading also embraced history, travels, poetry, philosophy and all natural sciences. His father, an anti-revolutionary justice of peace, perished at the scaffold. This event produced a profound impression on the youth's susceptible mind. He was married in 1799 and moved to Bourg in 1801 to become a professor of physics and chemistry. His wife died in 1804 and he never recovered from the blow. In 1809 he was elected a professor of mathematics at the polytechnique school in Paris.

He died at Marseilles. The great amiability and childlike simplicity of Ampère's character are well brought out in his '*Journal et Correspondance*' (Paris, 1872).

1820–1836 CE **John Frederic Daniell** (1790–1845, England). Chemist and meteorologist. Invented (1820) a *dew-point hygrometer* (a device that indicates atmospheric humidity) which came into widespread use. Daniell (1823) revealed his findings on the behaviour of the atmosphere and on trade winds, in addition to giving details of improved meteorological equipment. Improved the voltaic cell by devising a cell giving steady current (1836). This 'Daniell-cell' was used as a source of energy in the electromagnetic telegraphy built by **W.F. Cooke** and **C. Wheatstone**. It gave new impetus to electric research and found many commercial applications.

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[3] Ampère discovered the following experimental facts concerning the magnetic fields produced by an electric current:

- A small coil (or loop) of current $\boldsymbol{J}$ behaves like a magnetic dipole of moment $\boldsymbol{m}$.
- $\boldsymbol{m}$ is perpendicular to the plane of the coil.
- $\boldsymbol{m}$ forms a right-handed screw with the flow of the current round the coil.
- $|\boldsymbol{m}|$ is proportional to the current J flowing in the coil.

With a proper choice of the unit of current (e.m.u), one can write $\boldsymbol{m} = \boldsymbol{J}\,\mathrm{d}\boldsymbol{s}$, where $d\boldsymbol{s}$ is the vectorial area element of the *magnetic shell*.

Daniell was born in London. In 1831 he became the first professor of chemistry in Kings College.

1820–1847 CE **John Herapath** (1790–1868, England). Natural philosopher. One of the first to discover that heat was not a substance but a form of motion. His *kinetic theory* was presented in an ambitious *Mathematical Inquiry into the Causes, Laws and Principal Phenomena of Heat, Gases, Gravitation etc.*, submitted to the Royal Society (1820) but rejected as being too speculative. It was however studied by **James Joule**, who in his own work on heat (1843–7), almost certainly followed Herapath.

Herapath was born in Bristol, and entered his father's profession as a maltster, but left business (1815) to open a mathematical academy in Bristol. He gave up teaching (1832), moved to Kensington, and began to write about the growing British railway network. This in turn aroused his interest in heat engines and hence modeling steam-gas physics in terms of elastic collisions between self-repulsive particles.

1820–1875 CE **Christian Gottfried Ehrenberg** (1795–1876, Germany). Naturalist. One of the first explorers of marine life. He was born at Delitzsch in Saxony, and studied at Leipzig and Berlin, where he took the degree of doctor of medicine in 1818. He was appointed professor of medicine at the University of Berlin in 1827.

Ehrenberg traveled widely, exploring the natural history of Egypt, Abyssinia, Arabia, and Russia, all the way to the frontier of China. After returning from these voyages, he examined his collections under the microscope. He discovered that many of the rock samples he had brought back were not inorganic products, as he had thought, but consisted of the remains of countless microscopic animals. In 1836 he showed that many silicate rocks were similarly composed of the remains of diatoms, sponges, and radiolaria.

Next he showed that living organisms similar to the ones that make up rocks still inhabit the sea. He reasoned that these rocks are continually forming as a result of the constant rain of dead organisms to the sea bottom. Ehrenberg also showed that the phosphorescence of the sea is due to the presence of microscopic organisms. Thus life in the sea extends from the largest living animal, the whale, to microscopic organisms which are so numerous that their accumulated remains make up thick layers of rock.

1820–1910 CE The European *Period of Romanticism* in music; Its leading composers are:

- Mauro Giulliani 1781–1828
- Nicolo Paganini 1782–1840

- Carl Maria von Weber 1786–1826
- Gioachino Rossini 1792–1868
- **Franz Schubert 1797–1828**
- Abraham Niedermeyer 1802–1861
- Hector Berlioz 1803–1869
- Mikhail Glinka 1804–1857
- Felix Mendelssohn 1809–1847
- Robert Schumann 1810–1856
- Frederic Chopin 1810–1849
- Franz Liszt 1811–1886
- **Giuseppe Verdi 1813–1901**
- Cesar Franck 1822–1890
- Edouard Lalo 1823–1892
- Bedrich Smetana 1824–1884
- **Johannes Brahms 1833–1897**
- Alexander Borodin 1834–1887
- Camille Saint-Saëns 1835–1921
- Georges Bizet 1838–1875
- Max Bruch 1838–1920
- Modest Mussorgsky 1839–1881
- Peter Ilytch Tchaikovsky 1840–1893
- Antonin Dvorak 1841–1904
- Edvard Grieg 1843–1907
- Nicolai Rimsky-Korsakov 1844–1908
- Gabriel Fauré 1845– 1924
- Francesco Paolo Tosti 1846–1916
- Charles Hubert Parry 1848–1918
- **Giacomo Puccini 1858–1924**
- Mikhail Ippolitov-Ivanov 1859–1935
- Gustav Mahler 1860–1911
- Claude Debussy 1862–1918
- Alexander Glazunov 1865–1936
- **Jean Sibelius 1865–1957**
- Sergei Rachmaninoff 1873–1943
- Maurice Ravel 1875–1937

1821 CE **Jean Francois Champollion** (1790–1832, France). Egyptologist. The first to decipher hieroglyphic writing. The Rosetta stone, discovered in 1799 during Napoleon's campaign in Egypt, provided the key to the language of ancient Egypt.

1821 CE **Joseph Maria Wronski** (1776–1853, Poland and France). Originated the '*Wronskian determinant*' in the theory of linear ODE. This was his

only contribution to mathematics. He is the sole Polish mathematician of the 19^{th} century, remembered today. Wronski, who had an impecunious and erratic personality, spent most of his life in France.

The Emergence of Dynamic Elasticity Theory[4]

In 1787 *the German physicist* **Ernst Florens Friedrich Chladni** *(1756–1827) studied vibration of plates by means of sand figures (1827). In his experiments he poured fine sand on top of a glass plate. He then rubbed a bow against the plate, causing a vibration. The sand bounced away from regions that vibrated and collected at* nodes *(places that remained still). Within a second the plate was covered with a series of sandy curves. The patterns were symmetric and spectacular: circles, stars and other geometric figures. The character of the pattern depended on the shape of the plate, the position of the supports and the frequency of vibration.*

Since this phenomenon could not be explained by contemporary German and Swiss mathematicians, Chladni visited Paris in 1808 *and presented his experiments before mathematicians and physicists of the Institute of France, a section of the French Academy of Science. Chladni's experiments so astounded the scientists that they asked him to repeat his demonstration before Napoleon. The Emperor was impressed, and he agreed that the Academy should award a medal of one kilogram of gold to anyone who could devise a theory that explained Chladni's experiments. The contest was announced in* 1809 *and a deadline of two years was set for all entrees.*

In 1811, **Sophie Germain** *(1776–1831, France) was the only entrant in the contest: she tried to explain the behavior of elastic plates by applying the methods that* **Euler** *(1751) had used: Euler had suggested that a force applied*

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4 For further reading, see:

- Love, A.E.H., *A Treatise on the Mathematical Theory of Elasticity*, Dover Publications: New York, 1944, 643 pp.

- Todhunter, I. and K. Pearson, *A History of the Theory of Elasticity from Galileo to Lord Kelvin*, 2 Volumes, Dover Publications: New York, 1960.

to a rod is counteracted by an internal force of elasticity that is proportional at any point along the rod to the curvature of the rod. Similarly, she proposed that at any point on the surface, the force of elasticity is proportional to the sum of the major curvatures of the surface at that point. (The major curvatures are the maximum and minimum values of curvature out of all the curves formed when planes cut through the surface perpendicularly).

Germain's work did not win the award; she had not derived her hypothesis from principles of physics nor could she have done so at the time, because she lacked knowledge of analysis and the calculus of variations.

But her work did spark new insights: ***Lagrange****, who was one of the judges of the contest, corrected the error's in Germain's calculations and came up with the equation*

$$\frac{\partial^2 z}{\partial t^2} + K^2 \left(\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} + 2\frac{\partial^4 z}{\partial x^2 \partial y^2} \right) = 0,$$

where $z(x, y)$ *is the amplitude of the vibration for small* z, t *is time and* (x, y) *represent a point on the surface.*

In 1811, *the Academy extended the contest deadline by two years, and again Germain submitted the only entry. She demonstrated that Lagrange's equation did yield Chladni's patterns in several simple cases. But still, she could not devise a satisfactory derivation of Lagrange's equation from physical principles. For this work, she received honorable mention from the Academy.*

At about the same time ***Simeon Dennis Poisson*** *approached the subject of elasticity with all the resources available to a* 19^{th} *century mathematician and physicist. In* 1812, *he was elected to the Academy and was therefore ineligible to compete for the prize. In* 1814 *Poisson published an article on elastic plates which sought to arrive at Lagrange's equation by applying a Newtonian model in which the plate consists of molecules that mutually repel and attract each other. By modern standards, Poisson's assumptions seem absurd.*

In her third entry in the contest (1815) Germain proposed to regard the work done in bending as proportional to the integral of the square of the sum of the principal curvatures taken over the surface. From this assumption and the principle of virtual work she deduced the equation of flexural vibration in the form now generally admitted. Later investigations have shown, however, that the formula assumed for the work done in bending was incorrect.

The judges in the contest were, at that time, ***Legendre****,* ***Laplace*** *and* ***Poisson****. They could not accept all of her assumptions, and with this reservation she won the prize. Germain did not attend the award ceremony.*

At the end of 1820, *no one knew yet how to combine the Newtonian conception of the constitution of bodies with Hooke's law. In the words of A.E.H. Love in the* 1892 *edition of his Treatise on the Mathematical Theory of Elasticity:*

"the fruit of all the ingenuity expanded on elastic problems might be summed up as — an inadequate theory of flexure, an erroneous theory of torsion, an unproved theory of the vibrations of bars and plates, ...".

Yet, very little was needed to combine the older researches: the recognition of the distinction between shear and extension (strain) was there. So was the recognition of forces across the elements of a section of beam (stress). Also, there was an awareness that deflection of a bent beam and vibrations of rods and plates are expressible in terms of differential equations. Finally, the generalization of the principle of virtual work in the Mécanique Analytique of Lagrange *threw open a broad path in this as in every other branch of mathematical physics. Physical science had emerged from its incipient stages with definite methods of hypothesis and induction and of observation and deduction, with the clear aim to discover the laws by which phenomena are connected with each other, and with a fund of analytical processes of investigation.*

This was the hour for production of general theories, and the men were not wanting: in a span of seven years, 1821–1828, ***Navier***, ***Cauchy*** *and* ***Poisson*** *established the modern theory of elasticity and applied the general theory to special problems. It was further applied by* ***Lamé*** *and* ***Clapeyron*** *(1831–1833) to numerous problems of vibrations and of static elasticity, and thus means were provided for testing its consequences experimentally.*

After the equations of elasticity had been formulated, little advance seems to have been made in the treatment of problems of shells and plates. Only in 1860 *did* ***Kirchhoff*** *succeed formulating the equation of small motion of a plate in a correct way. He deduced it from the principle of virtual work and applied it to the problem of the flexural vibrations of a circular plate. Chladni's experiments were finally explained.*

1821–1823 CE **Claud Loui Marie Henri Navier** (1785–1836, France). Engineer and applied mathematician. Worked on topics such as engineering, elasticity and fluid mechanics. He was first to develop a theory of suspension bridges which hitherto had been built on empirical principles. Presented a molecular theory of an elastic body, giving the equation of motion for the

displacement of a particle in an elastic solid[5]. Led by formal analogy with the theory of elasticity, he succeeded in setting up the differential equation of motion of a viscous fluid[6] (*Navier-Stokes equations*, 1823).

Navier was born in Dijon and educated at the École Polytechnique. In 1819 he became professor of applied mechanics at the École des Ponts et Chaussées in Paris, and in 1831 a professor of analytical mechanics at the École Polytechnique. He was a disciple and a friend of **Fourier**.

1821–1831 CE **Augustin Louis Cauchy** (1789–1857, France). The most outstanding analyst of the first half of the 19^{th} century. Pioneered the study

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[5] The field equations for linear elastic media are derived from three fundamental physical principles applied to the elastic continuum: conservation of mass, linear momentum and angular momentum. The conservation of linear momentum leads to the *Cauchy equation of motion*

$$\operatorname{div}\mathfrak{T} + \rho\boldsymbol{F} = \rho\frac{\partial^2\boldsymbol{u}}{\partial t^2},$$

where $\mathfrak{T}$ is the *stress tensor*, $\boldsymbol{F}$ is a *body force* per unit mass, ρ is the density, and $\boldsymbol{u}(\boldsymbol{r}, t)$ is the displacement field. [The conservation of mass manifests itself in the relation $\delta\rho = -\rho\operatorname{div}\boldsymbol{u}$, reflecting changes of density due to wave motions, which are usually negligible in many applications. The conservation of angular momentum leads to the *symmetry* of the stress tensor.] For linear elastic solids, the stress-strain relation (*Hooke's Law*) is of the form

$$\mathfrak{T} = \overset{4}{C} : \frac{1}{2}(\nabla\boldsymbol{u} + \boldsymbol{u}\nabla),$$

where $\overset{4}{C}$ is the fourth-order *tensor of elastic moduli*. The special case of an *isotropic homogeneous solid* renders

$$\mathfrak{T} = \lambda I \operatorname{div}\boldsymbol{u} + \mu(\nabla\boldsymbol{u} + \boldsymbol{u}\nabla).$$

The substitution of this relation into the Cauchy equation yield the *Navier elastodynamic equation* for the unknown displacement field,

$$(\lambda + 2\mu)\operatorname{grad}\operatorname{div}\boldsymbol{u} - \mu\operatorname{curl}\operatorname{curl}\boldsymbol{u} + \rho\boldsymbol{F} = \rho\frac{\partial^2\boldsymbol{u}}{\partial t^2},$$

where (λ, μ) are the *Lamé parameters* of the elastic solid.

[6] **Navier** derived the proper form of these equations although he had no conception of shear stress in a fluid and despite not fully understanding the physics of the situation which he was modeling. He rather based his work on modifying **Euler**'s equation to take into account forces between molecules in the fluid.

of analysis and the theory of permutation groups. Researched the convergence and divergence of infinite series, differential equations, determinants, probability and mathematical physics. He wrote extensively and profoundly on both pure and applied mathematics and can be ranked next to Euler in volume of output[7] [27 large quarto volumes of collected works, including 789 papers]. His work, however, is of uneven quality.

His greatest contributions to the mathematical sciences are couched in the rigorous methodology which he introduced since 1821, banishing formal manipulations and intuition from analysis. These are mainly embodied in his 3 great treatises: *Cours d'analyse de l'Ecole Polytechnique* (1821), *Le Calcul infinitésimal* (1823), *Lecons sur les applications du calcul infinitesimal a la geometrie* (1826–1828). In these volumes he developed the theory of convergence and divergence of infinite series [root test, ratio test, integral test, product of series, absolute convergence] and of infinitesimal calculus [limits, continuity, differentiability, definite integral as a limit of a sum, etc.].

He was the first to prove Taylor's theorem rigorously, establishing his well-known form of the remainder. He was also first to properly define length of arcs and surface areas by integrals [the question of defining surface areas independent of integrals was taken up somewhat later, but this posed many difficult problems that were not properly solved until the 20^{th} century].

It is true that the first known appearance of infinite series occurred in the work of Archimedes and that infinite series were freely used in the late 17^{th} century by **Newton**, **Leibniz** and others, but little or no attention was given to the general question of establishing rigorous tests for their convergence or divergence.

During 1823–1828 Cauchy laid the foundations of today's mathematical theory of elasticity. He introduced the notions of stress and strain tensors, derived the 3D stress-strain relations and also correctly established the number of elastic constants: two for an isotropic solid and 21 for a crystal.

In 1831 he developed his complex function theory where one encounters *Cauchy's inequality*[8], *Cauchy's integral theorem*, *Cauchy's integral formula*,

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[7] In 1835 the Academy of Sciences began publishing its *Comptes Rendus*. So rapidly did Cauchy supply this journal with articles that the Academy became alarmed over the mounting printing bill, and accordingly passed a rule, still in force today, limiting all published papers to a maximum length of 4 pages.

[8] *Cauchy inequality*: Let $f(x)$ and $g(x)$ be two functions, not identically equal to zero, given on the interval (a, b). We choose arbitrary numbers λ and μ and form the expression

$$\int_a^b |\lambda f(x) - \mu g(x)|^2 dx \geq 0, \quad \text{i.e.} \quad 2\lambda\mu C \leq \lambda^2 A + \mu^2 B,$$

Cauchy-Riemann equations and *Cauchy's power series expansion* of an analytic function.[9]

Cauchy was born in Paris. He received his early education from his father Louis Francois Cauchy (1760–1848), who held several minor public appointments and counted Lagrange and Laplace among his friends.

Later, at the École du Panthéon, he excelled in ancient classical studies. In 1805 he entered the École Polytechnique and won the admiration of Lagrange and Laplace. In 1807 he enrolled at the École des Ponts et Chaussées, where he prepared for a career as a civil engineer. In 1810 he left Paris for Cherbourg and returned in 1813 on account of his health, whereupon Lagrange and Laplace persuaded him to renounce engineering and devote himself to mathematics.

He obtained an appointment at the École Polytechnique but was forced to give up his professorship after the revolution of 1830, and was excluded from

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where

$$\int_a^b f(x)g(x)dx = C, \qquad \int_a^b f^2(x)dx = A, \qquad \int_a^b g^2(x)dx = B.$$

Since the above inequality is valid for arbitrary values of λ and μ, we may choose $\lambda = \sqrt{\frac{C}{A}}$, $\mu = \sqrt{\frac{C}{B}}$. Substituting these values of λ and μ in the inequality, we obtain $\frac{C}{\sqrt{AB}} \leq 1$, or the Cauchy inequality:

$$\int_a^b f(x)g(x)dx \leq \left\{ \int_a^b f^2(x)dx \int_a^b g^2(x)dx \right\}^{1/2}.$$

[9] To dig deeper, see:

- Titchmarsh, E.C., *The Theory of Functions*, Oxford University Press: London, 1939, 452 pp.
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public employment for 18 years. A short sojourn at Freiburg in Switzerland was followed by his appointment, in 1831, to the newly-created chair of mathematics and physics at the University of Turin. In 1833 the deposed King Charles X, living in exile in Prague, summoned Cauchy to tutor his grandson, the 13-year-old Duke of Bordeaux. For five years the great analyst served as a baby-sitter of sorts to the pampered youth, and Charles made him a baron for this martyrdom.

Finally, in desperation, Cauchy escaped to Paris, saying he had to attend the celebration of his parents' golden anniversary. Once back in France, he was permitted to resume his post at the Académie. In 1848, after teaching in some church schools in Paris, he was allowed to return to the École Polytechnique without having to take the oath of allegiance to the new government. Cauchy was disliked by most of his colleagues. He displayed self-righteous obstinacy and an aggressive religious bigotry. His last words were: "*Men pass away, but their deeds abide*".

The New Mathematics

A tidal wave of mathematical inventiveness and novelty began to sweep the European continent at the dawn of the 19^{th} *century. This movement began with a quintet of brilliant minds:* ***N.H. Abel*** *(1824),* ***W.R. Hamilton*** *(1818),* ***C.G.J. Jacobi*** *(1829),* ***E. Galois*** *(1829) and* ***P.G.L. Dirichlet*** *(1829).*

With them the centroid of mathematical activity shifted from France to Germany. During the post-Newtonian era (ca 1740–1819)[10]*, the leading French school of mathematics was strongly tied to problems of Newtonian mechanics and astronomy. The French revolution, with its ideological break from the past and its many sweeping changes, created favorable conditions for the growth of mathematics. Thus in the* 19^{th} *century mathematics underwent a great forward surge, first in France and later, as the motivating forces spread over Northern Europe, in Germany, and still later in Britain.*

The new mathematics began to free itself from its narrow interest in mechanical problems, and a more general and abstract outlook evolved. The great mathematicians of the 19^{th} *century seem to be almost of a different species from their predecessors. They were not content with* special problems*, but attacked and solved* general problems *whose solutions yielded those of a multitude of problems which, in the* 18^{th} *century, would have been considered one by one.*

The 60 *years starting with the pivotal* 1829*, witnessed a most extraordinary phenomenon in the history of human thought: mathematicians prepared for the physicists of the* 20^{th} *century most of the mathematical tool needed to model the world of* 20^{th} *century physics: groups, matrices, vectors, quaternions, sets, non-Euclidean geometry, integral transforms, integral equations, partial differential equations, special functions and symbolic algebras.*

Indeed, during 1828–1893*, these weapons were forged by armies of enthusiastic workers that wheeled into the front ranks of analysis, geometry, algebra, theory of functions, theory of numbers and applied mathematics, under the leadership of* ***G. Green*** *(1828)*[11]*,* ***N.I. Lobachevsky*** *(1832),* ***J. Bolyai*** *(1832),* ***A.F. Möbius*** *(1832),* ***J.J. Sylvester*** *(1834),* ***J. Liouville*** *(1837),* ***H.G. Grassmann*** *(1844),* ***L. Kronecker*** *(1845),* ***J.B. Listing***

[10] The discovery of electromagnetism (1820), which marked a new era in *physics*, is contemporary with the rise of the new *mathematics* whose harbingers were **Poncelet** (1822) in geometry and **Abel** (1824) in *analysis*.

[11] Year in parenthesis indicates *beginning* of activity.

(1847), ***K.T.W. Weierstrass*** *(1848),* ***E.E. Kummer*** *(1850),* ***G.F.B. Riemann*** *(1851),* ***A. Cayley*** *(1854),* ***G. Boole*** *(1854),* ***S. Lie*** *(1870),* ***J.W.R. Dedekind*** *(1872),* ***G.F.L.P. Cantor*** *(1872),* ***W.K. Clifford*** *(1873),* ***J.W. Gibbs*** *(1876),* ***O. Heaviside*** *(1881),* ***G. Darboux*** *(1887),* ***V. Volterra*** *(1887),* ***G. Ricci-Curbastro*** *(1887),* ***A.M. Lyapunov*** *(1892) and* ***K. Pearson*** *(1893).*

The very foundations of mathematics were re-examined, and fundamental principles were worked out anew. The terms number, function, limit, continuity, infinity and integral were given more precise meaning. One of the phases of the quest for rigor was the re-defining of mathematics. An old idea which goes back to Aristotle defined mathematics as the science of quantity.

Auguste Comte *(1798–1857, France), philosopher and mathematician, defined mathematics as "the science of indirect measurement". These definitions had to be abandoned, however, since the modern branches of mathematics, such as the theory of groups, topology, projective geometry, theory of numbers and the algebra of logic, have no a priori relation to quantity and measurement. Moreover, the continuum, the central supporting pillar of modern analysis, as constructed by K. Weierstrass, R. Dedekind and G. Cantor, did not bear any reference whatsoever to quantity. In this light we understand the definition of* ***Benjamin Peirce*** *(1870) that "mathematics is the science which draws necessary conclusions".*

Thus, reasoning which seemed absolutely conclusive to one generation, no longer satisfied the next.

1822 CE The first meeting of the association of German speaking scientists and doctors (*Deutscher Naturforscher Versammlung*), was held at Leipzig. Some 20 scientists who had published work attended, together with 60 guests. First viewed with suspicion[12], these congresses later became instruments for increasing national unity in Germany; subsequent meetings, which were held

[12] Being both liberal and nationalistic in tone, the first congresses drew the suspicion of the rulers of the German states. Members attending the 1^{st} meeting refused to allow their names to be recorded, for fear their governments should find out. **Metternich** is Austria suggested to Viennese scientists applying for passports that it would be contrary to their own interests to go, with the result that the Austrian scientists were not represented until 1832, when the annual meeting was held at Vienna.

annually in one or other of the main German cities, gradually grew larger: some 600 attending the 1828 meeting at Berlin, and a 1000 the 1842 meeting at Mainz. Quite early on, the Prussian government saw that the national science congresses could become a controlled force for German unity, and it extended patronage to the meetings from 1828 on. Thereafter the congresses came more under the control of the German governments, with the state acting as host for a particular annual congress, appointing the president of the congress for that year and the secretary who organized and ran the meetings.

The idea of the association was conceived and realized by **Lorentz Oken** (Ockenfuss, 1779–1851), a Swabia-born German naturalist (who in 1806 elaborated on Goethe's theory that the skull in vertebrates evolved from enlargement and fusion of Vertebrae; this theory was disproved in 1858 by **Thomas Huxley**).

1822–1826 CE **Jean Victor Poncelet** (1788–1867, France). A mathematician and engineer. One of the founders of modern *projective geometry*. Born at Metz. From 1808–1810 he attended the École Polytechnique and was a pupil of **Monge**. In 1812 he became lieutenant of engineers and took part in the Russian campaign, during which he was taken prisoner and confined at Saratov on the Volga. It was during his imprisonment that he began his researches on projective geometry, which led to his great treatise on that subject: "*Traité des propriétés projectives des figures*" (1822), which is a study of those properties which remain invariant under projection. It contains fundamental ideas such as the *cross-ratio*, *perspective*, *involution*, and circular points at infinity. In 1826, Poncelet discovered the *principle of duality*. It was applied in 1826 by **Joseph Diaz Gergonne** (1771–1859) to the theorem of **Desargues**, and proved by **Plücker** in 1829.

From 1815 to 1825 Poncelet was occupied with military engineering at Metz and from 1825 to 1835 he was professor of mechanics at the École d'Application there. From 1838 to 1848 he was professor at the faculty of science at Paris and from 1848 to 1850, director of the École Polytechnique.

1822 CE Chemist **Georges-Simon Serulas** discovered iodoform and its antibacterial action.

1822 CE **Thomas Johann Seeback** (1770–1831, Germany). Physicist. Invented the *thermocouple* and discovered *thermoelectricity*; he showed, in his Berlin laboratory, that a temperature difference between the junctions of two different metals in a closed circuit can create an electric current. But, because of a mistaken interpretation of what was involved, he did not do any practical application for it. Thermoelectricity lay undisturbed for over a hundred years like a Sleeping Beauty. The Prince that awoke her was the semiconductor.

1822 CE **Joseph Nicèphore Niépce** (1765–1833, France). Physicist. Was the first person to make a permanent photographic image[13]. He exposed a light-sensitive metal plate in a camera, and then used an engraving process to "fix" the image to obtain what could be called a "photograph". A photograph he made in 1826 still exists today.

In 1839, **Louis Jacques Mandé Daguerre** (1787–1851, France) produced the first popular form of photography, the *Daguerreotype*. He based his process on Niépce's work, exposing a light-sensitive metal plate and developing the image with mercury vapor. He then 'fixed' the image with common salt. Also in 1839, **William Henry Fox Talbot** (1800–1877, England) invented a light-sensitive paper. This paper, coated with salt and silver-nitrate, produced a negative image from which positive prints could be made. This was the first negative-positive system of making photographs. The astronomer **John Frederick William Herschel** (1792–1871, England) named the invention *photography* and suggested the use of sodium thiosulphate (hypo) as a fixing agent.

1822–1842 CE **Eilhard Mitscherlich** (1794–1863, Germany). Physical and organic chemist. Discovered the phenomenon of *isomorphism* ("the same shape"), namely — that substances with similar chemical composition may have the same shape of crystal (1822). He demonstrated it with crystals of potassium arsenate and potassium phosphate and with some of the sulphates. He further noticed that sulphur forms either rhombic or monoclinic crystals, and this led him to the discovery of *dimorphism*, the capacity of some elements to occur in two distinct forms. He synthesized nitrobenzene in the laboratory (1832), which he termed *benzine* (1834). He also synthesized artificial minerals by fusing silica with various metallic oxides. He showed that yeast (which in 1842 he identified as a microorganism) can invert sugar in solution.

Mitscherlich was born in Jever, Lower Saxony, and entered Heidelberg University to study Oriental languages, but had to abandon it with the fall of Napoleon. He instead studied science at Göttingen and then worked with the Swedish chemist **Jöns Berzelius** in Stockholm for two years. He became professor at the Friedrich Wilhelm Institute in Berlin (1825).

1823 CE **Michel Eugène Chevreul** (1786–1889, France). Chemist. One of the founders of modern organic chemistry. Elucidated the true nature of

[13] Niépce "almost" discovered *radioactivity*! He knew that uranium salt may darken a metal plate but failed to understand what he was doing.

soap and explained clearly for the first time the reaction of *saponification*[14] (soap formation) in a classical paper (1823): "*Recherches chimiques sur les corps gras d'origine animale*". In this work he established the fact that soap is formed by a combination of alkali with an acid constituent of the fat[15], the other constituent (glycerol) being set free. It was one of the first works ad-

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14 *Saponification* is the reaction between an ester (fat; fatty acid) and a base (such as NaOH or KOH) in aqueous solution to form an alcohol and a salt (soap), e.g.,

$$\mathrm{H_3CCH_2{-}O{-}\overset{\overset{\displaystyle O}{\|}}{C}{-}CH_3} + (\mathrm{Na^+})_{aq} + (\mathrm{OH^-})_{aq} \rightarrow \mathrm{H_3CCH_2OH} + \mathrm{CH_3{-}C}\begin{matrix} \mathrm{/\!/\,O} \\ \mathrm{\backslash\, O^-Na^+} \end{matrix}$$

ethyl acetate — sodium hydroxide — ethyl alcohol (ethanol) — sodium acetate (soap)

Another typical soap molecule is *sodium laurate* $CH_3(CH_2)_{10}COO^-Na^+$. The long hydrocarbon chain in this molecule is *nonpolar* and does not dissolve well in water. This end (tail) of the molecule is called *hydrophobic* ("water hating"). On the other hand, the polar end of the molecule (head) $-COO^-Na^+$ is a salt, and dissolves well in water. It is *hydrophilic* ("water loving") and therefore gives soap its stability.

The hydrocarbon tails of the molecules can mingle with the grease, which is typically a mixture of fats and oils. As a result, the surface of a grease droplet becomes surrounded by a sheath of head groups, which do not mix with the grease. The head groups form hydrogen bonds with water, which lifts them and the droplet and washes them away from the object. Thus, soap acts as an enveloping coating over the grease and dirt particles.

Detergent is the name used for synthetic substances that are not soaps but have similar properties. In the most common type, the polar carboxylate end of the soap molecule is replaced by a sulfate group. A typical detergent molecule is sodium dodecylsulfate

$$\mathrm{CH_3(CH_2)_{10}CH_2{-}O{-}\underset{\underset{\displaystyle O}{\|}}{\overset{\overset{\displaystyle O}{\|}}{S}}{-}O^-Na^+}$$

Whereas soaps tend to precipitate in hard water (i.e., water containing bicarbonates of such divalent metals as Mg or Ca), leaving a ring of Mg soap or Ca soap around a bathtub, most detergents do not, and are therefore superior for use in hard water. Most detergents are not *biodegradable*, however, so in recent years their environmental impact has become the source of increasing concern.

15 This fact was asserted already by the apothecary and medical chemist **Otto**

dressing the issue of the fundamental structure of a large class of compounds. In 1825, Chevreul and Gay-Lussac patented a method of making candles from fatty acids; these candles were a great improvement on tallow (fats of cattle and sheep) candles, then commonly in use.

Chevreul was born at Avigers, a son of a physician. He came to Paris (1803) and studied under **Vauquelin**. He later occupied technical positions in Paris, including the directorship of the famous Goblin tapestry works. Chevreul became (1826) member of the Academy of Sciences and foreign member of the Royal Society of London. He subsequently became professor of organic chemistry in the Natural History Museum (1830). As a result of the researches of Gay-Lussac, Vauquelin and Chevreul, Paris became a center of work and research in the new science of organic chemistry. His completion of his 100^{th} year was celebrated with public rejoicing and after his death he was honored with a public funeral.

Soap-making is one of the oldest chemical syntheses: *soap* both as medicinal and as a cleansing agent was known to **Pliny**, who mentions it as originally a Gallic invention for giving a bright hue to the hair. Thus, there is reason to believe that soap came to the Romans from Germany. *Detergents*, however, were in use in earlier times and mentioned as soap in the Bible (*Jer* **2**, 22; *Mal* **3**, 2; etc.). These refer to the alkali-rich ashes of plants and other such purifying agents.

Phosphates are added to detergents to provide the optimal acidity for the functioning of the *surfactant* (surface-active agents like soap) molecules, to remove calcium and magnesium ions by wrapping around them and hiding them away from other ions with which they precipitate and form a scum, and to attache to dirt particles that have been washed off the fabric to prevent them from redepositing. Unfortunately, since phosphates are fertilizers, the waste water from a load of wash is highly nutritious and can promote the growth of microorganisms in rivers and lakes. This can lead to *eutrophication*, or overnourishment, which leads to clogging by organic growth, perhaps to the point of transforming a lake into a swamp.

1823–1828 CE **Niels Henrik Abel** (1802–1829, Norway). A great mathematician of the 19^{th} century with a meteoric career. Abel was born in Findo, Norway. His father was a country minister of considerable culture. His mother's outstanding characteristic seems to have been her beauty. Burdened with the support of his mother and five brothers and sisters when he was only 18, Abel struggled to take care of them and pursue his mathematical studies at the University of Christiania.

Tachenius (1620–1699, Germany) in his book *Hippocrates Chimicus* (1666). In this book he also stated that every salt is composed of acid and alkali.

The first explicit appearance of *integral equations* in the history of mathematics[16] was in Abel's thesis (1823) on tautochrones. His first notable work was a proof of the impossibility of solving the *quintic equation* by radicals (1824–1826).

State aid enabled him to visit Germany and France in 1825. In Freiburg he made his brilliant researches in the *theory of elliptic, hyperelliptic and Abelian functions*. He came to Paris in 1826, and during a ten months' stay he met the leading French mathematicians, but he was little appreciated, for his work was scarcely known. Pecuniary embarrassments, from which he had never been free, finally compelled him to return to Norway. When he realized that he was dying of tuberculosis, he praised the good qualities of his fiance Crelly to his friend Kielhan, and indeed Kielhan did marry Crelly after Abel's death.

In 1829, August Crelle (1780–1855) was able to secure for him an appointment as professor of mathematics at the University of Berlin, but the offer did not reach Norway until after his death. His premature death at the age of 26 cut short a career of extraordinary brilliance and promise.

1823–1839 CE **Jan Evangelista Purkyne** (**Purkinje**) (1787–1869, Bohemia). Physiologist. Known for observations and discoveries in physiology and microscopic anatomy. Professor in Breslau (1823–1850) and Prague (1850–1869). Director (1839) of the first institute of physiology in Breslau (Wroclaw, Poland).

His main contributions:

- Developed first system for classifying *fingerprints* (1823) and recognized fingerprints as means of identification.
- Discovered sweat glands (1833).
- Noted that animal tissues are made from cells (1835).
- Outlined the key features of the *cell* theory (to be more fully propounded by **Schwann** in 1839). Discovered ciliary movements in vertebrates; a class of pear-shaped cells in the middle layer of the cerebellar cortex are known as *Purkyne's cells* (1837).
- Observed *cell division* under the microscope (1838).

[16] There is an opinion that the first appearance of an integral equation was marked by **Laplace** (1782), when he introduced the *Laplace transform*.

- Discovered networks of fibers made up of large muscle cells in cardiac muscles, known as *Purkyne's network*, system or tissue (1839). Promoted the word *protoplasm* in the modern sense.
- Discovered ganglionic bodies in the brain.

1823–1855 CE **Justus von Liebig** (1803–1873, Germany). Chemist. Discovered (1823) the concept of chemical *isomers* (compounds with the same chemical composition but very different properties). The name *isomer* (Greek for *equal parts*) was coined in 1830 by **Jöns Jacob Berzelius** (1779–1848, Sweden), who also discovered *silicon* in 1823.

Liebig discovered the composition of many organic compounds. Made a systematic organization of organic chemistry, based on the radicals (radicals in organic chemistry act analogously to the elements in mineral chemistry, with the same general principles of combination and reaction). Discovered *chloroform* ($CHCl_3$) in 1831, and explained the theory of exchange of carbon and nitrogen in plants and animals (1840). Founded agricultural chemistry (chemistry of fertilization).

Liebig was born in Darmstadt. Studied at the Universities of Bonn and Erlangen and graduated as Ph.D. in 1822. He then went to Paris and practiced chemistry under **Gay-Lussac**. In 1826 he became a professor of chemistry at Giessen. In this small town his most important work was accomplished. There he established a new chemical laboratory, unique of its kind at the time, that soon rendered Giessen the most famous school in the world. It gave a great impetus to the progress of chemical education throughout Germany. In 1852, Liebig accepted the chair of chemistry at Munich University.

1824–1844 CE **Friedrich Wilhelm Bessel** (1784–1846, Germany). Astronomer. As director of the Königsberg observatory (1810–1846) he inaugurated the modern era of precision astronomy. Determined the positions and proper motions of over 50,000 stars and discovered the parallax of 61 Cygni. In 1838 Bessel made the first authentical measurement of a star's distance from the sun, using stellar parallaxes. This was achieved by employing an instrumental technique that enabled measurements of angles to a fraction of an arc second. In 1831–1832 Bessel measured an arc of the meridian in East Prussia and deduced for the earth's figure, in 1841, an ellipticity of $\frac{1}{299}$.

In 1844 he discovered the companion of Sirius[17] (called Sirius B), a star of the type known later as a *white dwarf*. Bessel *deduced* the binary nature of

[17] In 1914, **Walter Sydney Adams** (1876–1956, U.S.A.) succeeded in taking the spectrum of Sirius B, from which he inferred that it was a 'white' star, not very

Sirius by noticing that the star was moving back and forth slightly (wobbling), as if orbited by an unseen object. Only in 1863 was the companion first seen.

Bessel (1835) used dates of disappearances of the *Saturn Rings*[18] (as viewed from earth) in an effort to determine the orientation of Saturn's pole (w.r.t. to its own orbital plane about the sun). To this end he surveyed the astronomical

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different from its companion. He found that it had a surface temperature of ca 10,000 °C, which according to its luminosity, would have been quite *small*. Recent satellite observations at ultraviolet wave lengths showed that the surface temperature of Sirius B is about 30,000 °K.

[18] *Saturn*, whose equatorial diameter is 9.44 earth diameter, orbits at a mean distance of 9.539 AU from the sun with a mean orbital velocity of 9.6 $\frac{\text{km}}{\text{sec}}$ (compared to 29.8 $\frac{\text{km}}{\text{sec}}$ for earth). The Saturnian year is 29.46 earth years and its equatorial rotation period is $10^h 13^n 59^s$. The inclination of its orbit to the ecliptic plane is $2°29'$.

Earth-based views of the Saturnian ring-system (first identified by Huygens in 1655) change dramatically as Saturn orbits slowly about the sun. This change is observed because the rings (which are in the plane of Saturn's equator) are tilted $26°44'$ from the plane of Saturn's orbit. Thus, over the course of Saturn's year, the rings are viewed from various angles by earth-based observers. At one time, the observers look "down" on the rings; one-half of a Saturnian year later, the "underside" of the rings is exposed to view from earth. At intermediate times, the rings are seen *edge-on*, and they then disappear entirely from the view of the earth-based observer.

Saturn turns ringless as the earth passes the plane of the planet's razor-thin rings. Most of the time, however, the ring-plane does not cut the earth's orbit. It does so with a period of 14.7 years, and on that year, there are 3 ring-plane crossings (i.e., 3 disappearances of the rings at unequal intervals) due to the approximate ratio 3: 1 of the orbital velocities of earth and Saturn. Thus, the four last ring-plane crossings were {May 22, 1855; Aug. 10, 1985; Feb. 12, 1986; May 22, 1995}. Not until 2009 will the rings be aligned directly toward us once more. The ring-plane takes about 12 to 28 minutes to sweep across the earth.

Prior to the mid-19^{th} century, observers often imagined the rings to be a thin, solid, opaque disc, divided into two concentric portions by the dark gap of the *Cassini Division*. This was despite **Laplace**'s demonstration (1785) that such broad rings, if truly solid (and thus in a state of uniform rotation) would be torn apart by Saturn's tidal forces. A series of many narrow solid rings could evade Laplace's objection, but **Maxwell** showed (1857) that even such narrow rings would be unstable. Only then did most astronomers reluctantly accept that the rings must be composed of myriads of independently orbiting moonlets. Arguments about *fluid* rings persisted until **Harold Jeffreys**' definitive work (1946) showed that the moonlets must be separate bodies.

literature to collect all reliable ring-plane crossing observations between 1714 and 1833. He even attempted to determine the precession period of the pole, obtaining a value of 340,000 years (the modern estimate is 1.7 million years).

Bessel was born at Minden. Early work on Comet Halley in 1804, which he communicated to **Olbers**, and an investigation of the comet of 1807, enhanced his reputation and he was summoned by the King of Prussia in 1810 to help establish a royal observatory.

In 1819 Bessel introduced into an investigation of *Kepler's problem* the solutions of the differential equation

$$y'' + \frac{1}{x}y' + \left(1 - \frac{m^2}{x^2}\right) y = 0,$$

which now bear his name: '*Bessel functions*'[19].

These functions were known earlier to **Daniel Bernoulli** (1732) [obtained the Bessel function of order zero as a solution to the problem of oscillations of

[19] **Bessel** (1824) discovered the intriguing continued-fraction expansion of a ratio of the modified functions

$$\frac{I_1(2)}{I_0(2)} = \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\cdots}}}}} = \frac{\sum_{k=0}^{\infty}\frac{1}{k+1}\frac{1}{(k!)^2}}{\sum_{k=0}^{\infty}\frac{1}{(k!)^2}}$$

$$= \frac{1.59063685\ldots}{2.27958530\ldots} = 0.697\ 774\ 658\ldots$$

It is interesting that the continued fraction converges, thus differing from the harmonic series in this regard.
To prove the above relationship we write the n^{th} convergent for the continued fraction as $\frac{P_n}{Q_n}$ where

$$P_n = nP_{n-1} + P_{n-2}, \qquad Q_n = nQ_{n-1} + Q_{n-2}, \qquad n \geq 3$$
$$P_1 = 1, \qquad Q_1 = 1, \qquad P_2 = 2, \qquad Q_2 = 3.$$

Now, the modified Bessel functions K_n and I_n obey the recurrence relations

$$\begin{aligned} K_{n+1}(x) &= \tfrac{2n}{x}K_n(x) + K_{n-1}(x) \\ I_{n+1}(x) &= -\tfrac{2n}{x}I_n(x) + I_{n-1}(x). \end{aligned}$$

a chain[20] suspended at one end] and **Leonhard Euler** (1764), who employed

Hence $A_n = K_{n+1}(2)$ and $A_n = (-)^{n+1} I_{n+1}(2)$ are independent solutions of $A_n = nA_{n-1} + A_{n-2}$. Consequently

$$\begin{aligned} P_n &= \alpha K_{n+1}(2) + \beta(-)^{n+1} I_{n+1}(2) \\ Q_n &= \gamma K_{n+1}(2) + \delta(-)^{n+1} I_{n+1}(2). \end{aligned}$$

The values of $(\alpha, \beta, \gamma, \delta)$ are found from the initial conditions and the Wronskian relation $K_{n+1}(x)I_n(x) + K_n(x)I_{n+1}(x) = \frac{1}{x}$ namely,

$$\alpha = 2I_1(2), \qquad \beta = 2K_1(2), \qquad \gamma = 2I_0(2), \qquad \delta = 2K_0(2).$$

But $I_n(2)$ tends to zero as n tends to infinity, while $K_{n+1}(2) \sim \frac{1}{2}n!$ as n tends to ∞. This yields $\frac{P_\infty}{Q_\infty} = \frac{I_1(2)}{I_0(2)}$.

20 In 1781, the problem was taken up by **Euler**, who formalized it as follows (modern notation): A chain (or a massive thread, devoid of flexural rigidity), with line density ρ and total length L, is suspended at one end. Take the origin o at that point, with the x-axis pointing downward along the undisturbed chain, and the y-axis pointing to the right.

Let the chain execute a motion with *small transverse* (y) *amplitude*. At a general point A, the tension makes an angle ψ with ox. The y-component of this tension is $T \sin\psi \approx T\frac{\partial y}{\partial x}$. An adjacent point B, at $x + dx$, experiences a tension differing by $\frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right) dx$. The mass of the element AB is $\rho\, dx$ and its acceleration is $\frac{\partial^2 y}{\partial t^2}$. Moreover, since the oscillations are small, it is sufficiently accurate to take the tension T as the weight of the chain below A; hence $T = \rho g(L - x)$. This renders the equation of motion

$$\rho\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left[\rho g(L - x)\frac{\partial y}{\partial x}\right].$$

Assuming a harmonic motion $y = ue^{i\omega t}$ and denoting $L - x = z$, the equation for the amplitude u becomes

$$\frac{d^2 u}{dz^2} + \frac{1}{z}\frac{du}{dz} + k^2\frac{u}{z} = 0, \qquad k^2 = \omega^2/g.$$

Its general solution is

$$u(z) = aJ_0(2k\sqrt{z}) + bY_0(2k\sqrt{z}).$$

The free-end conditions implies $b \equiv 0$. At the fixed end, $u(x = 0) = 0$ yields $J_0(2k\sqrt{L}) = 0$, which furnishes an equation for the eigenfrequencies $\omega_n = \omega_n(L, g)$. By an extremely ingenious analysis Euler proceeded to obtain the values 1.445795, 7.6658 and 18.63 for the three smallest roots of the period equation [more accurate values are 1.4457965; 7.6178156; 18.7217517].

And here is how he did it: first, he *assumed* that all zeros are distinct and real

$$0 < \alpha_1 < \alpha_2 < \alpha_3 \dots$$

and that it is possible to express it as the infinite product

$$J_0\left(2\sqrt{x}\right) = \prod_{n=1}^{\infty}\left(1 - \frac{x}{\alpha_n}\right).$$

If it is differentiated logarithmically, then

$$-\frac{d}{dx}\log J_0\left(2\sqrt{x}\right) = \sum_{n=1}^{\infty}\frac{1}{\alpha_n - x} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{x^m}{\alpha_n^{m+1}}$$

provided that $|x| < \alpha_1$, and the last series then converge absolutely.

Put

$$\sum_{n=1}^{\infty}\frac{1}{\alpha_n^{m+1}} = \sigma_{m+1}$$

and change the order of summations; then

$$-\frac{d}{dx}J_0\left(2\sqrt{x}\right) \equiv J_0\left(2\sqrt{x}\right)\sum_{m=0}^{\infty}\sigma_{m+1}x^m.$$

Replace $J_0\left(2\sqrt{x}\right)$ on each side by its series expansion

$$1 - \frac{x}{1^2} + \frac{x^2}{1^2 \cdot 2^2} - \frac{x^3}{1^2 \cdot 2^2 \cdot 3^2} + \cdots,$$

multiply out the product on the right, and equate coefficients of the various powers of x in the identity; we thus obtain the system of equations

$$\begin{aligned}
1 &= \sigma_1,\\
-\frac{1}{2} &= \sigma_2 - \sigma_1,\\
\frac{1}{12} &= \sigma_3 - \sigma_2 + \frac{1}{4}\sigma_1,\\
-\frac{1}{144} &= \sigma_4 - \sigma_3 + \frac{1}{4}\sigma_2 - \frac{1}{36}\sigma_1\\
\frac{1}{2880} &= \sigma_5 - \sigma_4 + \frac{1}{4}\sigma_3 - \frac{1}{36}\sigma_2 + \frac{1}{576}\sigma_1\\
-\frac{1}{86400} &= \sigma_6 - \sigma_5 + \frac{1}{4}\sigma_4 - \frac{1}{36}\sigma_3 + \frac{1}{576}\sigma_2 - \frac{1}{14400}\sigma_1,
\end{aligned}$$

whence

$$\sigma_1 = 1, \quad \sigma_2 = \frac{1}{2}, \quad \sigma_3 = \frac{1}{3}, \quad \sigma_4 = \frac{11}{48}, \quad \sigma_5 = \frac{19}{120}, \quad \sigma_6 = \frac{473}{4320}, \dots$$

Since $0 < \alpha_1 < \alpha_2 < \alpha_3 \ldots$, it is evident that

$$\frac{1}{\alpha_1^m} < \sigma_m, \qquad \sigma_{m+1} < \frac{\sigma_m}{\alpha_1},$$

and so

$$\sigma_m^{-1/m} < \alpha_1 < \frac{\sigma_m}{\sigma_{m+1}}.$$

By extrapolating from the following table:

m	$\sigma_m^{-1/m}$	σ_m/σ_{m+1}
1	1.000 000	2.000 000
2	1.414 213	1.500 000
3	1.442 250	1.454 545
4	1.445 314	1.447 368
5	1.445 724	1.446 089
6	1.445 785	—

Euler inferred that $\alpha_1 = 1.445795$, whence

$$\frac{1}{\alpha_1} = 0.691661, \qquad 2\sqrt{\alpha_1} = 2.404824.$$

By adopting this value for α_1, writing

$$\sum_{n=2}^{\infty} \frac{1}{\alpha_n^m} = \sigma'_m,$$

and then using the inequalities

$$\frac{1}{\alpha_2^m} < \sigma'_m, \qquad \sigma'_{m+1} < \frac{\sigma'_m}{\alpha_2},$$

Euler deduced that $\alpha_2 = 7.6658$, and hence that $\alpha_3 = 18.63$, by carrying the process a stage further.

Poisson (1833) calculated α_1 by solving the quadric equation obtained by equating to zero the first five terms of the series for $J_0(2\sqrt{x})$:

$$1 - \frac{x}{1^2} + \frac{x^2}{2^2} - \frac{x^3}{6^2} + \frac{x^4}{24^2} = 0,$$

obtaining $\alpha_1 = 1.446796491$. **Rayleigh** (1874) used the method of Euler to calculate the smallest zero of $J_\nu(x)$.

the functions of both zero and integral orders in an analysis of the vibrations of a stretched membrane. Later, **Schlömilch** (1857) defined these functions as the coefficients of the power of t in the expansion of $\exp\left\{\frac{1}{2}z(t-t^{-1})\right\}$, namely:[21]

$$e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(z)t^n.$$

History of Thermometry (1593–1848)

In Aristotelian philosophy, hot and cold were associated with the terrestrial elements, earth, water, air and fire, which conveyed the sensations of coldness *(earth-water),* wetness *(water-air),* heat *(air-fire), and* dryness *(fire-earth). Therefore, it was natural for Greek scientists, although not too keen on the experimental approach, to investigate the qualitative differences between these elements.*

The first thermoscope *was developed by* ***Philo of Byzantium*** *in* 250 *BCE. This rudimentary device was able to distinguish between a balloon filled with cool air and the same balloon exposed to the heat of the sun.* ***Heron of Alexandria*** *later constructed a more refined thermoscope, but in the social conditions of the Hellenistic world and the Middle Ages that followed, these devices remained solely objects of amusements for more than* 1500 *years.*

The first step in the development of the science of heat was of necessity the invention of the thermometer — an instrument for indicating temperature and measuring its changes[22]*. The first requisite for such an instrument is that it should always give, at least approximately, the same indication at the same temperature. The invention of such a device is generally attributed to* ***Galileo***

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21 To dig deeper, see:

- Watson, G.N., *A Treatise on the Theory of Bessel Functions*, Cambridge University Press: Cambridge, 1966, 804 pp.

22 The invention of the thermometer preceded by some 260 years the notion that temperature is linearly related by the *mean molecular kinetic energy* of a certain kind of system in thermodynamic equilibrium.

at about 1593*. An improved version was made by him in* 1612*. It consisted of alcohol hermetically sealed in a glass bulb, with an attached graduated fine tube. In order to render the readings of such instruments consistent with each other, it was necessary to select a fixed point (standard temperature) at the zero starting-point of the graduations.*

It was soon found preferable to take two fixed points *and to divide the interval between them into the same number of degrees. It was natural in the first instance to take the temperature of the human body as one of the fixed points.*

In 1701*,* ***Newton*** *proposed a scale in which the freezing-point of water was taken to be zero, and the temperature of the human body as* 12°*.*

In 1714*,* ***G.D. Fahrenheit*** *proposed to take as zero the lowest temperature obtainable with a freezing mixture of ice and salt, and to divide the interval between this temperature and that of the human body into* 12°*. To obtained finer graduations, the number was subsequently increased to* $96^\circ = 8 \times 12^\circ$*. The freezing point of water was at that time supposed to be somewhat variable, because as a matter of fact, it is possible to cool water several degrees below its freezing-point, in the absence of ice.*

Fahrenheit showed, however, that as soon as ice began to form, the temperature always rose to the same point, and that the mixture of ice or snow with pure water, always gave the same temperature. At a latter date he also showed that the temperature of boiling water varied with the barometric pressure, but that it was always the same at the same pressure, and might therefore be used as a second fixed point — provided that a definite pressure, such as the average atmospheric pressure, were specified. The freezing and boiling points on one of his thermometers came out in the neighborhood of 32° *and* 212° *respectively, giving an interval of* 180° *between the points. Shortly after his death (1736), the freezing and boiling points of water were generally recognized as the most convenient fixed points to adopt, but different systems of subdivision were employed.*

Fahrenheit's scale, with its small degrees and its zero below the freezing-point, possesses undeniable advantages for meteorological work, and is still retained in most English-speaking countries. But for general scientific purposes, the centigrade system, in which the freezing-point is marked at 0° *and the boiling-point at* 100°*, is now almost universally employed, on account of its greater simplicity from the arithmetical point of view. For work of precision the fixed points have been more exactly defined, but no change has been made in the fundamental principle of graduation.*

In general, any property of a suitable substance which varies as the temperature is changed, can be used to compare temperature differences with

the fundamental interval. For example: the volume of a liquid enclosed in a vessel, the volume of a fixed mass of gas maintained at constant pressure, the pressure of a fixed mass of gas maintained at constant volume, the electrical resistance of a piece of metal, the saturated vapor pressure of a liquid, are among the many measurable physical properties which alter reproducibly as the temperature changes. Any one of these can be made the basis of a temperature scale[23].

It was soon observed that thermometers constructed with different liquids (such as oil, alcohol and mercury) did not agree precisely in their indications at points of the scale intermediate between the fixed points, and diverged even more widely outside these limits.

In 1802 *the research of* ***Gay-Lussac*** *showed that the laws of expansion of gases are much simpler than those of liquids, and that almost all gases expand nearly equally such that the differences between them cannot be detected without the most refined observations. This affords a strong a priori argument for selecting the scale given by the expansion of gas as the standard scale of temperature. Among liquids, mercury is found to agree most nearly with the gas scale, and is therefore used as a secondary standard to replace the gas thermometer within certain limits.*

In 1848, ***Lord Kelvin*** *proposed to take advantage of the fact that the efficiency*[24] *of a reversible Carnot ideal engine is independent of the nature of the working substance and dependent on temperature alone. This, he argued, could serve as a basis for an* absolute temperature scale. *The defining equation* $Q_1/Q_2 = T_1/T_2$ *does not, however, prescribe the numbering of the scale, nor indicate how the scale is to be realized in practice.*

If, however, T_0 *is the temperature of the ice-point and* $T_0 + 100$ *that of the steam-point, then if* Q_0 *and* Q_{100} *are the quantities of heat absorbed and ejected by the Carnot engine between the ice-point and the steam-point*

$$(T_0 + 100)/T_0 = Q_{100}/Q_0.$$

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[23] Consider any one quantity, the magnitude x of which changes linearly with temperature. Then $T = 100\frac{x_T - x_0}{x_{100} - x_0}$, where x_0, x_{100}, x_T are the respective values of x at the ice-point, the steam-point and the unknown temperature T. In a mercury thermometer, for example $x = \ell$, where ℓ is the length of the mercury column above the bulb. For an ideal gas $x = PV$ (P = pressure, V = volume). This leads to the *ideal gas temperature scale.*

[24] If a Carnot engine takes in a quantity of heat Q_1 at temperature T_1, and ejects a quantity of heat Q_2 at a lower temperature T_2, its efficiency is $\eta = \frac{Q_1 - Q_2}{Q_1} = \frac{T_1 - T_2}{T_1}$ and hence $\frac{Q_1}{Q_2} = \frac{T_1}{T_2}$. This defining equation is the basis upon which the Kelvin ($^\circ$K) scale of temperature is founded.

Assuming that both Q_{100} and Q_0 can be measured, this equation enables us to find T_0, the thermodynamic absolute temperature of the ice point, and the grading of the scale is thus settled.

To realize the absolute scale in practice, the working substance of the Carnot engine is chosen as a unit mass of an approximately ideal gas. It is then shown that the Kelvin Absolute Thermodynamic Scale of temperature and the ideal gas scale of temperature are identical — both giving the one truly absolute scale of temperature.

1824 CE **Henri J. Paixhans** (1783–1854, France). Artillerist. Introduced the *shell gun*, a revolutionary invention in the history of warfare.

1824 CE **Sadi Nicolas Léonhard Carnot** (1796–1832, France). A French engineer. The founder of the science of thermodynamics. An original and profound thinker of the foremost rank, whose full stature was not recognized until pointed out by **Lord Kelvin** in 1848.

The only work he published was *Réflection sur la puissance motrice du feu et sur les machines propres a developer cette puissance* (Paris, 1824) [Reflections on the Motive Power of Heat and on Proposed Machines to Develop that Power].

In this manuscript Carnot described a working cycle, now called a *Carnot cycle*, that is of great importance from both a practical and a theoretical viewpoint. Paving the way to the forthcoming Second Law of Thermodynamics, he gave the first substantial theory of heat engines.[25]

The steam engine was well known to Carnot[26]. He knew that it had been made increasingly efficient over the years, and he wondered whether there

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[25] For further reading, see:

- Noakes, G.R., *A Text-Book of Heat*, Macmillan, 1945, 469 pp.
- Rocard, Y., *Thermodynamics*, Pitman and Sons: London, 1961, 681 pp.
- Sommerfeld, A., *Thermodynamics and Statistical Mechanics*, Academic Press: New York, 1955, 401 pp.
- Bruhat, G., *Thermodynamique*, Masson and C^{ie}, 1947, 428 pp.

[26] The first real steam *engine* was invented by **Newcomen** in 1705. It was an inefficient contraption until **James Watt** introduced, during 1769–1774, a number of innovations which made the steam engine a practical device. **Fulton**'s steamboat was driven by one of Watt's engines in 1807. Another heat engine that

was some limit to its improvement. He appreciated that real steam engines leaked steam and that friction reduced their efficiency. So he imagined the *ideal engine*, one that we call reversible, on the basis of which he formulated the problem in exactly the right way.

Carnot showed that such a heat engine operating in an ideal, reversible cycle between two heat reservoirs, would be the most efficient engine possible. Such an ideal engine, called a *Carnot engine*, establishes an upper limit on the efficiencies of all engines. That is, the net work done by the working substance taken through the Carnot cycle is the largest possible for a given amount of heat supplied to the working substance. This efficiency is found to be $\eta = 1 - T_l/T_h$, where $\{T_l, T_h\}$ are the low and high temperature limits of the working substance [Carnot's theorem].

In addition to the frustration encountered by any inventor who tries to contravene the law of conservation of energy, another fundamental limitation governs all engines designed to convert heat into mechanical work or other forms of useful energy: it is impossible to construct an engine that will do work by extracting heat from a *single* heat reservoir.

For example, universal frustration is met by inventors who want to construct a device consisting of a box which, when immersed into the ocean, will via some mechanism inside the box convert the heat content of the ocean into some other form of energy. Such a hypothetical device, which is no way violates the law of conservation of energy, is called a perpetual motion engine "of the second kind", to distinguish it from perpetual motion engines "of the first kind" (which do not conserve energy).

The origin of this impossibility can be traced to the fact that to transfer heat from a reservoir at lower temperature to a reservoir at higher temperature requires *additional* work. Indeed, if the perpetual engine were feasible one could, in principle, use the work drawn from the box to boil some water taken from the ocean. The net effect would resemble an experiment in which the water in a kettle placed on a stove would freeze by transfer of heat to the stove, which would become hotter. This sort of thing does not happen in nature.

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influenced Sadi Carnot was due to the French engineer, physicist and inventor **Charles Cagniard de la Tour** (1777–1859) who reported (via Lazare Carnot) to the Academy of Sciences in 1809 on his novel invention: his "buoyancy engine" relied on air expanding in a liquid to produce work. Cagniard was born in Paris and studied at the École Polytechnique and the École du Genie Geographie. Between 1809 and 1838 he made several inventions (the *cagniardelle*, a forced-draft blowing machine and a *siren* for acoustical studies) and worked on crystallization and fermentation.

The most efficient conversion of heat into work is effected by a *reversible* engine, operating between *two* reservoirs of temperatures T_1 and T_2. Such an engine can convert at best only a part W of the amount of heat Q_1 drawn from the reservoir at T_1 into work; the balance $Q_2 = Q_1 - W$ must be transferred as heat to the other reservoir at T_2. An engine is "reversible" if none of its moving parts generates any heat by friction, and if all heat transfers between different parts of the engine take place "isothermally", i.e., only between parts that do not differ in temperature by more than an infinitesimal amount.

The classic example of such an ideal device is provided by Carnot's cyclic engine, which does work in four strokes of a frictionless piston with an ideal gas as working substance. Carnot noticed that the efficiency of such an engine depends on the two temperatures *only*, and is independent of the working substance. This observation is now generally known as "Carnot's theorem". It enables one to *define* an absolute temperature scale by writing $W = \frac{T_1-T_2}{T_1}Q_1$ or $Q_1 = \frac{T_1}{T_1-T_2}W$, where all temperatures are measured in the absolute scale. The factor $(T_1 - T_2)/T_1$ is called the ideal efficiency and is, obviously, always ≤ 1. By measuring this efficiency for an engine operating between various heat reservoirs one can, in principle, establish the absolute temperature scale experimentally. Indeed, the efficiency of a reversible engine approaches 1 as the temperature of the colder reservoir approaches the absolute zero of temperature ($0\,^\circ\mathrm{K} = -273\,^\circ\mathrm{C}$).

Another instructive way of looking at the above definition of the absolute temperature scale is obtained if one draws from engineering experience the following inference: the amount of heat Q_1 extracted by a given reversible engine from the hotter reservoir does not depend on what happens to the heat later, and is thus the same for given T_1 independent of the value of T_2. Similarly, the amount Q_2 of heat delivered by a given engine to the reservoir at T_2 does not depend on the value of T_1.

If this is accepted as empirically obvious, then one can arrive at an absolute temperature scale by first introducing as standard $T_S = 1\,^\circ$, the temperature of a reservoir that accepts from the given engine a standard amount of heat $Q_S = S$ (say), and then defining T_0 as the temperature of a reservoir from which the engine draws T times the standard amount S, namely, $Q = TS$. Since, by definition, this is true for all temperatures, one has, specifically, $\frac{Q_1}{T_1} = \frac{Q_2}{T_2} = \frac{Q_S}{T_S} = S$.

The law of conservation of energy as applied to a reversible engine, $Q_1 = Q_2 + W$, allows one to write the above equation in the form $\frac{T_2}{T_1}Q_1 = Q_1 - W$ or $W = \frac{T_1-T_2}{T_1}Q_1$. One then obtains an expression for Q_2 in terms of W, $Q_2 = \frac{T_2}{T_1-T_2}W$ or $W = \frac{T_1-T_2}{T_2}Q_2$.

Since these relations must hold even if the engine is run in reverse, so that it acts as a heat pump, the above equation can be used to define the factor $(T_1 - T_2)/T_2$ as the ideal inefficiency (sometimes called "performance" coefficient) of a heat pump. The lower the temperature T_2 of the colder reservoir, the larger the unavoidable inefficiency of the pump; it will take more and more work W to transfer the amount Q_2 out of T_2 into T_1 as T_2 is made lower and lower.

For example, a steam engine operating between a reservoir at $T_1 = 127\,^\circ\mathrm{C} = 400\,^\circ\mathrm{K}$ and outside air at $T_2 = 27\,^\circ\mathrm{C} = 300\,^\circ\mathrm{K}$ cannot exceed the efficiency $[(400 - 300)/400] = \frac{1}{4}$. A refrigerator operating between an ice box at $T_2 = -3\,^\circ\mathrm{C} = +270\,^\circ\mathrm{K}$ and the kitchen at $T_1 = 27\,^\circ\mathrm{C} = 300\,^\circ\mathrm{K}$ cannot have an inefficiency less than $[(300 - 270)/270] = \frac{1}{9}$.

The beauty of Carnot's result is that it does not depend on the particular design of the reversible engine. All reversible engines must have the same efficiency, independent of the working substance, which may be a gas as in the case of the steam engine, or consist of electrons as in the case of a thermoelectric device. To see this, suppose the assertion were not true. Then there should be an engine 1 giving work W and another engine 2 giving work $\widetilde{W} > W$. By running engine 1 backward with the work W delivered by engine 2, one would have as net result an engine that delivers the work $\widetilde{W} - W$ by drawing this amount of heat from a single reservoir T_2, contrary to the observed impossibility of perpetual motion of the second kind. To avoid this inconsistency one must conclude $\widetilde{W} \leq W$, which proves the assertion[27].

Thus all heat engines convert only part of their heat intake into work, and discard the remainder into the surrounding medium. This limitation is not contained within the First Law of Thermodynamics, nor does it result from imperfections in the engines. This suggested that there must be a Second Law of Thermodynamics which imposes limits not expressed by the First Law. Indeed, the Second Law of Thermodynamics is *implied* here in the assumption that energy cannot flow spontaneously of its own accord from a colder to a hotter body [such a flow *happens* in refrigerators, but at the expense of additional *external* electrical energy and is thus *not* spontaneous]. This law was stated explicitly by **Rudolf Clausius**, 25 years later.

[27] **Maxwell** found his kinetic theory of gases to be in conflict with the ideas of Carnot and he postulated a hypothetical situation where *intelligence* could contradict Carnot's principle (Maxwell's demon). Maxwell then correctly concluded that the second law is of *statistical nature*. In the twentieth century, it has been realized (**Szilard** and **Brillouin**) that the doings of the hypothetical sorting demon involve information processing, itself requiring some energy input.

Sadi Carnot was born in Paris. He entered the École Polytechnique in 1812. Later he served as officer in the French army, but was unemployed in his profession because of his father's political activity. He then devoted himself to mathematics, chemistry, natural history, technology, music, art and athletic sports. He became Captain in the Engineers in 1827, but left the service altogether in 1828.

His naturally feeble constitution, further weakened by excessive study, finally broke down in 1832. An attack of scarlatina led to brain fever, and he had scarcely recovered when he fell victim to cholera, of which he died in Paris at the mere age of 36.

A quotation from his memoir is appropriate:

"*The steam engine works our mines, impels our ships, excavates our ports and our rivers, forges iron... Notwithstanding the work of all kinds done by steam engines, notwithstanding the satisfactory condition to which they have been brought today — their theory is very little understood*".

1824–1859 CE **Johann Franz Encke** (1791–1865, Germany). Astronomer. Used the data from a Venus transit to deduce a sun-earth distance of 153,200,000 km[28] (1824). Encke studied the comets of 1680 and 1812, the orbit of a comet that now bears his name (1818) and the motion of asteroids. He expounded a method of determining an elliptic orbit from 3 observations (1849).

Encke was born at Hamburg. Graduating from the University of Göttingen in 1811, he devoted himself to astronomy under **Carl Friedrich Gauss**. He enlisted in the Hanseatic Legion for the campaign of 1813–1814 and became lieutenant of artillery in the Prussian service in 1815. He returned to Göttingen in 1816 to start his astronomical observations in the Seeberg Observatory near Gotha. He visited England in 1840.

1825 CE **Thomas Drummond** (1797–1840, England). Engineer and administrator. Invented the limelight lamp: an intense beam of light focused by a parabolic mirror and produced by burning lime in an alcohol flame enriched by addition of oxygen.

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[28] This was too high by a little over 3.2 million km. In 1931 the asteroid *Eros* was scheduled to approach earth to within a distance of only about $\frac{2}{3}$ that of Venus. Since Eros held no atmosphere to fuzz its outlines, its position could be determined with great accuracy. An international project was set up to determine the position and parallax of Eros and it was found that the *average sun-earth distance* is just a bit less than 149,079,000 km [at perihelion 146,514,000 km and at aphelion 151,644,000 km].

1825–1836 CE **William Sturgeon** (1783–1850, England). Electrical engineer and inventor. Built the first *electromagnet* capable of supporting more than its own weight (1825). This device led to the invention of the telegraph, the electric motor, and numerous other devices basic to modern technology. He built an electric motor (1832) and invented the *commutator*, an integral part of most modern electric motors. He invented the first *suspension coil galvanometer* (1836), a device for measuring current.

Sturgeon was born in Whittington, Lancashire, England.

1826 CE **August Leopold Crelle** (1780–1855, Germany). Civil engineer and mathematical enthusiast who made various discoveries in the geometry of the triangle (1816). A unique figure in the annals of mathematics. Founded (1826) a new periodical devoted exclusively to mathematics, the *Crelle's Journal der Mathematik*. He started it off by publishing a whole series of papers by **Abel**, including the great one that proved the unsolvability of the general 5^{th}-degree equation. Thus Crelle was able to give an international circulation to Abel's first important results, while Abel could supply papers of a quality that ensured the success of the new journal.

Crelle constructed most of the Prussian highroads (1816–1826) and planned the Berlin-Potsdam railway. He published a German translation of **Legendre**'s geometry (1822) and **Lagrange**'s mathematical work (1823–1824).

1826 CE The last recorded *auto-da-fé* of the *Spanish Inquisition* took place in Valencia, Spain, where a Jew and a Quaker were tortured to death. Between 1481 to 1826, there were 2000 autos-da-fé with about 30,000 persons (mostly *Jews*) burned alive at the stakes.

1826–1837 CE Repeated outbreaks of *cholera* ravaged Europe; millions perished; Ca 900,000 in 1831 alone.

1826–1837 CE **Henri** (**René Joachim**) **Dutrochet** (1776–1847, France). Physician and physiologist. First to discover the quantitative dependence of the *osmotic pressure* on the difference of *concentrations* over the two sides of the membrane (1826). In 1837 he discovered that carbon dioxide is absorbed only by those plant cells that contain green pigment and only in the presence of light.

He was born at Chateau de Néon (Indre). In 1802 he began the study of medicine at Paris, and was subsequently appointed chief physician to the hospital at Burgos. He returned to France in 1809 and dedicated himself to the natural sciences.

1826–1842 CE **Peter Gustav Lejeune Dirichlet** (1805–1859, Germany). One of the eminent German mathematicians of the 19^{th} century. At **Gauss**' death in 1855 he was appointed his successor at Göttingen, a fitting honor for a mathematician who was Gauss' former student and a lifelong admirer of his mentor.

Dirichlet was born in Duren, Germany. As a young man he attended a Jesuit college in Cologne, where one of his teachers was **Georg Simon Ohm** (1787–1854). In 1822 he went to Paris to learn from the great French masters Laplace, Legendre, Fourier and Cauchy. In particular, he found the work of Fourier appealing. In 1828 Dirichlet moved to Berlin to teach mathematics at the military academy. In 1831 he was made a member of the Berlin Academy and married Rebecca Mendelssohn, the sister of the composer. While at Göttingen, he hoped to finish Gauss' incomplete works, but his early death in 1859 prevented this.

At the time when the hegemony in mathematics moved from France to Germany, Dirichlet, being fluent in both German and French, served as liason between mathematicians of the two nationalities.

In 1829, Dirichlet found a sufficient condition for the convergence of Fourier series which suffices for practical purposes and covers a wide class of functions, including functions with discontinuities. This theorem is of paramount importance in harmonic analysis of physical signals. The undertaking led him to generalize the classical function concept [through what we call today the *Dirichlet function*[29] $\{\sin \lambda t/\pi t\}$] and derive the *Dirichlet conditions*.

The name of Dirichlet is associated with a number of other topics: *Dirichlet test* for uniform convergence, *Dirichlet theorem* (1826) on primes [every arithmetic progression in which the first term and the common difference are primes contains an infinite number of primes] and the *Dirichlet boundary value problem* for the Laplace equation [solve $\nabla^2\psi = 0$ inside (outside) V such that ψ takes prescribed values f on the boundary of V]. The Dirichlet problem is related to the calculus of variations because the solution of the Dirichlet interior problem minimizes the integral $I = \int_V |\nabla\psi|^2 d\tau$. This is known as the *Dirichlet principle*.

Weierstrass later disagreed with **Riemann** about the automatic existence of a function which makes this integral minimum, but **Hilbert** later showed that provided certain conditions on f are satisfied, Dirichlet's variational problem always possesses a solution. The value of the method lies in the fact that in certain cases "direct methods", (i.e. methods which do not reduce the variational problem to one in differential equations), may produce

[29] A hundred years later, the limit of this function as λ tends to infinity, became known as one of the useful representations of the 'delta-function'.

a solution of the variational problem more easily than the classical methods could produce a solution of the corresponding interior Dirichlet problem.

The variational method is also of great value in providing approximate solutions, especially in certain physical problems in which the minimum value of I is the object of some interest[30].

Dirichlet contributed notably to number theory. In 1832 he provided a proof for the special case $n = 14$ of Fermat's conjecture. He studied the *Dirichlet series* [$\sum a_n n^{-s}$, including the Riemann zeta-function as a special case] which are of great importance in applications of analysis to the theory of numbers.

1827 CE **Felix Savary** (1797–1841, France). Astronomer. Showed that the motion of binary stars is in full accord with Newton's theory of universal gravitation.

1827–1865 CE **August Ferdinand Möbius** (1790–1868, Germany). Astronomer and mathematician. He is known and appreciated for his work in five fields:

(1) In his book '*Barycentrische Calcül*', Möbius presented, 20 years ahead of **Grassmann** and **Hamilton**, the ideas of vectors and quaternions (1827).

(2) Introduced the '*Möbius function*' and the '*Möbius transform*' (1832) into the theory of numbers. After Euler's totient function, these are among the most important tools of number theory[31].

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[30] e.g. in electrostatic problems, I is closely related to the *capacity* of the system. In this connection, one notices that $\frac{1}{8\pi}\int_V |\nabla\psi|^2 d\tau$ is the potential energy of the field.

[31] The Möbius function: $\mu(n) = 1$ for $n = 1$, $\mu(n) = 0$ if n is divisible by a square and $\mu(n) = (-)^k$ if n is a product of k distinct primes. The Möbius transform of $f(x)$ is defined as

$$F(x) = \sum_{n=1}^{\infty} f(nx)n^{-s}$$

and its inverse is

$$f(x) = \sum_{n=1}^{\infty} \mu(n)F(nx)n^{-s}.$$

For $x = 1$ (Dirichlet series), $f(n) = 1$ yields for $F(1)$ the Riemann zeta function $\zeta(s)$ and therefore

$$[\zeta(s)]^{-1} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

(3) Discovered the 'Möbius bilinear mapping' in the complex plane $\left[w = \frac{az+b}{cz+d}\right]$. It is linked to projective geometry via the cross-ratio $\frac{(w-w_1)/(w-w_3)}{(w_2-w_1)/(w_2-w_3)}$, which is invariant under a projective transformation (1840).

(4) Originated the concept of *homogeneous coordinates* (1827).

(5) Constructed a one-sided, one-edged surface, the '*Möbius strip*' (1865). This appeared in a paper in which a polyhedral surface was viewed as a collection of joint polygons, which in turn introduced the concept of 2-complexes into topology.

Möbius was born near Naumberg, Germany. Through his father, a dance teacher, he was a descendant of **Luther**. In 1809 he entered the University of Leipzig to study law, but ultimately devoted himself to mathematics and astronomy and was a student of **Gauss** at Göttingen. In 1816 he became an associate professor at Leipzig University, and later was chosen as the director of the university observatory. He waited 28 years to become a full professor (1844).

The Dawning of Topology (1735–1914)

"A Geometry is defined by a group of transformations, and investigates everything that is invariant under the transformations of this given group".

Felix Klein (1849–1925)

The Möbius function is also tied up with the *prime zeta-function*

$$\mathcal{P}(s) = \sum_p p^{-s},$$

p prime, since

$$\mathcal{P}(s) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log_e \zeta(ks).$$

"It has been said that geometry is the art of applying good reasoning to bad diagrams. This is not a joke but a truth worthy of serious thought. What do we mean by a poorly drawn figure? It is one where proportions are changed slightly or even markedly, where straight lines become zigzag, circle acquire incredible humps. But none of this matters.

An inept artist, however, must not *represent a closed curve as if it were open, three concurrent lines as if they intersected in pairs, nor must he draw an unbroken surface when the original contains holes".*

Henri Jules Poincaré, 1895

In the first half of the 19^{th} century there began a completely new development in geometry that was soon to become one of the great forces in modern mathematics. The new subject, called analysis situs *or topology, has as its object the study of the properties of geometrical figures that persist even when the figures are subjected to deformations so drastic that all their metric and projective properties are lost.*

The major actors in this drama were ***L. Euler*** *(1707–1783),* ***Lhuilier*** *(1750–1840),* ***Gauss*** *(1777–1855),* ***Möbius*** *(1790–1868),* ***Listing*** *(1808–1882),* ***Betti*** *(1823–1892),* ***Kirchhoff*** *(1824–1887),* ***Riemann*** *(1826–1866),* ***C. Jordan*** *(1838–1922),* ***F. Klein*** *(1849–1925)* ***H. Poincaré*** *(1854–1912),* ***Hausdorff*** *(1868–1942),* ***Lebesgue*** *(1875–1941),* ***Frechet*** *(1878–1973),* ***F. Riesz*** *(1880–1956),* ***Veblen*** *(1880–1960),* ***Brouwer*** *(1881–1966),* ***Lefschetz*** *(1884–1972) and* ***Alexander*** *(1888–1971).*

In Euclidean geometry, the allowed movements are rigid motions (translations, rotations, reflections), in which the distance between any two points of the figure are not changed. Thus the geometric properties are those which are invariant under the group of rigid motions. Since the transformations are the rigid motions, two figures are considered equivalent if they are congruent.

In projective geometry*, two figures are considered equivalent if one may be projected into the other. As the projections include parallel projection, planar, projective geometry includes Euclidean geometry as a special case. Here, not only are similar figures equivalent, but* any *two triangles are equivalent. This means that all triangles may be placed in the same equivalence class and may therefore be treated alike as being congruent.*

Although distances are no longer invariant under projection, collineation is preserved. [i.e. if three points lie on a line in one configuration, their transformed images will likewise be on a line; if three lines go through a single point, their images go also through a single point.]

Whereas Euclidean transformations may be expressed by polynomials of the first degree, projective transformation may be expressed as the ratio of polynomials of degree one.

In algebraic geometry *(which includes projective geometry as a subset) the class of transformations is widened to that of algebraic transformations. All such transformations also constitute a group. Thus we have extension upon extension in geometry, each extension comprising all the other lower in the hierarchy as special cases. This systematized categorization of the geometries in due to* ***Felix Klein*** *(1849–1925) and is known as his Erlangen Program.*

In topology, the allowed movements are continuous invertible deformations that might be called elastic motions. *We imagine that our figures are made of perfectly elastic rubber and, in moving the figure, we can stretch, twist, pull and bend it at pleasure. We are even allowed to* cut such a rubber figure and tie it in a knot, *provided that we later sew up the cut exactly as it was before, so that points which were close together before we cut the figure, are close together after the cut is sewed up. We are not allowed to force two different points to coalesce into just one point. Two figures are* topologically equivalent *iff one figure can be made to coincide with the other by such an elastic motion.*

The topological properties of a figure are those which are invariant under all such continuous transformation. *So, topology is not concerned with the issues of Euclidean geometry — the measurements of lengths, areas, volumes, angles, the making of scale drawings or enlargements. Moreover, topology is not limited to the rules of* projective geometry, *where straight lines may change in position but may never be distorted into curves, and circles may be transformed into ellipses and vice versa, but may not acquire humps or altogether arbitrary closed contours.*

Thus, topology might be described as the general study of continuity, *that concept whose various aspects have challenged philosophers and mathematicians from Pythagoras' day.*

One of the first topological observations is due to ***Descartes****, who as early as* 1640 *deduced an equation relating the numbers of vertices, edges and faces of simple polyhedra. This formula was rediscovered by* ***Euler*** *in 1752. [The typical character of this relation as a topological theorem, became apparent much later, after* ***Poincaré*** *has recognized 'Euler's formula' and its generalizations as one of the central theorems of topology.]*

Euler's name is linked to the subject through another problem which deserves to be considered as the beginning of topology: In 1736 *he presented a memoir to the St. Petersburg academy, in which he solved the problem of the Königsberg bridges:*

The river Pregelarme has two islands linked by a bridge. One island has one bridge crossing from it to each bank; the other has two bridges to each bank. Can the citizens of Königsberg cross all seven bridges (each being traversed only once) in a continuous walk? Euler showed that to be impossible[32]*, and he solved the most general problem of the same type.*

His method is based on the observation that it is the way the bridges connect, not their precise positions or sizes, that matters. It is evident that the question will not be affected if we suppose the islands to diminish to points and the bridges to lengthen out. In this way one ultimately obtains a geometrical figure of a network*. Euler's problem therefore consisted in finding whether a given network can be described by a point moving so as to traverse every line in it once and only once.*

Euler then proved the rule $V - E + F = 1$ *for planar networks in contradistinction to* $V - E + F = 2$ *for polyhedra. [This problem marks the origin of today's* graph theory *which in itself is part of* combinatorial topology*. It is applied to electrical networks, perturbative Feynman graphs, industrial management science, linear programming, game theory, statistical mechanics, social psychology and other behavioral sciences.]*

Not much happened in topology during the next 100 years[33]*. However, in the middle of the* 19^{th} *century, topology began to assert itself as a separate branch of geometry (known then as '*analysis situs*'), soon to become one of the main themes in modern mathematics. About 1850* ***Francis Guthrie*** *adduced a conjecture concerning the 4-color problem: that any map on a plane or on a sphere can be colored with at most 4 colors. The problem was later taken up by* ***Augustus de Morgan****,* ***Arthur Cayley*** *and others.*

One of the great geometers of the time was ***A.F. Möbius****, a man whose lack of self-assertion destined him to the career of an insignificant astronomer in a second-rate German observatory. Möbius probably did not think of himself as a topologist, because at that time there was no general subject called* topology*; nevertheless his ideas have had a profound influence on the development of the subject. At the age of* 68 *he submitted to the Paris Academy*

[32] In 1875, an eighth bridge was built. The addition of this bridge made it possible to solve the problem.

[33] **Gauss** made several contributions to topology. Of the several proofs that he furnished of the fundamental theorem of algebra, two are explicitly topological. His first proof of this theorem employs topological techniques and was given in his doctoral dissertation in 1799 when he was 22 years old. Later, Gauss briefly considered the theory of knots, which today is an important subject in topology. Although he added little beyond these few abstractions, much has been achieved by his students: **Möbius**, **Listing**, **Kirchhoff** and **Riemann**.

a memoir on 'one-sided' non-orientable surfaces that contained some of the most surprising facts of this new kind of geometry. Like other important contributions before it, his paper lay buried for years in the files of the Academy until it was eventually made public.

*Independently of Möbius, the astronomer **J.B. Listing** (one of Gauss's students) in Göttingen had made similar discoveries, and at the suggestion of Gauss, had published in* 1847 *a little book,* Vorstudien zur Topologie. *In this book, the first devoted to the subject, Listing introduced the term* topology. *The theory of Euler's networks is included as a particular case among the propositions proved by Listing in his book. [In* 1857, ***W.R. Hamilton**, using combinatorial analysis and group theory, solved some special problems in network theory.]*

***G.R. Kirchhoff**, another of Gauss' students, employed (1847) the topology of linear graphs in his study of electrical networks.*

*But of all of Gauss's students, the one who contributed by far the most to topology was **Bernhard Riemann**, who, in his doctoral thesis of 1851, introduced topological concepts into the study of complex-function theory.*

When Bernhard Riemann came to Göttingen in 1847 *as a student, he found the mathematical atmosphere of that university town filled with keen interest in these strange new geometrical ideas. Soon he realized that therein was the key to the understanding of some deep properties of analytic functions of a complex variable. His was the first major contribution to* analysis situs *since Möbius, namely the so-called* Riemann surfaces *or* Riemann sheets *which enable to set up a one-to-two, or more, correspondence between the function* $w(z)$ *and its argument* z *[e.g. two* z*-sheets for* $w^2 = z$ *and an infinite number of* z*-sheets for* $w = \log z$*].*

He pictured the sheets as attached at certain special points ("branch points"), and also envisioned abstract "bridges" enabling passage from one sheet to another. In this way he established the homeomorphism *or* topological equivalence *of the* w*-plane to the many-sheeted* z*-plane (Riemannian surface[34]). Nothing, perhaps, has given more impetus to the later developments of topology than the structure of Riemann's theory of functions, in which topological concepts are absolutely fundamental.*

J.C. Maxwell** (1873) used the topological theory of connectivity in his study of electromagnetic fields. Others, such as **H. Helmholtz** and **Lord

[34] Such a surface is topologically equivalent to a sphere to which several *handles* have been added or, to put it another way, to a plane with several *holes*.

Kelvin, *can be added to the list of physicists who applied topological ideas with success*[35].

The next mathematician in chronological order, as far as combinatorial topology is concerned, was ***H. Poincaré***, *who developed many of his topological methods while studying ordinary differential equations which arise in the study of certain astronomy problems. Indeed he was led to topology through his efforts to solve the n-body problem for the case $n = 3$. This involves the determination of all-time orbits for sun, earth and moon, for example. [The 3-body problem involves the solution of a system of 9 differential equations.]*

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[35] **Helmholtz** (1858), building on the ideas of **Riemann** (1851, 1857), introduced topological considerations into hydrodynamic theory. He defined *vortex lines* as lines integrating the local directions of the axes of rotation of the fluid, and *vortex tubes* as bundles of vortex lines through infinitesimal elements of area. Helmholtz showed that the vortex tubes had to close up and also that the particles in a vortex tube at any given instant would remain in the tube indefinitely. So, no matter how much the tube was distorted, it would retain its topological shape. Helmholtz was aware of the topological ideas in his paper, particularly of the fact that the region *outside* a vortex tube was multiply connected, which led him to consider many-valued potential functions.

Tait (1867) verified Helmholtz's theoretical claims regarding two circular vortex rings via experiments with smoke rings. Curiously enough, Helmholtz's topology, driven by physical ideas of fluid-flow, impacted the Scottish mathematical physicists, **Kelvin** (Thomson), **Maxwell** and **Tait**, each in a different way: Kelvin concocted an elaborate theory according to which atoms, viewed as knotted vortex tubes in the ether, interact (chemically) as fluid vortices do. Tait and Maxwell wished to model the interaction of linked current circuits after the dynamics of vortex rings in a fluid, using the integral formula counting the linking number of two closed curves which **Gauss** had discovered (1833). Thus, these physicists became involved in topological concepts, in particular *knot theory*, because it entered their physical considerations in a natural way.

Of the three, **Maxwell** was ahead of his time by some 50 years. Although his approach lacked mathematical rigor, he defined (1868) the "*Reidemeister moves*" which, later (1920's), would be shown to be the fundamental moves in modifying. Moreover, Maxwell considered a region of 3D space bounded by one external surface of genus n, and m internal surfaces of genera $n_1, n_2, \dots n_m$ and showed that the region possessed $N = n + n_1 + n_2 + \cdots + n_m$ cycles. Now in modern terminology, Maxwell was claiming that the first **Betti** number of the region was N. Again we should note that **Maxwell** did not give precise mathematical definitions of the concepts he was dealing with, so no rigorous proof was possible. It is reasonable to ask how, then, did he find the correct answer. The answer is that he reached his correct results using correct physical understanding, rather than mathematical intuition.

Poincaré showed that if two of the bodies have masses that are small compared to the third, periodic *solutions exist. In 1912 he proved that certain orbits could be periodic, provided that a simple geometric theorem, topological in nature, is true [the problem being to prove that when a certain topological transformation of the annular area between 2 concentric circles, is carried out, 2 of its points must remain fixed].*

It is hard to believe that a serious issue of dynamical astronomy can depend on such an apparently simple question that sound like an exercise in high school geometry.

Poincaré also put on a completely rigorous basis (1895) the concept of connectivity, *elaborated earlier by* ***Listing***, ***C. Jordan*** *and* ***Betti***. *He introduced the concepts of* homology *and* homotopy, *gave a more precise definition of the* Betti numbers, *and generalized Euler's convex polyhedra formula to* p*-dimensional space.*

At the same time, topology developed along another route through the generalization of the ideas of convergence. *This process began already in 1817 when* ***B. Bolzano*** *removed the association of convergence with a sequence of numbers and associated convergence with any bounded infinite set of real numbers.* ***Cantor*** *(1872) introduced the concept of a* set of limit points, *defined closed subsets of the real line as subsets containing their set of limit points, and introduced the idea of an* open set *— a fundamental concept in point set topology.*

Weierstrass *(1877) introduced the concept of a* neighborhood *of a point.* ***Hilbert*** *(1902) used this concept when he stated that a continuous transformation group is differentiable.* ***Frechet*** *(1906) extended the concept of convergence from Euclidean space by defining* metric spaces, *and showed that Cantor's ideas of open and closed subsets extended naturally to metric spaces.*

Riesz *(1909) disposed of the metric altogether, and proposed a new axiomatic approach to topology based on the definition of a set of limit points,* with no concept of distance. ***Hausdorff*** *(1914) followed suit by defining neighborhoods via four axioms devoid of metric considerations. These contributions allow the definition of* abstract topological spaces.

Topological concepts also entered mathematics via functional analysis, *pioneered by* ***Volterra*** *(1887). This topic arose from mathematical physics and astronomy because the methods of classical analysis were somewhat inadequate in tackling certain types of problems in the calculus of variations. Further advances in the theory of functionals were made by* ***Hadamard*** *(1903),* ***Frechet*** *(1904), and* ***Schmidt*** *(1907).*

However, the pioneers, like Poincaré, were forced to rely largely upon geometrical intuition. Recent work has brought topology within the framework

of rigorous mathematics, where intuition remains the source but not the final validation of truth. During this process, starting by **L.E.J. Brouwer** *(1881–1966), the significance of topology has steadily increased, and the collection of methods developed by Poincaré was built into a complete topological theory.*

Today, topology, together with abstract algebra, is at the root of almost all of modern pure mathematics. In fact, topology has penetrated into other mathematical subjects and pervades current activity. The subject of topology itself consists of several different branches such as point-set topology, algebraic topology, differential topology, analytic topology and combinatorial topology[36]. *We know today that some very fundamental features of our physical reality are topological.*

1827 CE **Georg Simon Ohm** (1787–1854, Germany). Physicist. Born in Erlangen and educated at the university there. In 1817 he became a professor of mathematics in the Jesuits' college at Cologne. In a pamphlet published in Berlin in 1827 with the title "*Die galvanische kette mathematisch bearbeitet*" he stated that a current flowing in a wire is proportional at each point to the gradient of the potential $(E = RI)$. This he modeled after the flow of heat over a temperature gradient, mutatis mutandis. This phenomenological law, albeit approximate, had a most notable influence on the whole development of the theory and applications of dynamical electricity.

However, when the law was announced it seems too good to be true, and was not believed(!). Ohm was considered unreliable because of this, and was so badly treated that he resigned his professorship at Cologne and lived for several years in obscurity and poverty before it was recognized that he was right. So, in 1833, Ohm returned to become a professor at the polytechnic school in Nuremberg, and in 1852 he was appointed a professor of experimental physics at the University of Munich. He died soon thereafter, in 1854.

Peter Dirichlet was one of Ohm's pupils at Cologne.

1827–1828 CE **Robert Brown** (1773–1858, England). Scottish botanist. Discovered the erratic microscopic movement of small inorganic particles suspended in fluids. While investigating the pollen of several different plants, he

[36] In many cases a problem originally conceived as number-theoretic, algebraic, analytic, or geometric has eventually turned out to be *combinatorial*. Recent progress with electronic computers is playing an important role in the solution of various combinatorial problems arising in large systems, and the study of combinatorial topology is now quite active in terms of both theoretical development and applications.

observed that pollen dispersed in water in a great number of tiny particles exhibit uninterrupted irregular zig-zag motion. This phenomenon, which can also be observed in gases, is referred to as *Brownian motion*. Although it soon became clear that Brownian motion is an outward manifestation of the molecular motion postulated by the atomic theory of matter, it was not until 1905 that **Albert Einstein** first advanced a satisfactory theory.

Brown was one of England's greatest botanists. He is best known for his discovery of the nuclei of plant cells, as well as for classifying a large number of unfamiliar plants which he brought back from an Australian expedition in 1801–1805. In 1810 he became librarian to the Royal Society. Though offered a university chair he preferred to retain this job, where he had the use of valuable collections. In 1828 Brown wrote a pamphlet entitles "*A brief account of microscopical observations made in the months of June, July and August, 1827, on the particles contained in the pollen of plants and on the general existence of active molecules in organic and inorganic bodies*".

Brown was born at Montrose, a son of a Scottish Episcopalian clergyman. He studied medicine at the Universities of Aberdeen and Edinburgh and spent 5 years (1795–1800) in the British army as an assistant surgeon. He gained international reputation and was elected member in many learned foreign societies. He died in London.

The motions of microscopic particles in fluids were observed by biologists long before Brown, but were considered to be of *organic character*. [**John Turberville Needham** (1713–1781, in 1767); **Lazzaro Spallanzani** (1729–1799, in 1767) and others before 1800.] What Brown showed was that this phenomenon was not biological but *physical* in nature, thus removing the subject from the realm of biology to the realm of physics.

Brown had other claims to fame, and Brownian motion is not mentioned in his biography in the *Encyclopaedia Britannica*'s 9^{th} edition, 1878. In the 13^{th} edition of 1925, it merited a few words in passing.

1827–1861 CE **Anyos Istvan Jedlik** (1800–1895, Hungary). Physicist and inventor. Invented the first prototype of a *dynamo*.

Jedlik was born in Zemna (the Kingdom of Hungary, now Slovenia). Became a Catholic priest (1817). Lectured on physics (1839–1879) at the Budapest University of Science.

In 1827 he started experimenting with electromagnetic rotating devices. Discovered the principle of the *self-excited generator* (1856–1861), at least six years ahead of Ernst Werner von Siemens. His idea was that the performance

of current generators[37] could be perfected by using the current produced by the machine to feed their magnets. In fact, he observed that the very slight remnant magnetization present in the iron core of the electromagnets was sufficient to start the process of induction.

This principle of self-excitation (i.e. the dynamo) was financially exploited a few years later by Siemens and Wheatstone, to whom Jedlik had not been known.

1828–1832 CE **Friedrich Wöhler** (1800–1882, Germany). Chemist. Synthesized the first artificial product which is created in nature within a living being[38], thus shattering the popular belief that some mysterious *vital* principle was at work in organic chemicals (1828).

Wöhler was born at Eschersheim, near Frankfurt-on-Main. He took his degree in medicine and surgery at Heidelberg in 1823, but was persuaded to devote himself to chemistry. He studied in Berzelius' laboratory at Stockholm. He later taught chemistry in Berlin (1825–1831), Cassel (1832–1836) and Göttingen (1836–1882), where he held the position of professor of chemistry in the medical faculty.

Wöhler maintained lifelong friendships with both **Berzelius** and **Liebig**. With the latter he carried out a number of joint researches.

1828–1835 CE **Lambert Adolph Jacques Quetelet** (1796–1874, Belgium). Statistician, astronomer and meteorologist. The father of statistics.

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[37] **Faraday**'s discovery (1831) of *electromagnetic induction* opened up the possibility of generating *electric currents* by the mechanical movement of a conductor in a magnetic field: The reversal of the process makes it possible to obtain *mechanical* work by the action of a magnetic field upon and electric current.
Accordingly, we may classify electrical machines into:

- *Current generators*, by which mechanical work is transformed into electrical energy.
- *Motors*, by which electrical energy is transformed into mechanical work.

In 1867 **W. von Siemens** introduced the self-excited generator (dynamo) in which the magnetic field is established not by permanent magnets but by the generator itself. (It is the residual magnetization of the iron that makes it possible for a machine to excite itself once set running.)

[38] Prepared urea, $CO(NH_2)_2$, by evaporating the inorganic isomer ammonium cyanate NH_4OCN. People were astounded in his time because they thought that organic compounds can be produced only by living organisms.

Demonstrated the use of probability models in describing social and biological phenomena. Conducted statistical research on the development of the physical and intellectual qualities of man, formulating a theory of the "average man" as a basic type. First to use the *normal curve* other than as an error law.

Quetelet was born in Ghent. Became professor of mathematics at the University of Brussels (1819). Was the founder and director of the new Royal Observatory at Brussels. Conducted (1829) the first statistical breakdown of a national census, examining for the Belgian census the correlations of death with age, sex, occupation and economic status.

Quetelet studied briefly with **Laplace** and the latter's influence on him was unmistakable. He traveled through Europe in the ensuing years, spreading with fervor the statistical "gospel". (By 1870, "modern" mathematical statistics was poised and ready for its debut.)

His book *Sur l'homme et le développement de ses facultés, ou Essai de physique sociale* (1835) was the first attempt to apply mathematical analysis to the study of man — not only of his body but of his behaviour and morality, his mind and soul. In this respect he may be considered the first modern sociologist[39]. The rest of his life was devoted to the consolidation of his initial effort. He laid a great stress on the universal applicability of the binomial distribution.

The roots of Quetelet's thought must be sought outside the statistical literature. There are two main sources:

- The calculus of probabilities which originated (1654) in a correspondence between **Pascal** and **Fermat** and reached its climax in **Laplace**'s *Théorie analytique des probabilités* (1812). (Quetelet was introduced to him in Paris, in 1823.)
- The "*political arithmetic*" which was developed in England during 1662–1683 by **John Graunt** (1620–1674), **William Petty** (1623–1687) and **Edmund Halley** (1656–1742) who estimated mortality rates drawn from tables of births and funerals with an attempt to ascertain the price of annuities upon lives.

[39] **Auguste Comte** (1798–1857) was probably the first to speak of social physics (as early as 1822) and of sociology (1839). But Comte wrote on these matters with prolixity and conceit whereas Quetelet was not only saying what need be done, but actually doing it, and much better than Comte could imagine. For the real difficulties and crucial points only appear when one is tackling concrete problems.

1828–1837 CE **George Green** (1793–1841, England). One of the most brilliant and original mathematical physicists of the 19^{th} century; created a number of ideas that were far ahead of his time. Made major contributions to *potential theory.*

In 1828 he printed privately in Nottingham the tract "*An essay on the application of mathematical analysis to the theories of electricity and magnetism*". In this manuscript he presented a theorem, named after him, which concerns the relationship between a line integral over a simple closed curve in the plane and a double integral over a region bounded by the curve.

Green's theorem in the plane (1828): Arose in connection with gravitational and electric potential theory. States the following: Let D be a region in the xy-plane bounded by a simple closed curve C which consists of a finite number of smooth arcs. Then, if $P(x,y)$, $Q(x,y)$ are continuous functions with continuous first partial derivatives we have

$$\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_C (Pdx + Qdy),$$

where the circuit integral is taken in the *positive sense*: (a person making the circuit will always have the region D on his left). Green's theorem has the *vector form*

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \int_D (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{e}_z dA$$

where

$$\boldsymbol{F} = (P, Q), \qquad d\boldsymbol{s} = (dx, dy) = \text{tangent vector line element},$$

$\boldsymbol{e}_z$ is a unit vector normal to D, and $dA = dxdy$. If we choose $\boldsymbol{F} = (-Q, P)$, with $\boldsymbol{n}ds = (dy, -dx) =$ normal vector line element to the curve C, the above vector form changes into

$$\int_C \boldsymbol{F} \cdot \boldsymbol{n}ds = \int_D \operatorname{div} \boldsymbol{F} dA.$$

Green's theorem is very useful because it relates a line integral around a boundary of a region to an area integral over the interior of that region and in many cases it is easier to evaluate the line integral than the area integral, e.g. if we know that $P(x,y)$ *vanishes* on the boundary, we can conclude that $\int_D \frac{\partial P}{\partial y} dxdy = 0$ even though $\frac{\partial P}{\partial y}$ need not vanish in the interior.

The latter form of Green's theorem generalizes – for the case of three-dimensional curves, surfaces and vector fields – to *Stokes' theorem* (1850). It relates the line-integral of a vector-field around a simple curve C to an integral

over a surface S for which C is a boundary. If $f(\boldsymbol{r})$ is a continuously differentiable vector function over a two-sided piecewise smooth oriented surface S, spanning the closed curve C, then

$$\int_S \operatorname{curl} \boldsymbol{f} \cdot d\boldsymbol{S} = \oint_C \boldsymbol{F} \cdot d\boldsymbol{s}, \quad d\boldsymbol{S} = \boldsymbol{n} dS, \quad d\boldsymbol{s} = \boldsymbol{t} ds,$$

where $\boldsymbol{n}$ is the positive normal to S and $\boldsymbol{t}$ is the tangent to C in the positive sense.

The theorem states that the *circulation* of a vector field around the contour of some surface is equal to the *flux* of the curl of the vector field through this surface. Otherwise stated: the integral of the *normal* component of the curl of a vector-field over a surface is equal to the integral of the *tangential* component of the same vector-field around the boundary of that surface.

In 1837, Green defined the *elastic strain-energy density*, derived the elastodynamic equation of motion in anisotropic media from the principle of virtual work, and correctly established the boundary conditions at the surface.

Green was primarily self-taught, and the tract was published with the aid of a patron who later helped him enter Cambridge as an undergraduate in 1833. The essay contained not only his theorem, but many other important results. Green graduated from Cambridge in 1837 and although he continued his research, none of his subsequent work had the depth or importance of his essay.

Green's essay went largely unnoticed until it was discovered by **Lord Kelvin** (1845), who arranged to have it printed. It is now regarded as one of the great classics of mathematical physics.

1828–1843 CE **William Rowan Hamilton** (1805–1865, Ireland). A great mathematician of the 19^{th} century, and Ireland's greatest claim to fame in the field of mathematics.

In 1828 he transformed the Lagrange 2^{nd} order equations of a dynamical system to a set of canonical first order equations, with twice as many variables, considering the position coordinates and the momenta as independent variables. These $2n$ first order differential equations are called *Hamilton's equations* for the system, and can replace those of Lagrange in giving the solution of a given problem.

Hamilton gave the first exact formulation of the *principle of least action*.

He also realized that problems in mechanics and geometrical optics can be tackled from a united viewpoint, where the *characteristic function* satisfies the same partial differential equation. He was first to grasp the concept of *group velocity* (1839).

With the rapid development of Newtonian dynamics and the geometric representation of complex numbers, the vector concept and its applications were coming of age. In 1843, Hamilton originated *quaternions*[40]and coined the words: *scalar*, *vector* and *tensor*. [His quaternions consist of four terms, three of which correspond to *vector* components and the fourth being a *scalar*. The term *tensor* was used by Hamilton to define the root of the sum of squares of the four elements of the quaternion. This definition has nothing whatsoever to do with the later use of this word.] During 1843–1850, Hamilton developed for the first time the underlying concepts of vector analysis (e.g. scalar and vector products, ∇ operator[41]) within the framework of his quaternion theory.

It is told of him that on the evening of Oct. 16, 1843, while walking along the Royal Canal in Dublin, the algebra of Quaternions dawned upon him and he carved on a stone on Brougham Bridge the formulas $i^2 = k^2 = j^2 = ijk = -1$.

Hamilton was born in Dublin of a branch of a Scottish family which had settled in the north of Ireland in the times of James I. He was early orphaned and his upbringing was entrusted to an uncle who gave the boy a strenuous but lopsided education with strong emphasis on languages. William proved to be a prodigy and when he reached the age of 13 he acquired, besides modern European languages, Persian, Hebrew, Arabic, Hindustani, Sanskrit and Malay. At 16 he mastered a great part of Newton's *Principia* and at 17 he read Laplace's *Mécanique céleste*. Hamilton's career at Trinity College, Dublin, was unexampled, for in 1828, when he was still a 23 year old graduate student, the university electors unanimously appointed him Royal Astronomer of Ireland and a professor of astronomy.

Two unhappy love affairs (he attempted to drown himself after the first one), a hypochondriac wife and alcoholism marred the personal life of this great Irish mathematician. Although it is thought by many that these difficulties lowered the quality of his mathematical thought, Hamilton's output continued unabated to the end of his life.

Hamilton was inspired by the necessity for appropriate mathematical tools to enable the application of Newtonian mechanics to various aspects of astronomy and physics. The theory of quaternions was too complicated in structure and was not able to survive in its original form as a tool of classical mathematical physics. However, quaternions can be used to represent complex variables, vectors and rotations in 3-space and 4-space. Moreover, the algebra of quaternions is of deep significance and its importance has been emphasized in recent

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[40] The origin of the name comes from the New Testament *Acts 12*, 4.

[41] In his quaternion theory, he introduced (1853) the symbolic operator $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$, which Heaviside later named "*nabla*".

decades by applications in group representation theory, quantum mechanics of spinors and special and general relativity.

Fifty years later, **Oliver Heaviside** said of Hamilton: "*Vector analysis without quaternions could have been found by any mathematician, but to find out quaternions required a genius*".

The highest tribute to Hamilton was perhaps paid by **Erwin Schrödinger**:
"*I dare say not a day passes — and seldom an hour — without somebody, somewhere on this globe, pronouncing or reading or writing or printing Hamilton's name. His famous analogy between mechanics and optics virtually anticipated wave mechanics, which did not have to add much to his ideas, only had to take them seriously — a little more seriously than he was able to take them, with the experimental knowledge of a century ago. The central conception of all modern theory in physics is 'the Hamiltonian'*".

The Principle of Least Action

Consider n particles of masses m_j, located at points $\boldsymbol{r}_j(t)$, and acted upon by resultant external and internal forces $\boldsymbol{F}_j$. By d'Alembert's principle, we write $\sum_{j=1}^{n}(m_j\ddot{\boldsymbol{r}}_j - \boldsymbol{F}_j)\cdot\delta\boldsymbol{r}_j = 0$ for arbitrary variations $\delta\boldsymbol{r}_j$. Call $\delta W = \sum_{j=1}^{n}\boldsymbol{F}_j\cdot\delta\boldsymbol{r}_j$ the virtual work done by all the forces. We have $\frac{d}{dt}(\dot{\boldsymbol{r}}_j\cdot\delta\boldsymbol{r}_j) \equiv \dot{\boldsymbol{r}}_j\cdot\frac{d}{dt}(\delta\boldsymbol{r}_j) + \ddot{\boldsymbol{r}}_j\cdot\delta\boldsymbol{r}_j$, which together with $\frac{d}{dt}(\delta\boldsymbol{r}_j) = \delta\dot{\boldsymbol{r}}_j$ permits us to conclude that for the j^{th} particle

$$m_j\ddot{\boldsymbol{r}}_j\cdot\delta\boldsymbol{r}_j = \frac{d}{dt}(m_j\dot{\boldsymbol{r}}_j\cdot\delta\boldsymbol{r}_j) - \delta T_j,$$

where T_j = kinetic energy of j^{th} particle $= \frac{1}{2}m_j\dot{\boldsymbol{r}}_j^2 = \frac{1}{2}m_jv_j^2$. Summing over all particles, we finally have

$$\frac{d}{dt}\sum_{j=1}^{n} m_j\dot{\boldsymbol{r}}_j\cdot\delta\boldsymbol{r}_j = \delta T + \delta W.$$

Consider two times t_0 and t_1 at which we assume $\delta\boldsymbol{r}_j = 0$, i.e., two times at which actual and virtual paths coincide. Integrating the last equation between

t_0 and t_1, we have $\sum_{j=1}^{n} m_j \dot{\boldsymbol{r}}_j \cdot \delta \boldsymbol{r}_j|_{t_0}^{t_1} = \int_{t_0}^{t_1} (\delta T + \delta W) dt$. But the l.h.s. is zero by virtue of our restriction on $\delta \boldsymbol{r}_j$; hence $\int_{t_0}^{t_1} (\delta T + \delta W) dt = 0$. If we assume that W is a work function arising from a potential energy such that $W = -V(\boldsymbol{r_1}, \ldots, \boldsymbol{r_n})$, then $\delta W = -\delta V$ leads to

$$\delta \int_{t_0}^{t_1} (T - V) dt = \delta \int_{t_0}^{t_1} L dt = 0,$$

which is Hamilton's principle for conservative dynamical systems. L is known as the Lagrangian function of the system.[42]

We may enunciate the principle as follows: *a system moves from one configuration to another in such a way that the variation of the integral $\int_{t_0}^{t_1} L dt$ between the actual path taken and a neighboring virtual path, co-terminous in space and time with the actual path, is zero. In other words, $\int_{t_0}^{t_1} L dt$, called the action – a functional of the virtual trajectory in the system's configuration space — is stationary at the actual trajectory, if all virtual trajectories are constrained to evolve between the same two spatial points at the end-times t_0 and t_1.*

This principle is equivalent to the Lagrange equations. Indeed: for a single particle in one dimension x $\left(\dot{x} = \frac{dx}{dt}\right)$,

$$\begin{aligned}
\delta \int_{t_0}^{t_1} L\left(x, \frac{dx}{dt}; t\right) dt &= \int_{t_0}^{t_1} \left\{ \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta(\dot{x}) \right\} dt \\
&= \int_{t_0}^{t_1} \left\{ \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \frac{d}{dt} (\delta x) \right\} dt \\
&= \int_{t_0}^{t_1} \left\{ \frac{\partial L}{\partial x} \delta x - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) \delta x + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \delta x \right) \right\} dt \\
&= \int_{t_0}^{t} \left\{ \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right\} \delta x \delta t + \left[\frac{\partial L}{\partial \dot{x}} \delta x \right]_{t_0}^{t_1}
\end{aligned}$$

The last term vanishes (δx being zero at t_0 and t_1), and, since δx may otherwise vary arbitrarily in the time interval (t_1, t_2), we deduce the Euler-Lagrange equation:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

as a necessary and sufficient condition for the functional $\int_{t_0}^{t_1} L dt$ to be stationary at the trajectory $x(t)$, $t_0 \le t \le t_1$. When there are several dependent

[42] For further reading, see:

- *The Feynman Lectures on Physics*, Volume II, Addison-Wesley, 1964.

variables (i.e. the physical system is multi-dimensional or has multiple degrees of freedom for another reason), there results a system of (generally coupled) Euler-Lagrange equations, one for each dependent variable. Note that the Lagrangian of a closed system does not depend explicitly on time.

Denoting the system's generalized coordinates *by q_i, $i = 1, \dots, n$, the Lagrangian is a function of $q_i(t)$, their first time derivatives, and (possibly) also depends explicitly on time. Thus, the total differential of L is*

$$dL = \sum_i \frac{\partial L}{\partial q_i} dq_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt = \sum_i \dot{p}_i dq_i + \sum p_i d\dot{q}_i + \frac{\partial L}{\partial t} dt$$

with $\frac{\partial L}{\partial \dot{q}_i} = p_i$ by definition (generalized canonical momenta) and $\frac{\partial L}{\partial q_i} = \dot{p}_i$ by the Euler-Lagrange equations. The above equation then yields

$$d\left[\sum_i p_i \dot{q}_i - L\right] = -\sum_i \dot{p}_i dq_i + \sum_i \dot{q}_i dp_i - \frac{\partial L}{\partial t} dt.$$

The argument of the differential is called the Hamiltonian *of the system: $H(p, q; t) = \sum_i p_i \dot{q}_i - L$, and $dH = -\sum_i \dot{p}_i dq_i + \sum_i \dot{q}_i dp_i - \frac{\partial L}{\partial t} dt$, with the first two sums numerically canceling each other since $dq_i = \dot{q}_i dt$, $dp_i = \dot{p}_i dt$. Note that it is tacitly assumed here that the relations $\dot{p}_i = \frac{\partial L(q,\dot{q})}{\partial q_i}$ can be uniquely inverted to yield $\dot{q}_i = \dot{q}_i(p, p)$[43]. From this follows that[44] $-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$ and:*

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i},$$

which are known as Hamilton's equations *(1835). These constitute a system of first order ODE's in (q_i, p_i). They represent the simplest and most desirable form into which the differential equations of the variational problem can be brought. Hence the name* canonical equations *by which* **Jacobi** *designated them.*

The total time derivative of the Hamiltonian is

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum \frac{\partial H}{\partial q_i} \dot{q}_i + \sum \frac{\partial H}{\partial p_i} \dot{p}_i = \frac{\partial H}{\partial t},$$

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[43] The cases where the inversion is *not* unique — termed *constrained dynamical system* — encompass some of the most important applications of the least-action principle to quantum field theory, including *non-abelian gauge theories*, *quantum gravity*, and *string theory*.

[44] $\frac{\partial H}{\partial t}$ is $\frac{\partial}{\partial t} H(p, q; t)$ with $\{p_j, q_j\}_{j=1}^n$ held fixed, while $\frac{\partial L}{\partial t}$ means $\frac{\partial}{\partial t} L(q, \dot{q}; t)$ with $\{\dot{q}_j, q_j\}_{j=1}^n$ held fixed.

on account of Hamilton's equations. For a conservative closed system, neither $L(q,\dot{q})$ nor $H(p,q)$ depend explicitly upon time and the Hamiltonian – equal to $T+V$ – is, numerically, the conserved system energy.

The Hamilton-Jacobi equation (1828–1837) is the most important first-order PDE *that occurs in mathematical physics. It is derived as follows: the action integral $S=\int_{t_1}^{t_2} Ldt$ is taken along a path between two given positions which the system occupies at given instants t_1 and t_2. In varying the action, we compare the values of this integral for neighboring paths sharing the same values of the vectors $q^{(1)}=q(t_1)$ and $q^{(2)}=q(t_2)$ and find that generally only one of these paths corresponds to the actual motion, namely the path for which the integral has its minimum (or, more generally, extremum) value.*

Consider now the action integral S on the true path as a function of the value of the vector $q(t_2)$ at the upper limit of integration t_2 [$q(t_1)=q^{(1)}$ is held fixed]. In general

$$\delta S=\left[\sum_{i=1}^{n}\frac{\partial L}{\partial\dot{q}_i}\delta q_i\right]_{t_1}^{t_2}+\int_{t_1}^{t_2}\sum_{i=1}^{n}\left(\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i}\right)\delta q_i dt,$$

but since paths of actual motion satisfy Lagrange's equations, the integral in δS is zero. Since $\delta q(t_1)=0$ and $\frac{\partial L}{\partial\dot{q}_i}=p_i$, we have for $\delta S=\sum_i p_i\delta q_i$, where we have denoted $\delta q(t_2)$ by δq. From this relation it follows that $\frac{\partial S}{\partial q_i}=p_i$.

Now the action may similarly be regarded as an explicit function of time (even for a closed system) by considering paths starting at a given instant t_1 and at a given point $q^{(1)}$ and ending at a given point $q^{(2)}$ at various times $t_2=t$. Clearly if $q^{(2)}=q(t_2)$ is allowed to evolve with $t=t_2$ in accordance with actual motion,

$$\frac{dS}{dt}\equiv L=\frac{\partial S}{\partial t}+\sum_i\frac{\partial S}{\partial q_i}\dot{q}_i=\frac{\partial S}{\partial t}+\sum_i p_i\dot{q}_i.$$

Hence $\frac{\partial S}{\partial t}=L-\sum p_i\dot{q}_i$ or $\frac{\partial S}{\partial t}=-H$. Combining this result with $\frac{\partial S}{\partial q_i}=p_i$ we have

$$dS=\sum_i p_i dq_i-Hdt$$

for the total differential of the action as a function of independently-varied coordinates and time at the upper limit of integration.

The action $S(q,t)$ thus obeys the equation

$$\frac{\partial S}{\partial t}+H(p,q,t)=0.$$

Accordingly, replacing the generalized momenta p_i in the Hamiltonian by the partial derivatives $\frac{\partial S}{\partial q_i}$, we have the equation which must be obeyed by the function $S(q,t)$. This first order nonlinear PDE *is called the Hamilton-Jacobi equation:*

$$\frac{\partial S}{\partial t} + H\left(q_1, \ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}; t\right) = 0.$$

Like Lagrange's equations and the canonical equations, the Hamilton-Jacobi equation is the basis of a general method of integrating the equations of motion.

If the system is closed and conservative, H does not depend upon the time explicitly and its numerical value is time-independent along actual paths. Thus, the integration of $\frac{\partial S}{\partial t} = -H$ yields $S = S_0(q) - Et$, where E is the conserved total system energy and also equals the constant value of H. The Hamilton-Jacobi equation then assumes the somewhat simpler from[45]

$$H\left(q_1, \ldots, q_n; \frac{\partial S_0}{\partial q_1}, \ldots, \frac{\partial S_0}{\partial q_n}\right) = E.$$

A particular *solution to the latter energy equation can be obtained for the motion of a point particle in a field of potential energy V in rectangular coordinates.*

The energy-equation then takes the form

$$\frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + V(x,y,z) = E,$$

or, since $p_i = \frac{\partial S}{\partial q_i} = \frac{\partial S_0}{\partial q_i}$,

$$(\boldsymbol{\nabla} S_0)^2 = 2m(E - V),$$

i.e.

$$\left(\frac{\partial S_0}{\partial x}\right)^2 + \left(\frac{\partial S_0}{\partial y}\right)^2 + \left(\frac{\partial S_0}{\partial z}\right)^2 = 2m(E - V).$$

Given constants m, E and the function V, it is then required to find at any given field point $P(x,y,z)$ a surface *$S_0(x,y,z) = const.$, such that the modulus of the normal $\boldsymbol{\nabla} S_0$ is the scalar function $\sqrt{2m(E-V)}$; the Hamilton-Jacobi theory then guarantees that the geometric path of the moving point-mass is normal to this family of surfaces at every point. We obtain a family of possible paths by constructing the* orthogonal trajectories *to the*

[45] This simplification procedure is equivalent to applying the *separation of variables* technique to the full Hamilton-Jacobi PDE.

surface $S_0 = const.$ *These mechanical paths have the ray property, because they behave exactly like light rays in optics, the latter being orthogonal to the wave surfaces*[46].

The mechanical-optical analogy that follows from the Hamilton-Jacobi equations draws parallels between Fermat's principle of least time and the Hamilton principle of least action; between surfaces of equal time in optics to surfaces of equal action in mechanics. Moreover, the basic differential equation of geometrical optics:

$$\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 = \frac{n^2}{c^2}$$

$n =$ *inhomogeneous refractive index,* $c =$ *light speed in vacuum,*

$\phi =$ *wavefront-phase* $= \omega T,$ $\omega =$ *monochromatic ray angular frequency,*

$T =$ *ray travel time;* *Wavefronts are perpendicular to the rays,*

which expresses Huygens' principle in infinitesimal form (known as the eikonal equation), has the same form as the Hamilton-Jacobi equation with the correspondence

$$\phi = \alpha S_0, \qquad \frac{n}{c} = \alpha\sqrt{2m(E-V)},$$

α *being an arbitrary constant.*

[46] This orthogonality does not always involve orthogonality in the ordinary Euclidean sense, although it does in our above point-mass mechanical example; e.g. an electron moving in a magnetic field, does *not* cross the surfaces $S_0 = \text{const.}$ perpendicularly; nor do light rays in crystals, in general. This is because $p_i = \frac{\partial S}{\partial q_i}$ is not always parallel to the vector $\dot{q}_i$

Non-commutative systems: Quaternions and Polyadics

The efforts of **Gauss** *(1819) to build a 3-dimensional complex-number system within the framework of common algebra have failed. The isomorphism of complex numbers and two-dimensional vectors in a plane prompted* **Hamilton** *(1843) to extend 3-dimensional vector algebra to include both multiplication and division. But he soon noticed that if one tries to define vector-division by seeking a vector* $\boldsymbol{C}$ *such that* $\boldsymbol{B} \times \boldsymbol{C} = \boldsymbol{A}$ (or $\boldsymbol{C} \times \boldsymbol{B} = \boldsymbol{A}$) *for two* given *vectors* $\boldsymbol{A}$ and $\boldsymbol{B}$, *then this operation is well-defined only when* $\boldsymbol{A} \cdot \boldsymbol{B} = 0$. *It is however* non-unique *on account of the identity*

$$\boldsymbol{B} \times \boldsymbol{C} = \boldsymbol{B} \times (\boldsymbol{C} - \lambda \boldsymbol{B}).$$

Thus, Hamilton was led to invent a new division algebra for quadruples *of numbers (an analogue of 2-D complex numbers) at the price of* relinquishing *the commutative law of multiplication.*

Hamilton considered a 4-dimensional vector-space with abstract unit base elements $\{e_0, e_1, e_2, e_3\}$. *A general vector in this space, known a* quaternion *("four-fold" numbers) is written in the form*

$$q = q_0 e_0 + (q_1 e_1 + q_2 e_2 + q_3 e_3) = q_0 e_0 + \boldsymbol{q}.$$

Quaternions obey the rules of common algebra w.r.t. addition and multiplication by a scalar. The number q_0 *is the* scalar *of the quaternion and* $\boldsymbol{q}$ *is its* vector.

Multiplication is defined by the table:

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	$-e_0$	e_3	$-e_2$
e_2	e_2	$-e_3$	$-e_0$	e_1
e_3	e_3	e_2	$-e_1$	$-e_0$

Clearly, the product of two unit quaternions with different indices is in general non-commutative, since $e_r e_s = -e_s e_r$ $(r, s = 1, 2, 3,\ r \neq s)$. *Using the table and the distributive law, one verifies that the product of two general quaternions is*

$$pq = (p_0 q_0 - \boldsymbol{p} \cdot \boldsymbol{q}) e_0 + p_0 \boldsymbol{q} + q_0 \boldsymbol{p} + (\boldsymbol{p} \times \boldsymbol{q}) \neq qp$$

where

$$\begin{aligned} \boldsymbol{p}\cdot\boldsymbol{q} &= p_1q_1 + p_2q_2 + p_3q_3 \\ \boldsymbol{p}\times\boldsymbol{q} &= (p_2q_3 - p_3q_2)e_1 + (p_3q_1 - p_1q_3)e_2 + (p_1q_2 - p_2q_1)e_3. \end{aligned}$$

The multiplication table shows that $e_0^2 = e_0$ and $e_0e_r = e_re_0$; therefore we may choose $e_0 = 1$. For $e_0 = 1$, $\{\pm e_1, \pm e_2, \pm e_3\}$ are the six square roots of -1. Two limiting cases are of interest:

- $p_2 = q_2 = p_3 = q_3 = 0$, $e_0 = 1$, $e_1 = \sqrt{-1}$, $p = p_0 + ip_1$, $q = q_0 + iq_1$. *Quaternions reduce to ordinary 2-D complex numbers with the corresponding algebra.*
- $p_0 = q_0 = 0$: $pq = -(\boldsymbol{p}\cdot\boldsymbol{q})e_0 + (\boldsymbol{p}\times\boldsymbol{q})$. *Neglecting the scalar piece, one may view the elements $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ as unit vectors in 3D. Then pq is the ordinary vector cross product which is* covariant *under coordinate rotation.*

Just as a complex number can be viewed as an ordered pair of numbers, a general quaternion can be viewed as an ordered pair of complex numbers. *Indeed, since $e_1 = e_2e_3$, we may write, identifying $e_0 = 1$ and $e_3 = i$:*

$$q = (q_0 + q_3e_3) + (q_2 - q_1e_3)e_2 = x + ye_2 \Rightarrow (x, y),$$

with $x = q_0 + iq_3$, $y = q_2 - iq_1$. Addition and multiplication are then defined as

$$\begin{aligned} (x, y) + (u, v) &= (x + u, y + v) \\ (x, y)(u, v) &= (xu - yv^*, xv + yu^*) \qquad {}^* = \text{complex conjugation} \end{aligned}$$

In analogy to the algebra of complex numbers, one defines the quaternion conjugate

$$q^t = (x^*, -y) = q_0 - q_1e_1 - q_2e_2 - q_3e_3,$$

the square of the norm

$$\|q\|^2 = qq^t = q^tq = q_0^2 + q_1^2 + q_2^2 + q_3^2,$$

and the inverse

$$q^{-1} = \frac{q^t}{\|q\|^2}.$$

One then finds a one-to-one correspondence between the complex matrices *$(e_1 \Rightarrow i)$*

$$q \to \begin{bmatrix} x & y \\ -y^* & x^* \end{bmatrix} = \begin{bmatrix} q_0 + q_3i & q_2 - q_1i \\ -q_2 - q_1i & q_0 - q_3i \end{bmatrix}$$

and the quaternions $q_0 + q_1e_1 + q_2e_2 + q_3e_3$, which preserves multiplication.

So far we have not specified the nature of the abstract *base elements $\{e_0, e_1, e_2, e_3\}$, except for some special cases. But, although obeying the same multiplication table, different* representations *of these elements may exist. To stress this important point we first recast the above 2×2 matrix representation of q in the form*

$$q \to q_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + q_1 \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} + q_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + q_3 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$= q_0E + q_1I + q_2J + q_3K.$$

We then 'discover' that the 2×2 matrices $\{E, I, J, K\}$ have the same properties as the basis quaternions $\{e_0, e_1, e_2, e_3\}$. In fact

$$E^2 = E, \quad I^2 = J^2 = K^2 = -E$$

$$IJ = K = -JI; \; KI = J = -IK; \; JK = I = -KJ.$$

The 2×2 matrix representations of the basis quaternions are intimately connected with the Pauli spin matrices (1925)

$$\sigma_1 = iI; \quad \sigma_2 = -iJ; \quad \sigma_3 = -iK$$

which were found to be of central significance in quantum mechanics!

It is easy to show that $\{e_0, e_1, e_2, e_3\}$ have yet another representation, as 4×4 matrices. To see this we begin with

$$q \to \begin{bmatrix} x & y \\ -y^* & x^* \end{bmatrix}$$

and then represent each of the four complex numbers by its own matrix representation with 1 and i represented as $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, respectively. This leads to a 4×4 real matrix for each quaternion

$$q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3 \to \begin{bmatrix} q_0 & q_3 & q_2 & -q_1 \\ -q_3 & q_0 & q_1 & q_2 \\ -q_2 & -q_1 & q_0 & -q_3 \\ q_1 & -q_2 & q_3 & q_0 \end{bmatrix},$$

such that the one-to-one correspondence again preserves multiplication.

For example:

$$(2 - 4e_1 + e_2 + 3e_3)(1 + 2e_1 + 3e_2 - e_3) = 10 - 10e_1 + 9e_2 + 13e_3$$

and

$$\begin{bmatrix} 2 & 3 & 1 & 4 \\ -3 & 2 & -4 & 1 \\ -1 & 4 & 2 & -3 \\ -4 & -1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 & -2 \\ 1 & 1 & 2 & 3 \\ -3 & -2 & 1 & 1 \\ 2 & -3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & -13 & 9 & 10 \\ 13 & 10 & -10 & 9 \\ -9 & 10 & 10 & 13 \\ -10 & -9 & -13 & 10 \end{bmatrix}$$

Again, if we write $q = q_0E_4 + q_1I_4 + q_2J_4 + q_3K_4$, *where*

$$E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad K_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$J_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad I_4 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

we find that the 4×4 *basis matrices* E_4, I_4, J_4, K_4 *obey the same algebra as* $\{e_0, e_1, e_2, e_3\}$.

Quaternions and finite rotations

Suppose that a rigid body is first rotated by a certain angle ϕ *in a given sense around the axis* OA *passing through a given point* O, *and that it is then rotated by an angle* ϕ_1 *around another axis* OB *passing through the same point. The question is: Around what axis and by what angle must the body be rotated in order to bring it from its first position directly to the third? This is the well-known problem of addition of finite rotations. True, it can be solved by means of the ordinary analytic geometry, as was done already by* ***Euler*** *in the 18th century. However, its solution assumes a far more lucid form by means of quaternions.*

Define

$$\boldsymbol{n} = \frac{1}{h}(q_1e_1 + q_2e_2 + q_3e_3), \quad \boldsymbol{n}^2 = -e_0,$$

$$N = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}, \quad h = \sqrt{q_1^2 + q_2^2 + q_3^2} = N \sin\frac{\phi}{2},$$

$$q_0 = N\cos\frac{\phi}{2}$$

Under this definition, every quaternion is reduced to the standard form

$$q = N(e_0 \cos\frac{\phi}{2} + \boldsymbol{n}\sin\frac{\phi}{2}); \quad q^{-1} = \frac{1}{N}(e_0\cos\frac{\phi}{2} - \boldsymbol{n}\sin\frac{\phi}{2})$$

It then follows that with $\boldsymbol{n} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3$

$$\begin{aligned} \boldsymbol{r}' &= q\boldsymbol{r}q^{-1} = \boldsymbol{r}\cos\phi + (1-\cos\phi)\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{r}) + \sin\phi(\boldsymbol{n}\times\boldsymbol{r}) \\ &= [I\cos\phi + (1-\cos\phi)\boldsymbol{n}\boldsymbol{n} + \sin\phi(\boldsymbol{n}\times I)]\cdot\boldsymbol{r} = \mathfrak{R}\cdot\boldsymbol{r}, \end{aligned}$$

where $\mathfrak{R}$ *describes an active rotation of space about an axis given by the vector* $\boldsymbol{n}$, *relative to the fixed axes* $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, *by an angle* $\phi = 2\tan^{-1}\{\frac{1}{q_0}\sqrt{q_1^2+q_2^2+q_3^2}\}$.

Applying a second rotation represented by a quaternion p, *the combined action is given by the expression*

$$p(q\boldsymbol{r}q^{-1})p^{-1} = (pq)r(pq)^{-1}$$

since $q^{-1}p^{-1} = (pq)^{-1}$ *by the* associative *law of multiplication. This means that the result of two successive rotations, characterized by the quaternions* q *and* p, *is the rotation characterized by the product quaternion* pq. *In other words the* addition *(or more precisely, the* composition*) of the rotations corresponds the* product *of the respective quaternions.*

The Pauli spin matrices, *mentioned earlier, tie in with the subject of finite rotation in the following way: we adopt the representation (*I *is the* 2×2 *unit matrix)*

$$e_0 = I, \quad e_1 = -i\sigma_1, \quad e_2 = -i\sigma_2, \quad e_3 = -i\sigma_3, \quad \boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$$

which leads us to the unit quaternion

$$q(\boldsymbol{n}, \phi) = I\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}(\boldsymbol{\sigma}\cdot\boldsymbol{n}),$$

where

$$\boldsymbol{\sigma}\cdot\boldsymbol{n} = \begin{bmatrix} n_3 & n_1 - in_2 \\ n_1 + in_2 & -n_3 \end{bmatrix}.$$

The matrices σ_k *are* Hermitian *(transpose = complex conjugate) and* traceless. *Moreover, they obey appropriate laws of multiplication.*

Since any such rotation can be decomposed into 3 successive rotations with Euler angles (α, β, γ) *about the respective fixed space axes* $\{\boldsymbol{e}_z, \boldsymbol{e}_y, \boldsymbol{e}_z\}$,

we find that

$$\begin{aligned} q(\boldsymbol{n},\phi) &= q(\boldsymbol{e}_z,\alpha)q(\boldsymbol{e}_y,\beta)q(\boldsymbol{e}_z,\gamma) \\ &= \begin{bmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{+i\frac{\alpha}{2}} \end{bmatrix} \begin{bmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{bmatrix} \begin{bmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{+i\frac{\gamma}{2}} \end{bmatrix} \\ &= \begin{bmatrix} \cos\frac{\beta}{2}e^{-\frac{i}{2}(\gamma+\alpha)} & -\sin\frac{\beta}{2}e^{\frac{i}{2}(\gamma-\alpha)} \\ \sin\frac{\beta}{2}e^{-\frac{i}{2}(\gamma-\alpha)} & \cos\frac{\beta}{2}e^{\frac{i}{2}(\gamma+\alpha)} \end{bmatrix} = \begin{bmatrix} q_0 - iq_3 & -(q_2+iq_1) \\ q_2 - iq_1 & q_0 + iq_3 \end{bmatrix} \end{aligned}$$

which is a unimodular unitary matrix. If we shift ϕ to $\phi + 2\pi$ (or by any odd multiple of 2π), $\mathfrak{R}(\boldsymbol{n},\phi)$ remains the same while $q(\boldsymbol{q},\phi)$ changes its sign. Thus, both $\pm q$ represent the same rotation and the correspondence between unimodular quaternions and 3D rotations is indeed established, though it is multi-valued.

Hamilton introduced the vector quaternion differential operator ($e_0 - 1$, $e_1 = i$, $e_2 = j$, $e_3 = k$)

$$\boldsymbol{\nabla} = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

(known today as the gradient operator). He then showed that when ∇ operates on the vector quaternion $\boldsymbol{v} = v_1 i + v_2 j + v_3 k$, their formal "product" yields the vector quaternion

$$\nabla \boldsymbol{v} = -\mathit{div}\ \boldsymbol{v} + \mathit{curl}\ \boldsymbol{v}.$$

Thus, $\nabla\boldsymbol{v}$ is a quaternion with a scalar $\{-\mathit{div}\ \boldsymbol{v}\}$ and a vector curl $\boldsymbol{v}$, where

$$\mathit{div}\ \boldsymbol{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z};$$

$$\mathit{curl}\ \boldsymbol{v} = (\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z})i + (\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x})j + (\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y})k.$$

Quaternions form a 4 dimensional non-commutative associative algebra over the reals (in fact a division algebra) and contain complex numbers, but do not form an algebra over the complex numbers. The quaternions, along with the complex numbers and real numbers, are the only finite dimensional skew fields over the field of real numbers.

Hermann Grassmann[47] *(1844) [and independently* ***Saint-Venant*** *(1832)*[48]*] invented the noncommutative algebra of polyadics in n-dimensional*

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47 Grassman, H.D., *Die Linear Ausdehnungslehre*, Leipzig, 1844.

48 In 1832 the French engineer **Adhémar, Comte de Saint-Venant** (1797–1866) exposed mathematical ideas similar to those which are present in the Grassmanian system. Among other things he defined the dyadic product of two vectors.

Euclidean space. Grassmann's work remained neglected until its resurrection by **James Clerk Maxwell** *(1871),* **Josiah Willard Gibbs** *(1881) and* **Oliver Heaviside** *(1893) who built upon its foundation the modern algebra and analysis of vectors and dyadics in 3-dimensional Euclidean space.*

Grassmann's ideas in Gibbs' notation are as follows: Let $\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3$; $\boldsymbol{b} = b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3$ *represent two vectors with the respective components* $\{a_1, a_2, a_3\}$ *and* $\{b_1, b_2, b_3\}$ *in an orthogonal Cartesian coordinate system with unit vectors* $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$*. One then defines three types of products between the two vectors:*

- *The scalar (inner) product* $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a} = a_1b_1 + a_2b_2 + a_3b_3$
- *The vector (outer) product* $(\boldsymbol{a} \times \boldsymbol{b}) = -(\boldsymbol{b} \times \boldsymbol{a}) = \lambda_1\boldsymbol{e}_1 + \lambda_2\boldsymbol{e}_2 + \lambda_3\boldsymbol{e}_3$

$$\lambda_1 = a_2b_3 - a_3b_2;\ \lambda_2 = a_3b_1 - a_1b_3;\ \lambda_3 = a_1b_2 - a_2b_1$$

$$(\boldsymbol{e}_i \cdot \boldsymbol{e}_j) = \delta_{ij}; \qquad (\boldsymbol{e}_i \times \boldsymbol{e}_j) = \sum_k \boldsymbol{e}_k\epsilon_{ijk},$$

with ϵ_{ijk} *the totally antisymmetric Levi-Civita symbol.*

- *The dyadic (indeterminate) product*

$$\boldsymbol{ab} = \sum_{i,j} a_ib_j\boldsymbol{e}_i\boldsymbol{e}_j \neq \boldsymbol{ba} \qquad i, j = 1, 2, 3$$

The dyadic product is a tensor of the second rank with 9 components.

An important scalar is the triple-product of the vectors $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$*:*

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

equal in absolute value to the volume of a parallelepiped constructed by the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ *and* $\boldsymbol{c}$.

The connection between the Hamilton and Grassmann algebras is the following: the Grassmann inner product of two vectors is equivalent to negative of the scalar of Hamilton quaternion product of two vectors; the Grassmann outer product is precisely Hamilton's vector of the quaternion product of two vectors. However, in a theory of quaternions, the vector appears as a subsidiary part of the quaternion, whereas in the Grassmann algebra the vector is a basic quantity.

While physicists ignored quaternions (up to 1928), Hamilton's work led mathematicians to the theory of linear associative algebras and beyond.

Grassmann's work, however, was redeemed sooner and became a powerful tool in exploiting the theory of the electromagnetic field. ***Maxwell*** *and* ***Clifford*** *separated Hamilton's* $\nabla \boldsymbol{v}$ *into a scalar* divergence *and the* curl *vector, establishing the identities:*

$$\textit{divcurl}\ \boldsymbol{v} \equiv 0, \qquad \textit{curlgrad}\ \boldsymbol{v} \equiv 0$$

and defining the Laplacian operator

$$\text{divgrad} \ = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Maxwell's work made clear that vectors *were real tool for physical thinking and not just an abbreviated notation. Thus, by Maxwell's time a great deal of* vector analysis *was created by treating the scalar and vector parts of quaternions separately. The formal break with quaternions and the inauguration of a new independent subject,* 3-dimensional vector analysis, *was made independently by* ***J. W. Gibbs*** *and* ***Oliver Heaviside*** *in the early 1880's. By the beginning of* 20^{th} *century, the physicists were quite convinced that vector analysis was what they wanted. The mathematicians finally followed suit and introduced vector methods in analytic and differential geometry. With the rise of quantum mechanics in the 1920's, physicists would return to embrace quaternions as representing a new physical reality.*

From the purely algebraic standpoint quaternions were exciting because they furnished an example of an algebra that had the properties of real numbers and complex numbers except for commutativity of multiplication. During the second half of the 19^{th} *century mathematicians explored other varieties of noncommutative algebras.* ***Clifford*** *(1873),* ***B. Peirce*** *(1881),* ***F. G. Frobenius*** *(1878),* ***C. S. Peirce*** *(1881) and* ***A. Hurwitz*** *(1898), made important contributions to the field of linear associative algebra.*

1828–1868 CE **Julius Plücker** (1801–1868, Germany). Distinguished mathematician and physicist. In a unique double career as geometer and experimental physicist, he was both the founder of *line geometry* (1830) and one of the first promoters of gas spectroscopy.

Plücker established Poncelet's *principle of duality* as a fundamental conceptual tool in projective geometry, and extended it to three dimensions where the duality is between points and planes, lines being unchanged (1828–1831).

Plücker introduced *line-geometry*, where straight lines are used as elements in 3-D space, rather than points (1830).

In 1839, Plücker established the field of algebraic geometry. He discovered six equations connecting the number of singularities of algebraic curves. He discovered *homogeneous coordinates* independently of **Möbius**, **E. Bobillier** and **Feuerbach**, defined *Plücker's coordinates* and *Plücker's equations*.

In 1846 he switched to experimental physics, and his research centered on spectroscopy of rare gases. After 1855, improved vacuum techniques enabled **Plücker** and **Crookes** to investigate the properties of the so-called '*Cathode-rays*', which led in 1897 to their identification with electron-streams by **J.J. Thomson**.

It is believed that Plücker was the first to identify three lines of hydrogen in the spectrum of the Solar Corona in 1858, and also first to invent the cathode-ray tube (1859).

Plücker was born at Elberfeld and was educated at the universities of Bonn, Heidelberg and Berlin. In 1823 he went to Paris and came under the influence of the school of French geometers established by **Monge**. He returned to Bonn in 1825 and stayed there until 1833, moving thereafter to Berlin and Halle. In 1836 Plücker returned to Bonn as a professor of mathematics, becoming in 1847 also a professor of physics.

1829 CE The word *technology* coined.

1829–1851 CE **Carl Gustav** (**Jacob Shimon**) **Jacobi** (1804–1851, Germany). One of the leading mathematicians of the 19^{th} century and the greatest mathematician in Germany after Gauss.

True to the spirit of his time, a spirit compounded of equal parts of faith and nearly incredible ingenuity, he derived in his magnum opus, *Fundamenta Nova Theoriae Functionum Ellipticum*, many elegant and intricate results by means of algebraic manipulations that surpassed even Euler and Gauss. He uncovered a treasure-house of results whose variety, aesthetic appeal and capacity for arousing our astonishment have not been equaled by research in any other area.

Jacobi was born of Jewish parents in Potsdam, Prussia and later converted to Christianity, without which he could not have pursued an academic career in the Germany of those days. He was introduced to mathematics at an early age by his maternal uncle Lehmann, who prepared him to enter the Potsdam Gymnasium in 1816. His unusual talents were recognized already at school and he left in 1821 to enter the University of Berlin. He taught himself algebra, calculus and number theory through the direct reading of the works of Euler and Lagrange. This earliest self-instruction was to give Jacobi's first

outstanding work — in elliptic functions — its definite direction, for Euler, the master of ingenious devices, found in Jacobi his brilliant successor. For sheer manipulative ability in tangled algebra, Euler and Jacobi have no rival, except perhaps Srinivasa Ramanujan in the 20^{th} century.

Jacobi's student days at Berlin lasted from 1821 to 1825. During the first two years, he divided his time about equally between philosophy, philology and mathematics. Mathematics, however, finally won him over, and in 1825 he obtained his degree and moved to the University of Königsberg, where he joined, amongst others, Friedrich Bessel. He soon rose to the rank of associate professor, due to his great talents as an inspiring teacher and his work on cubic reciprocity in number theory. The latter excited Gauss' admiration and with his recommendation, the Ministry of Education promoted Jacobi over the heads of his colleagues.

In 1829 he published his first masterpiece, *Fundamenta Nova* on the theory of elliptic functions and modular equations[49] and obtained his full professorship at the age of 25. This work is one of the greatest mathematical classics that has ever been written — a book perhaps never equaled in the annals of mathematics in the shear number of new and important results first given in it. He continued to work incessantly, with Gauss watching his phenomenal activity with more than a mere scientific interest — as many of Jacobi's discoveries overlapped some of his own youth, which he had never published. The two met in September 1839, when Jacobi, collapsing from 12 years of overwork, returned from a vacation in Marienbad.

In 1842 Jacobi met **Hamilton** at Manchester. It was one of Jacobi's greatest glories to extend Hamilton's work in dynamics, which Hamilton forsook in favor of his quaternions. Later in the year Jacobi became seriously ill with diabetes. Through the efforts of Dirichlet and von Humboldt, he was granted financial support to enable him to visit Italy for a few months and restore his health. On his return he moved to Berlin, where he lived as a royal pensioner. In February 1851 his health deteriorated again. He first contracted influenza and then, on the point of recovery, caught smallpox and died within a week.

Jacobi was the greatest university mathematical teacher of his generation, stimulating and influencing an unprecedented number of able students. He rejected the notion that before doing research, students should first master what has already been accomplished and held that young mathematicians "ought to be pitched into the icy water to learn to swim or drown by themselves, or else they never acquire the knack of independent work".

[49] Jacobi studied *modular equations* for elliptic functions. The equation $u^6 + v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0$ is fundamental for Hermite's 1858 solution of quintics.

His investigations of *elliptic functions*, the theory of which he established upon quite a new basis, and more particularly his development of the *theta functions*, constitute his greatest analytical discovery.[50]

He contributed to complex variable theory and was one of the early founders of the theory of *determinants*. He developed extensively the properties of the functional determinant formed by the n^2 partial derivatives of n given functions w.r.t. their n independent variables, which now bears his name — the *Jacobian* (1829) (although it was known to Cauchy already in 1815). This function plays an important role in differential geometry. The Jacobian of two functions $u(x,y)$, $v(x,y)$ is defined as the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \equiv (\nabla u \times \nabla v)_z.$$

It is represented by the symbol $\frac{\partial(u,v)}{\partial(x,y)}$ or $J(\frac{u,v}{x,y})$. A necessary and sufficient condition that two continuously differentiable functions $u(x,y)$ and $v(x,y)$ in a region R satisfy the relation $F(u,v) \equiv 0$ for some function F is that their Jacobian vanish in R.

Similarly, the Jacobian of three functions $u(x,y,z)$, $v(x,y,z)$ and $w(x,y,z)$ is defined by the determinant

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = (\nabla u \times \nabla v) \cdot \nabla w.$$

A necessary and sufficient condition that three continuously differentiable functions u, v, w satisfy an equation $F(u,v,w) \equiv 0$ in R is that their Jacobian vanish in this region.

A general differentiable transformation of coordinates $\bar{x}_i = f_i(x_1,x_2,x_3)$ in which the functions f_i are single-valued for all points in R can be solved to render $x_i = g(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ iff $J = \left|\frac{\partial \bar{x}}{\partial x}\right| = \det \frac{\partial \bar{x}_i}{\partial x_j} \neq 0$ everywhere in R. The *volume element* is altered by the transformation according to the relation

$$d\bar{V} = \left|J(\frac{\bar{x}_1,\bar{x}_2,\bar{x}_3}{x_1,x_2,x_3})\right| \, dV.$$

[50] On a higher order of originality is his discovery, of *Abelian functions*. Such functions arise in the inversion of an *Abelian integral*, in the same way that the elliptic functions arise from the inversion of an elliptic integral. Here he had nothing to guide him, and for long he wandered lost in a maze that yielded no clue. The appropriate inverse functions in the simplest case are functions of *two* variables having *four* periods; in the general case, the functions have n variables and $2n$ periods; the elliptic functions correspond to $n = 1$.

There are also the *Jacobi identity* for associative algebras, *Jacobi elliptic functions*, *Jacobi polynomials*, *Jacobi zeta function*, *Jacobi epsilon function*, and *Jacobi identity* for a triple infinite product.

Jacobi extended Hamilton's equations of motion via the *canonical transformations*, to what is known as the *Hamilton's-Jacobi equation*. In his formalism, geometrical optics, mechanics and wave mechanics [**Louis Victor de Broglie** (1892–1987, France, 1924) and **Erwin Schrödinger** (1887–1961, Germany, 1925)] meet on common ground: the geometrization of physical phenomena.

Jacobi's contributions to number theory were extensive. In 1827 he stated the law of cubic reciprocity. He applied elliptic functions to the theory of numbers, obtaining the Fermat-Lagrange four-square theorem. Furthermore, Jacobi's theory could determine the number of distinct ways in which each number can be represented.

Jacobi contributed to the theory of differential equations and to the calculus of variations. He introduced (1837) the concept (though not the term) of a *self-adjoint* differential equation: Using modern notation, we consider the general linear second-order PDE

$$\sum_{i,j=1}^{n} A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu + F = 0,$$

or in its index-free vector form,

$$\mathcal{L}[u] + F = \mathfrak{A}\colon \nabla\nabla u + \boldsymbol{b}\cdot\nabla u + cu + F = 0,$$

where $\mathfrak{A}$, $\boldsymbol{b}$, c, F are functions of the coordinates $(x_1, \ldots, x_n)$, one of which can be time.

Define the *adjoint operator* as

$$\overline{\mathcal{L}}[u] = \operatorname{div}\operatorname{div}(\mathfrak{A}u) - \operatorname{div}(\boldsymbol{b}u) + cu.$$

Using certain vector identities it is shown that if $\mathfrak{A}$ is a symmetric tensor and $\boldsymbol{b} = \operatorname{div}\mathfrak{A}$, the operators $\mathcal{L}$ and $\overline{\mathcal{L}}$ are identical. In that case we say that the original PDE is *self-adjoint*, and write it in the compact form

$$\operatorname{div}[\mathfrak{A}\cdot\nabla u] + cu + F = 0.$$

As an example, set $u = u(x_1, x_2, x_3, t)$, $\nabla u = \left\{\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \frac{\partial u}{\partial t}\right\}$ and choose $\mathfrak{A} = \begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & -\rho \end{bmatrix}$ with λ, ρ function of *position* only. The

self-adjoint equation then becomes

$$\rho \frac{\partial^2 u}{\partial t^2} = \operatorname{div}(\lambda \operatorname{grad} u) + cu + F,$$

which is recognized as the *wave-equation*, with div and grad as the usual operators in 3-dimensional space. On the other hand, the *diffusion equation* is *not* self-adjoint, and can be derived from the original equation, with the aid of the identity $\operatorname{div}[\mathfrak{A} \cdot \nabla u] = \operatorname{div} \mathfrak{A} \cdot \nabla u + \mathfrak{A} \colon \nabla\nabla u$. The equation then becomes

$$\operatorname{div}[\mathfrak{A} \cdot \nabla u] + (\boldsymbol{b} - \operatorname{div} \mathfrak{A}) \cdot \nabla u + cu + F = 0.$$

Choosing

$$\mathfrak{A} = \begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & 0 \end{bmatrix}, \qquad \boldsymbol{b} - \operatorname{div} \mathfrak{A} = (0, 0, 0, -\kappa),$$

we obtain

$$\kappa \frac{\partial u}{\partial t} = \operatorname{div}(\lambda \operatorname{grad} u) + cu + F,$$

which renders the diffusion equation.

Lagrange (1762 to 1765) was probably the first to present examples of adjoint differential equations. **Liouville** (1838) gave a special pair of adjoint differential systems. The term *adjoint* is due to **Fuchs** (1873). The theory of self-adjoint differential equations was further developed by **Frobenius** (1873 to 1878).

We note that $u\mathcal{L}[v] - v\overline{\mathcal{L}}[u] = \operatorname{div} \boldsymbol{P}$, where

$$\boldsymbol{P} = uv\boldsymbol{b} + u\mathfrak{A} \cdot \nabla v - v \operatorname{div}(\mathfrak{A}u).$$

When $\mathcal{L}$ is self-adjoint, $\boldsymbol{P} = (u\boldsymbol{\nabla} v - v\boldsymbol{\nabla} u) \cdot \mathfrak{A}$.

For $\mathfrak{A} = \mathfrak{I}$, the special case of *Green's identity*, $u\nabla^2 v - v\nabla^2 u = \operatorname{div}(u\boldsymbol{\nabla} u - v\nabla u)$, is obtained.

In a single spatial dimension, the self-adjoint PDE degenerates into the self-adjoint ODE

$$\frac{d}{dx}\left[p_0(x)\frac{du}{dx}\right] + p_2(x)u + F = 0,$$

where $p_0 \neq 0$ in the interval $a \leq x \leq b$.

One arrives at this result with the explicit requirement that

$$\mathcal{L}[u] = \left[p_0(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x)\right] u(x) + F$$

be made equal to the *adjoint operator*

$$\begin{aligned}\overline{\mathcal{L}}[u] &= \frac{d^2}{dx^2}(p_0 u) - \frac{d}{dx}(p_1 u) + p_2 u + F \\ &\equiv \left[p_0\frac{d^2}{dx^2} + (2p_0' - p_1)\frac{d}{dx} + (p_0'' - p_1' + p_2)\right] u + F,\end{aligned}$$

which happens whenever $p_0' = p_1$.

In this case, an operator can always be made self-adjoint upon its multiplication with

$$\frac{1}{p_0}\exp\left\{\int^x \left(\frac{p_1}{p_0}\right) dx\right\} = \mu(x),$$

leading to the self-adjoint

$$\mu(x)\mathcal{L}[u] = \frac{d}{dx}\left[p_0\mu\frac{du}{dx}\right] + p_2\mu u + F.$$

Also, for $F = 0$

$$u\mathcal{L}[v] - v\overline{\mathcal{L}}[u] = \frac{\partial}{\partial x}\left[p_0 u v' - v(p_0 u)' + p_1 u v\right].$$

To the Newton-Laplace-Lagrange theory of attraction Jacobi made substantial contributions, by his investigations on the functions which recur repeatedly in that theory and by the application of elliptic and Abelian functions to the attraction of ellipsoids.

Jacobi (1841) introduced the notation d and ∂ for total and partial derivatives (*differentialia partialia*), respectively, i.e. he was first to write $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$. This he generalized to a function of n variables $f(x_1, x_2, \ldots, x_n)$. The notation $\frac{\partial f}{\partial x}$ advocated by Jacobi did not meet with immediate adoption. It took half a century for it to secure a generally recognized place in mathematical writing.[51]By 1898, Jacobi's notation was accepted

[51] When **Cayley** (1857) abstracted Jacobi's paper, he paid no heed to the new notation and wrote all derivatives in the form $\frac{df}{dx}$, etc.

Partial derivatives appear in the writing of **Newton**, **Leibniz**, and the

in England, where Hamilton's gradient operator was written for the first time as $\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$.

The Elliptic Wonderland of Jacobi[52]

"God ever geometrizes"

Plato (427–347 BCE)

"God ever arithmetizes"

C.G.J. Jacobi (1829 CE)

A large number of important properties of elliptic integrals were observed by **Euler** *and* **Legendre** *before it was realized that the inverses of certain*

Bernoullis, but as a rule without any special symbolism. **Euler** (1776) used $\frac{\partial^\lambda}{p} \cdot V$ to indicate the λth derivative, partial w.r.t. the variable p, operating upon V. The use of the rounded letter ∂ in the notation for partial differentiation occurs again (1786) in an article by **Legendre**, but he himself soon abandoned his own notation in later papers.

52 For further reading, see:

- Lawden, D.F., *Elliptic Functions and Applications*, Springer-Verlag: New York, 1989, 334 pp.
- Dutta, M. and L. Debnath, *Elements of the Theory of Elliptic and Associated Functions* (With Applications), The World Press Private, 1965, 290 pp.
- Eagle, A., *The Elliptic Functions as They Should Be*, Gallaway and Porter: Cambridge, England, 1958, 508 pp.
- Oberhettinger, F. and W. Magnus, *Anwendung Der Elliptischen Functionen in Physik und Technik*, Springer-Verlag: Berlin, 1949, 126 pp.

standard types of elliptic integrals, rather that the integrals themselves, should be regarded as fundamental functions of analysis. This idea is due to ***Gauss****,* ***Abel*** *and* ***Jacobi****.*

Gauss inverted the lemniscate integral (1797)

$$u = \int_0^x \frac{dt}{\sqrt{1-t^4}}$$

and defined through it the "lemniscate sine function"

$$x = \mathrm{sl}(u).$$

He found that the function was periodic, like the sine, with period

$$2\widetilde{\omega} = 4\int_0^1 \frac{dt}{\sqrt{1-t^4}}.$$

From the relation

$$\frac{d(it)}{\sqrt{1-(it)^4}} = i\frac{dt}{\sqrt{1-t^4}}$$

he deduced $\mathrm{sl}(iu) = i\mathrm{sl}(u)$ *and hence that the lemniscate sine has a second period* $2i\widetilde{\omega}$*. Thus Gauss discovered double periodicity, one of the key properties of elliptic functions, though at first he did not realize its universality. However, the importance of elliptic functions became clear to him when he independently discovered (1799) an ingenious method to calculate numerically the values of complete and incomplete elliptic integrals by using the arithmetic-geometric mean.*

To grasp the revolutionary idea of inversion, consider, for example, the fundamental elliptic integral of the first kind

$$\begin{aligned} u(x) &= \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \\ &= \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \qquad t = \sin\theta, \qquad x = \sin\phi, \end{aligned}$$

where the parameter k *is known as the modulus and* $k' = \sqrt{1-k^2}$ *is the complementary modulus. In the trivial case* $k^2 = 0$ *we have*

$$u(x) = \phi = \sin^{-1} x = \int^x \frac{dt}{\sqrt{1-t^2}}$$

where u is a multivalued function of x. The inverse relation $x = \sin u$ is simple and represents a single-valued periodic function of period 2π. A similar situation occurs with $u = \log x = \int^x \frac{dt}{t}$ and $x = e^u$.

In light of this analogy, ***Jacobi*** *defined for all $k \leq 1$*

$$u(x) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \mathrm{sn}^{-1}x.$$

The inverse functions, single-valued and analytic, are then defined through the new symbols

$$x = \mathrm{sn}(u,k), \quad \phi = \mathrm{am}(u,k).$$

The fundamental new functions are related through the equations:

$$x = \mathrm{sn}(u,k) = \sin\phi = \sin[\mathrm{am}(u,k)]$$

$$\sqrt{1-x^2} = \mathrm{cn}(u,k) = \cos\phi = \cos[\mathrm{am}(u,k)]$$

$$\sqrt{1-k^2x^2} = \mathrm{dn}(u,k) = [1-k^2\sin^2(\mathrm{am}(u,k))]^{1/2}$$

$$\mathrm{am}(u,0) = u, \quad \mathrm{sn}(u,0) = \sin u, \quad \mathrm{cn}(u,0) = \cos u, \quad \mathrm{dn}(u,0) = 1$$

$$\mathrm{sn}^2u + \mathrm{cn}^2u = 1; \quad \mathrm{dn}^2u + k^2\mathrm{sn}^2u = 1$$

$$u = \mathrm{sn}^{-1}x = \mathrm{cn}^{-1}\sqrt{1-x^2} = \mathrm{dn}^{-1}\sqrt{1-k^2x^2}$$

Euler *(1761) has shown that if $R(\xi)$ is a rational polynomial in ξ of the 4^{th} order, then there exists an algebraic function $W(x,y)$ such that*

$$\int_0^x \frac{d\xi}{\sqrt{R(\xi)}} + \int_0^y \frac{d\xi}{\sqrt{R(\xi)}} = \int_0^{W(x,y)} \frac{d\xi}{\sqrt{R(\xi)}}.$$

Thus, for $R(\xi) = (1-\xi^2)(1-k^2\xi^2)$, Euler found

$$W(x,y) = \frac{x\sqrt{1-y^2}\sqrt{1-k^2y^2} - y\sqrt{1-x^2}\sqrt{1-k^2x^2}}{1-k^2x^2y^2}.$$

Using the notation of ***Jacobi***

$$u = \int_0^x \frac{d\xi}{\sqrt{R(\xi)}}, \quad x = \mathrm{sn} u, \quad v = \int_0^y \frac{d\xi}{\sqrt{R(\xi)}}, \quad y = \mathrm{sn} v$$

$$W = \mathrm{sn}(u+v);$$

one can recast Euler's result in the form of an addition theorem

$$\mathrm{sn}(u+v) = \frac{\mathrm{sn}\ u\ \mathrm{cn}\ v\ \mathrm{dn}\ v + \mathrm{sn}\ v\ \mathrm{cn}\ u\ \mathrm{dn}\ u}{1 - k^2 \mathrm{sn}^2 u \mathrm{sn}^2 v}.$$

In inverse-function notation this reads

$$\mathrm{sn}^{-1} x + \mathrm{sn}^{-1} y = \mathrm{sn}^{-1} W(x,y).$$

In the limit $k = 0$, the last two relations degenerate into the familiar trigonometric formulas

$$\sin(u+v) = \sin u \cos v + \cos u \sin v,$$

$$\sin^{-1} x + \sin^{-1} y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$$

The theory of elliptic functions includes two fundamental parameters: the modulus k and the complete elliptic integral of the first kind

$$K(k) = \mathrm{sn}^{-1}(1) = \int_0^1 [(1-t^2)(1-k^2t^2)]^{-1/2}\, dt = \frac{\pi}{2} {}_2F_1(\frac{1}{2}, \frac{1}{2}, 1, k^2),$$

$$\mathrm{am}\ K = \frac{\pi}{2}, \qquad k'^2 + k^2 = 1.$$

Associated with K is the quantity

$$K' = \int_0^1 [(1-t^2)(1-k'^2t^2)]^{-1/2}\, dt = K(k').$$

The elliptic functions reduce to circular functions with

$$k = 0, \quad k' = 1, \quad K = \frac{\pi}{2}, \quad K' = \infty$$

and to hyperbolic functions with

$$k = 1, \quad k' = 0, \quad K = \infty, \quad K' = \frac{\pi}{2}$$

in which case:

$$\mathrm{sn} x = \mathrm{th}\ x, \quad \mathrm{cn}\ x = \mathrm{dn}\ x = \frac{1}{\mathrm{ch}\, x}.$$

The elliptic functions are doubly periodic in the complex u-plane. To see this important feature, one effects the substitution

$$x = \frac{iy}{\sqrt{1-y^2}} = \sin\phi = i\tan\psi,$$

implying

$$\cos\phi = \frac{1}{\cos\psi}, \quad -\sin\phi\, d\phi = \sec\psi\tan\psi\, d\psi,$$

$$\sqrt{1-k^2\sin^2\phi} = \frac{1}{\cos\psi}\sqrt{1-k'^2\sin^2\psi}.$$

Then

$$u = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$= i\int_0^\psi \frac{d\alpha}{\sqrt{1-k'^2\sin^2\alpha}} = i\int_0^y \frac{d\xi}{\sqrt{(1-\xi^2)(1-k'^2\xi^2)}} \equiv iW.$$

This further implies $y = \sin\psi = \mathrm{sn}(W, k')$ *and*

$$\mathrm{sn}(u,k) = \mathrm{sn}(iW,k) = \sin\phi$$

$$\mathrm{cn}(u,k) = \mathrm{cn}(iW,k) = \cos\phi$$

$$\mathrm{dn}(u,k) = \mathrm{dn}(iW,k) = \sqrt{1-k^2\sin^2\phi}.$$

For $k = 0$ *(trigonometric functions)* $\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{2} = \frac{1}{4}T$ *where* $T = 2\pi$ *is the period. It is therefore natural to expect that* K *will assume the role of the quarter-period of the elliptic function. To see this we calculate*

$$\int_0^{\pi n+\beta} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \int_0^{\pi n} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} + \int_{\pi n}^{\pi n+\beta} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

$$= 2nK + \int_0^{\beta} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = 2nK + u.$$

The above relation can be translated into $\sin\beta = \mathrm{sn} u$, $\sin(\beta+\pi n) = \mathrm{sn}(2nK+u)$, *or* $\mathrm{sn}(u \pm 4K) = \mathrm{sn} u$. *In fact,* $4K$ *is the period of all three elliptic functions* $\mathrm{sn} u$, $\mathrm{cn} u$ *and* $\mathrm{dn} u$.

Similar manipulation involving k' *show that*

$$\mathrm{sn}(u+4K) = \mathrm{sn}(u+2iK') = \mathrm{sn}\ u$$

$$\mathrm{cn}(u+4K) = \mathrm{cn}(u+2K+2iK') = \mathrm{cn}\ u$$

$$\mathrm{dn}(u+2K) = \mathrm{dn}(u+4iK') = \mathrm{dn}\ u,$$

exhibiting the double periodicity *of the elliptic functions in the complex* x *plane.*

It can be shown that a function of complex variable cannot have two incommensurate periods in the same direction, *but if one of the periods is in a* different direction *in the complex plane (i.e. the ratio of the periods is not a real number), this is possible. Thus, instead of a* one-dimensional *sequence of periods (as in the case for the trigonometric functions) there will be a* two-dimensional *lattice of parallelograms, with the function repeating, in each parallelogram, its behavior in every other parallelogram. The smallest unit within which the function goes through all its behavior is called the* unit cell *for the function; each side of the unit cell is one of the fundamental periods for the function.*

Thus, for real W, *one has*

$$\{\mathrm{sn}\ iW,\ \mathrm{cn}\ iW,\ \mathrm{dn}\ iW\}$$

defined for purely imaginary argument $u = iW$ *in terms of the real Jacobi functions with real argument* W *and complementary modulus* k'.

Furthermore, the same reasoning that permitted us to define $\text{sn } u$ *and hence* $\text{cn } u$ *and* $\text{dn } u$ *as periodic functions with real period* $4K$ *shows that we can take* $\text{sn}(W, k')$, $\text{cn}(W, k')$ *and* $\text{dn}(W, k')$ *as real periodic functions with real period*

$$4K' = 4\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k'^2x^2)}}.$$

Clearly, $4iK'$ *is a second period, purely imaginary, for* $\text{sn } u$, $\text{cn } u$, *and* $\text{dn } u$, *that is*

$$\text{sn}(u + 4iK') = \text{sn } u$$

etc.

Jacobi elliptic functions are therefore doubly periodic functions of the complex variable u. *However, all told, the Jacobi elliptic function, although defined for real* u *and purely imaginary* u, *were never defined for complex* $u = \sigma + i\tau$. *Abel and Jacobi got around this by using the addition formula for* $\text{sn } u$, *etc.*

This procedure, unfortunately, breaks down when one wishes to use complex values of k.

Thus, if one wishes to define Jacobi elliptic functions as functions of a complex variable, by using the idea of inverting the elliptic integral of the first kind, [i.e., considering one limit of integration as a complex variable, and the value of the integral as a complex line integral over some curve], then one must use a thoroughly complex variable technique which takes into account all the difficulties of integrating a multi-valued function, with branch cuts in complex plane. The correct technique was discovered by ***Riemann***, *who introduced the notion of Riemann surface precisely to handle such problems.*

There are no functions of complex variable z *which have more than two independent periods.*

The elliptic function $y = \text{sn}(u, k)$ *has simple zeros at* $u = 2mK + 2nK'i$ $(m, n = 0, \pm 1, \pm 2, \ldots)$ *and has simple poles at* $u = 2mK + i(2n+1)K'$ *(the first pole at* $u = iK'$ *on the imaginary axis);*

It thus has a row of zeros along the real axis, spaced at distance $2K$ *apart and a row of poles along the line* $y = K'$, *each vertically above a zero on the real axis, and so on. [The residue at the pole* $u = iK'$ *is* $\frac{1}{K}$, *and the residue at* $u = 2K + iK'$ *is* $(-\frac{1}{K})$.*]*

This property makes the elliptic function useful in the solution of certain potential problem in electrostatics.

The 5-Fold Way

The number 5 has the following remarkable traits:

- *The geometry of art, aesthetics and life is associated with the pentagon, the pentagram and the Golden Section, that is inherent in both. Five is also the 4^{th} Fibonacci number. In Phyllotaxis (regular arrangement of leaves of a stems or petals in flowers), a pattern with 5 units occurs very frequent [whimsical: 5 fingers on human limbs]. Five is also the hypotenuse of the smallest Pythagorean triangle.*

- *There are 5 Platonic solids: the regular tetrahedron, cube, octahedron, dodecahedron and icosahedron (all but the cube were named after the Greek word for their number of faces). They were all known to the Greeks.* ***Euclid*** *showed that there are no more than 5.*

  Kepler *used them, with typical confidence in their mystical properties, to explain the relative sizes of the orbits of the planets.*

- *The 'worst' close-regular-packing of spheres*[53] *in any dimension is at dimension 5.*

- *The smallest integer n for which $F_n = 2^{2^n} + 1$ (Fermat number) is composite: $F_5 = 2^{32} + 1 = 641 \cdot 6,700,417 = 4,294,967,297$.*

- *The n^{th} Fibonacci number is given by the formula*

$$F_n = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n].$$

- *The smallest Z_n group that cannot be a symmetry of a periodic lattice is $Z_5 = 5$-fold symmetry.* ***D. Schechtman*** *(1984, Israel) has demonstrated that this symmetry is nevertheless realized in non-periodic quasi-crystals.*

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[53] The volume of a *unit sphere* in n-dimensions is $V_n = \pi^{n/2}/\Gamma\left(\frac{n}{2}+1\right)$. It is largest at $n = 5$ ($V_1 = 2$, $V_2 = 3.14\ldots$, $V_3 = 4.19\ldots$, $V_4 = 4.93$, $V_5 = 5.26$, $V_6 = 5.17\ldots$, $V_\infty = 0$). In a 5-dimensional space the *density of packing* has a minimal value of $\frac{\sqrt{2}}{60}\pi^2 = 0.2325\ldots$.

- *The general algebraic equation of the* 5^{th} *or higher degree cannot be solved in terms of the coefficients by using only a finite sequence of arithmetical operations and radicals. This was first proved by Abel during* 1824–1826*. By* 1831*,* ***Galois*** *established the theory of algebraic solution of equations in its most complete form, associating it with subgroups of the group of permutation of the roots. The results of Galois are much deeper and more general than those of Abel. Moreover, Galois found that algebraic equations of orders* 5*,* 7 *and* 11 *are related to the modular equations in the theory of elliptic functions.*

  Thus, by shutting the door to the possibility of algebraic *solution of a class of polynomial equations, he simultaneously opened another door to* nonalgebraic *solutions of the same class, requiring an* infinite *number of arithmetical operations on the coefficients. Indeed in* 1858*,* ***Hermite*** *used this method in a very elegant manner to obtain all* 5 *solutions of the quintic equation in terms of elliptic functions [analogously to the trigonometrical solution of the cubic equation]. Finally, it was shown by* ***C. Jordan*** *in* 1870 *that the solutions of the general algebraic equation of degree higher than* 5 *are not expressible in terms of* elliptic functions *alone.*

- *Plays an unexpected part in the Rogers-Ramanujan identities (1894)*

$$1+\sum_{1}^{\infty}\frac{x^{m^2}}{(1-x)(1-x^2)\dots(1-x^m)}=\prod_{0}^{\infty}\frac{1}{(1-x^{5m+1})(1-x^{5m+4})}$$

$$1+\sum_{1}^{\infty}\frac{x^{m(m+1)}}{(1-x)(1-x^2)\dots(1-x^m)}=\prod_{0}^{\infty}\frac{1}{(1-x^{5m+2})(1-x^{5m+3})}$$

It is also reflected in Ramanujan's most bizarre result, obtained with the aid of the above identities (1913)

$$u=\cfrac{x}{1+\cfrac{x^5}{1+\cfrac{x^{10}}{1+\cfrac{x^{15}}{1+\cfrac{x^{20}}{1+\dots}}}}};\qquad v=\cfrac{x^{1/5}}{1+\cfrac{x}{1+\cfrac{x^2}{1+\cfrac{x^3}{1+\cfrac{x^4}{1+\dots}}}}}$$

$$v^5=u\frac{1-2u+4u^2-3u^3+u^4}{1+3u+4u^2+2u^3+u^4}$$

As in the previous item, the number 5 *seems to emerge out of the theory of elliptic functions. Yet, there is here a hidden connection between seemingly unrelated formulas: even after we understand what has been done, the feeling of bewilderment and wonder is not lifted.*

After x *is eliminated, one is left with a quadratic equation in* v^5*, namely*

$$\frac{(u^{-1}-1-u)^6}{u^{-5}-11-u^5} = \frac{1}{v^5} - 11 - v^5,$$

which upon simplification yields the desired result.

1829–1832 CE **Évariste Galois** (1811–1832, France).

"Down, down, down into the darkness of the grave
Gently they go, the beautiful, the tender, the kind;
Quietly they go, the intelligent, the witty, the brave.
I know. But I do not approve. And I am not resigned".

Edna St. Vincent Millay, '*Dirge Without Music*'

A most brilliant mathematician who, in a brief meteoric career, laid the foundations to the theory of groups and the theory of algebraic equations. His theory of equations is based upon concepts of group theory and supplies criteria for the possibility of solving an algebraic equation by radicals.

Galois resolved the deeper issues of solvability. His *group-theoretic* approach superseded the *algebraic* theories of Lagrange (1770), Ruffini (1799) and Abel (1824).

In 1815 **Gauss** gave an algebraic proof of the fundamental theory of algebra. The problem with this theorem is, however, that it does not tell us what the roots are.

After **Abel's** work (1826), the situation was as follows: Although the general equation of degree higher then four was known to not be solvable by radicals, there were special equations (e.g. $x^p = a$, p *prime*) that were

solvable by radicals. It remained to determine *which equations are solvable by radicals*. This task was successfully undertaken by **Evariste Galois**.

Galois' idea was to associate to any polynomial equation a group in such a way that the properties of the group and the nature of the solutions of the equations are closely related. In particular, he devised groups that reflect the symmetry properties of the roots of general polynomial equations.

To this end he introduced the concept of the *Galois group of an equation*: If $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is a polynomial over the rationals, then there are certain rational functions with rational-valued coefficients $H(\alpha_1, \dots, \alpha_n) = 0$, among the solutions $\alpha_1, \dots, \alpha_n$ of $f(x) = 0$. The group of all permutations that leave all the relations $H(\alpha_1, \dots, \alpha_n) = 0$ invariant, is called the *Galois group*[54] of the equation[55]. It can then be shown that *any* rational relation (in the above sense) left invariant by all permutations in the Galois group, is rational-valued (in either of the senses explained in the previous footnote).

The central theorem in Galois' theory then states that a polynomial equation is soluble by radicals if and only if its group is 'solvable'. When the Galois group for any equation has been found, a criterion devised by Galois will indicate whether or not the group is 'solvable'.

The association of the group concept with solutions of algebraic equations can be illustrated with aid of the following example:

The equation $x^3 - 2 = 0$ has the three roots: $x_1 = \sqrt[3]{2}$, $x_2 = \omega\sqrt[3]{2}$, $x_3 = \omega^2\sqrt[3]{2}$, where $\omega = -\frac{1}{2} + \frac{1}{2}i\sqrt{3}$ is a primitive cube root of unity. The three roots can be pictured in the plane of complex numbers as 3 points, equally spaced on a circle of radius $\sqrt[3]{2}$. There are 6 operations of permuting the roots, such that $x^3 - 2 = (x - x_1)(x - x_2)(x - x_3)$ remains invariant[56]. They are:

(1) Rotation of each root-vector by 120° counterclockwise:

$$\sqrt[3]{2} \to \omega\sqrt[3]{2} \to \omega^2\sqrt[3]{2} \to \sqrt[3]{2}.$$

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[54] For further reading, see:

- Maxfield, J.E. and M.W. Maxfield, *Abstract Algebra and Solutions by Radicals*, Dover Publications, 1992, 209 pp.

[55] Here "rational-valued" could mean either rational numbers, *or* rational functions of the coefficients $\{\alpha_j\}$.

[56] The coefficients of this polynomial, or of any polynomial, are totally symmetric polynomials of the roots, and furthermore, any rational function of the roots that is totally symmetric under root permutations, can be shown to be a rational function of these symmetric polynomials.

The corresponding permutation is

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

(2) Rotation of each root-vector by 240° counterclockwise:

$$\sqrt[3]{2} \to \omega^2 \sqrt[3]{2} \to \omega \sqrt[3]{2} \to \sqrt[3]{2}.$$

The permutation is

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

(3) Reflection about the horizontal (real) axis:

$$\sqrt[3]{2} \to \sqrt[3]{2}, \quad \omega \sqrt[3]{2} \to \omega^2 \sqrt[3]{2}, \quad \omega^2 \sqrt[3]{2} \to \omega \sqrt[3]{2},$$

corresponding to the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

(4) The identity

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

(5) Reflection about an axis going through the origin and $\omega \sqrt[3]{2}$, corresponding to the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

(6) Reflection about an axis going through the origin and $\omega^2 \sqrt[3]{2}$, the corresponding permutation being

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

These 6 permutations constitute the group of symmetries of the equation.

Consider next the particular equation

$$x^4 + px^2 + q = 0, \qquad (p, q \quad \text{rational numbers})$$

having the explicit roots

$$x_1 = \sqrt{\frac{-p + \sqrt{p^2 - 4q}}{2}}; \qquad x_2 = -\sqrt{\frac{-p + \sqrt{p^2 - 4q}}{2}}$$

$$x_3 = \sqrt{\frac{-p - \sqrt{p^2 - 4q}}{2}}; \qquad x_4 = -\sqrt{\frac{-p - \sqrt{p^2 - 4q}}{2}}$$

Let $\mathbb{Q}$ be the field of the rationals. Clearly

$$x_1 + x_2 = 0, \qquad x_3 + x_4 = 0$$

holds. Of the 24 possible permutations of the above 4 roots, the following 8 substitutions (permutations)

$$\begin{aligned}
E &= \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix} & E_1 &= \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_1 & x_3 & x_4 \end{pmatrix} \\
E_2 &= \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 & x_2 & x_4 & x_3 \end{pmatrix} & E_3 &= \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_1 & x_4 & x_3 \end{pmatrix} \\
E_4 &= \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_3 & x_4 & x_1 & x_2 \end{pmatrix} & E_5 &= \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_4 & x_3 & x_1 & x_2 \end{pmatrix} \\
E_6 &= \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_3 & x_4 & x_2 & x_1 \end{pmatrix} & E_7 &= \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}
\end{aligned} \tag{1}$$

leave the two relations true in $\mathbb{Q}$. One could show that these 8 are the *only* substitutions of the 24=4! which leave invariant *all* rational relations in $\mathbb{Q}$ among the roots. These 8 are *the Galois group of the equation in* $\mathbb{Q}$ and constitute a subgroup of the full group. That is, the group of an equation w.r.t. a field $\mathbb{Q}$ is the group of substitutions on the roots which leave invariant the rational relations with coefficients in $\mathbb{Q}$ (among the roots of the given equation).

One can say that the number of substitutions that leave all rational relations in $\mathbb{Q}$ invariant is a measure of our ignorance of the root because we cannot distinguish them under these 8 substitutions.

Now consider

$$x_1^2 - x_3^2 \equiv \sqrt{p^2 - 4q}. \tag{2}$$

We adjoin this radical to $\mathbb{Q}$, and form the field $\mathbb{Q}'$, which is the smallest fields containing $\mathbb{Q}$ and $\sqrt{p^2 - 4q}$. Since $x_1 + x_2 = 0$ and $x_3 + x_4 = 0$ we also have

$$x_1^2 = x_2^2, \quad x_3^2 = x_4^2.$$

We then notice that of the 8 substations listed in (1) only $E,\ E_1,\ E_2,\ E_3$ leave the $\mathbb{Q}'$-valued relation (2) invariant. Then these four substitutions, since they leave *every* true $\mathbb{Q}'$-valued rational relation among the roots invariant, are the Galois group of the original quartic equation over $\mathbb{Q}'$. These four comprise a subgroup of the eight-member Galois group of the equation.

Suppose next that we adjoin to $\mathbb{Q}''$ the quantity $\sqrt{\frac{1}{2}\left(-p - \sqrt{p^2 - 4q}\right)}$, thereby forming the field $\mathbb{Q}''$. Then

$$x_3 - x_4 = 2\sqrt{\frac{1}{2}\left(-p - \sqrt{p^2 - 4q}\right)} \tag{3}$$

is a rational relation in $\mathbb{Q}''$. This relation remains invariant only under the substitution E and E_1, but not under the rest of the eight. Thus, the group of the equation in $\mathbb{Q}''$ consists of these two substitutions, because every rational relation in $\mathbb{Q}''$ among the roots remains invariant under theses two substitutions. The two comprise the subgroup of the previous four-substitutions subgroup.

If we finally adjoint to $\mathbb{Q}''$ the quantity $\sqrt{\frac{1}{2}\left(-p + \sqrt{p^2 - 4q}\right)}$ we get $\mathbb{Q}'''$. In which we have

$$x_1 - x_2 = 2\sqrt{\frac{\left(-p + \sqrt{p^2 - 4q}\right)}{2}}$$

It is found that the only substitution leaving all the rational relations over $\mathbb{Q}'''$ invariant is just E (the trivial subgroup) – and this is the group of the equation over $\mathbb{Q}'''$.

Now Galois showed that when the group of an equation w.r.t. a given field is just E, then the roots of the equation are members of that field.

There is next a straightforward process for finding the roots by rational operations in $\mathbb{Q}'''$. Galois pointed out, however, that his work was not intended as an efficient practical method of solving equations. Yet the Galois theory shows that the general n^{th} - degree equation for $n > 4$ is not solvable by radicals whereas for $n \leq 4$ they are. (see the following essay).

Galois was born in the village Bourg-la-Reine near Paris, the son of the village mayor. Throughout his school years, Galois was hampered by teachers who discouraged his interest in mathematics.

Galois discovered mathematics with the reading of Legendre's textbook of Euclidean geometry at the age of 13. Finding his school algebra textbook boring, he started at the age of 14 to read the original memoirs of Lagrange and Abel, whose algebraic analyses were addressed to professional mathematicians. Galois tried twice (1827, 1829) to enter the École Polytechnique, but was refused admission for inability to meet the formal requirements of his examiners, who completely failed to recognize his genius. This failure drove him in upon himself and embittered him for the remainder of his short life.

In 1828, at the age of 17, Galois was already making discoveries of epochal significance in the theory of equations, discoveries whose consequences are not yet exhausted after almost two centuries. In 1829, he published his first paper, on continued fractions, and entered the École Normale to prepare himself to teach. At about this time he presented an abstract of his fundamental discoveries to **Cauchy** for presentation to the Academy of Sciences. Cauchy promised to present this, but forgot — and also lost the manuscript. Embittered and frustrated, Galois was drawn by democratic sympathy into the turmoil of the 1830 revolution. He was expelled from school and spent several months in prison. Shortly after his release, he was killed in a pistol duel with a friend: both men, having fallen in love with the same girl, decided the outcome by a gruesome version of Russian roulette. The night before, he wrote his scientific testament in the form of a letter to one of his friends. He was buried in the common ditch of the South Cemetery, so that today there remains no trace of the grave of Évariste Galois. His enduring monument is his collected works, 60 pages in all.

Hermann Weyl, a leading 20^{th} century mathematician, had this to say (1952):

"*If judged by the novelty and profundity of ideas it contains, it is perhaps the most substantial piece of writing in the whole literature of mankind*".

In 1846, **Joseph Liouville** published several of Galois' memoirs and manuscripts in his *Journal de mathématique*. The importance of Galois' ideas became apparent only after they were applied in 1870 by **Camille Jordan**, **Felix Klein** and **Sophus Lie**.

Galois and the Dawn of Abstract Algebra

*The first stirrings of modern abstract algebra began with investigations of the theory of equations, and studies of n-object permutations that arose in this theory. This line of work began with successive attempts to algebraically prove the fundamental theorem of algebra (**Euler**, **Lagrange**, **Laplace**).*

Lagrange's work, in particular (1771–3), introduced the use of symmetric functions of the roots of a general polynomial, and proved what later became known as Lagrange's theorem in group theory[57]*. Those early proofs of the Fundamental Theorem*[58] *suffered from the common flaw of assuming that the roots of any polynomial exist, in some sense.*

***Gauss** gave the first (almost) algebraic proof which escaped this apparent tautology, by means of the so-called "principle of continuation of identities"*[59]*. In the course of this proof (1815), Gauss introduced a congruence of polynomials modulo a given polynomial, thus paving the way to such abstract-algebra concepts as ideals, quotient rings, field extensions and splitting fields.*

***Cauchy** independently co-discovered a subset of these new concepts and methods (1815); and his own studies of symmetry permutation groups of*

[57] *Lagrange Theorem*: The order (# of elements) of a subgroup divides the order of the larger group.

[58] *Fundamental Theorem of Algebra*: every n-th order polynomial with real coefficients is factorizable into linear and quadratic factors.

[59] It states that if a function F of the n fundamental symmetric polynomials

$$\sigma_1 = \alpha_1 + \cdots + \alpha_n, \ldots, \qquad \sigma_n = \alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_n$$

is identically zero, $F(\sigma_1, \ldots, \sigma_n) \equiv 0$, then this also holds for *any* n reals $\sigma_j = s_j$. Here α_j are the n putative roots.

algebraic functions led to what later came to be known as Cauchy's theorem of group theory[60]*. Additionally, Gauss' earlier work on the roots of unity and cyclotomic fields, together with* ***Abel****'s extensions, pioneered many related concepts, including what were later recognized as* Galois groups *(the* Abelian, *or commutative case).*

Galois*, having read the relevant works by* ***Legendre*** *and* ***Gauss****, made a complete study of* finite fields, *and introduced, for the first time, explicit definitions of groups, normal subgroups, field extensions and related concepts (albeit with different names, in some cases, from later nomenclature in what came to be known as "Galois' theory") His efforts to characterize the hidden structure of polynomials in terms of groups – especially with regard to the solvability-by-radicals of polynomials – were so* complete *and successful that the old "theory of equations"* ended, *in effect, with him, while giving birth to modern* abstract algebra.

Later (1843), ***Hamilton*** *managed to finally* extend the field of complex numbers *into the non-commutative (yet still associative) field of* quaternions. *Hamilton's approach – developed with physical applications in mind – led him to do what Galois and his followers did: namely to generalize ordinary addition and multiplication (and sometimes division) into* abstract *operations among a priori-undefined* elements, *thus breaking ground for modern axiomatics. Galois' work was continued by* ***Cayley****,* ***Jordan****,* ***Serret*** *and others*[61]*; Hamilton's quaternions were subsumed by Gibbs'* vectors, *but remained a cornerstone of modern algebra and re-entered physics in the guise of the Pauli matrices – describing quantum-mechanical* spin *– and was generalized to* Clifford algebras *in the context of quantum field theories and GTR. The abstract algebra pioneered bay Galois, Hamilton and their predecessors & followers, exercises a unifying effect within modern mathematics itself, and has also led to many important applications in modern physics.*

Here, we shall present a simplified modern synopsis of that part of Galois' theory dealing with the solvability by radicals of algebraic equations[62].

Let $f(x)$ be any n-th order polynomial over the field of rational numbers[63]*, with the coefficient of the highest power normalized to unity:*

60 *Cauchy's theorem*: a group of order $n = p \cdot m$, p prime, has a subgroup of order p

61 Including **F. Klein**, **E. Moore**, **Hölder**, **L. Kronecker**, and **S. Lie**.

62 Another branch of his work – that dealing with finite fields – has found modern applications in digital logic design and cryptography

63 Galois theory applies to polynomials over *any* algebraic field, not just $\mathbb{Q}$.

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad a_j \in \mathbb{Q}$$

where $\mathbb{Q}$ denotes the field of (real) rational numbers. By the fundamental theorem of algebra, $f(x)$ has n complex roots (not all of which need be distinct):

$$f(x) = (x - d_1) \cdot (x - d_2) \cdot \cdots \cdot (x - d_n), \qquad d_j \in \mathbb{C}$$

with $\mathbb{C}$ the field of complex numbers. Expanding this product, we find that the coefficients a_j are symmetric polynomials *in the roots $\{d_k\}$:*

$$a_0 = (-1)^n d_1 \cdot d_2 \cdot \cdots \cdot d_n, \ \ldots \qquad a_{n-1} = -d_1 - d_2 - \cdots - d_n.$$

These n-variable *polynomials are 'symmetric' in the sense that they remain invariant under arbitrary* permutations *of the n roots of $f(x)$. Some of these roots might be rational (i.e. lie in $\mathbb{Q}$).*

Denote a maximal, linearly-independent (over $\mathbb{Q}$) subset of the irrational *roots, if any, by $\{c_1, \ldots, c_m\}$ (we only include* distinct *irrational roots in this set). Note that if* all *roots of $f(x)$ are in $\mathbb{Q}$, $m = 0$ and the set is empty.*

It can be shown that any *symmetric rational function of the roots $\{d_j\}$ can be expressed as a rational function of the coefficients $\{a_j\}$ (which are themselves symmetric polynomials, as seen above.) Since a_j are in $\mathbb{Q}$, it follows that any symmetric rational $R(d_1, \ldots, d_n)$ has a numerical value in $\mathbb{Q}$. Galois posed the following question: are there any rational functions of $\{d_j\}$ which are* not *fully symmetric, yet nevertheless assume rational numerical values for a given polynomial $f(x)$? The answer is clearly in the affirmative; for example, if $f(x)$ has even a single rational root (say d_1), the rational function $R(d_1, \ldots, d_n) \equiv d_1$ is not invariant under all permutations, yet assumes a rational numerical value.*

Galois was thus led to associate to each polynomial a group: *The group[64] $\mathbb{G}(f)$ of all permutations of the n roots such that* a generic rational function *$R(d_1, \ldots, d_n)$ assumes a rational numerical value if, and only if, it is invariant under the permutation belonging to $\mathbb{G}(f)$. Clearly, $\mathbb{G}(f)$ is a subgroup of the group of* all *possible permutations of the n roots; the latter group is known as the* symmetric group *of n objects, and denoted S_n. It can be shown that, in fact, $\mathbb{G}(f) = S_n$ for a* generic *n-th order polynomial over $\mathbb{Q}$; but $\mathbb{G}(f)$ is a proper subgroup (i.e. smaller than S_n) for special choices of the coefficients a_j, or of algebraic relations among them. If all the d_j are rational, $R(d_1, ..., d_n)$*

[64] To be more precise, this group should be denoted $\mathbb{G}_{\mathbb{Q}}(f)$ and referred to as the Galois group of $f(x)$ over $\mathbb{Q}$. However, we shall employ the simpler notation.

is a rational number for any rational function R, so clearly $\mathbb{G}(f)$ is the trivial group[65] *$\{\mathbf{1}\}$, consisting only of the trivial (identity) permutation*

$$\begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix} \equiv \mathbf{1}.$$

There is another, equivalent way of determining the Galois group – an abstract-algebraic way, as follows. One defines an extension E of the field $\mathbb{Q}$, such that E is also a field and all roots d_j belong to E. The basis irrational roots $\{c_1, \dots, c_m\}$ and their powers generate E over $\mathbb{Q}$ (if $m = 0$, we simply have $E = \mathbb{Q}$). This minimal extension field, denoted as $E = \mathbb{Q}(c_1, \dots, c_m)$, is a finite-dimensional vector space, because $(c_j)^n$ and higher powers can be expressed as linear combinations of lower powers by using $f(c_j) = 0$.

An automorphism of the field E is a one-to-one mapping σ of E onto[66] *itself, $x \to \sigma(x)$, which preserves addition and multiplication:*

$$\sigma(x + y) = \sigma(x) + \sigma(y), \quad \sigma(xy) = \sigma(x)\sigma(y), \quad x \in E,\ y \in E.$$

It can be proven that the subfield $\mathbb{Q}$ is invariant under any automorphism σ: $\sigma(x) = x$ when $x \in \mathbb{Q}$. Clearly, σ maps any root d_j of $f(x)$ into another such root[67]*, d_j. And since σ is one-to-one, it acts on the set of roots $\{d_j\}$ by permuting them. The set of all automorphisms of E is readily seen to be group, called $Aut(E)$, with the map $x \to x$ being the identity element ($\sigma = \mathbf{1}$) and group multiplication being defined by map composition:*

$$(\sigma_1 \cdot \sigma_2)x \equiv \sigma_1(\sigma_2(x)), \quad x \in E, \quad \sigma_1 \in Aut(E), \quad \sigma_2 \in Aut(E)$$

Since any $x \in E$ is a linear combination of powers of roots, the permutation induced by any mapping $\sigma \in Aut(E)$ completely determines the action $\sigma(x)$ on all elements of E. It is thus possible to identify the group of automorphisms of the extension field with a subgroup of the permutations group S_n. It can be shown that this subgroup is exactly the Galois group:

$$Aut(E) = \mathbb{G}(f)$$

This, then, is the abstract-algebraic way of defining the Galois group. It is very useful, because abstract entities involved in this approach – field extensions, automorphisms and the riches of finite-group theory – obey numerous,

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65 In this case the set $\{c_1, \dots, c_m\}$ is an empty set.

66 That σ is "onto" means that for any $y \in E$, there exits an $x \in E$ that maps into it: $\sigma(x) = y$.

67 Since $0 = \sigma(f(d_i)) = f(\sigma(d_i)) = 0$.

quite powerful theorems. Using these tools – which he pioneered himself – Galois was able to answer questions concerning the solvability of a polynomial equation $f(x) = 0$ *by precisely mapping these questions into corresponding ones about the group* $\mathbb{G}(f)$ *and its subgroups.*

For a general finite group G*, let* $G^{(1)}$ *denote the set of commutators*

$$xyx^{-1}y^{-1}, \quad x \in G, \quad y \in G.$$

Clearly the unit element $\mathbf{1}$ *of* G *is also a member of* $G^{(1)}$*,since we may choose* $x = y$ *and then* $xyx^{-1}y^{-1} = \mathbf{1}$*. It is immediately seen that* $G^{(1)}$ *is a subgroup of* G*. One can repeat the procedure to form the subgroup* $G^{(2)}$ *of* $G^{(1)}$ *consisting of all commutators of element pair* $x \in G^{(1)}, \quad y \in G^{(1)}$*. If the repeated application of this procedure eventually yields the trivial, subgroup* $G^{(r)} = \{\mathbf{1}\}$ *after finite number of steps* r*, the original group* G *is said to be solvable.*

Galois' two main theorems concerning polynomial solvability (specialized to the case of rational coefficients) can now be stated[68]*:*

(i) *A non-constant, n-th order polynomial* $f(x)$ *over* $\mathbb{Q}$ *is solvable by radicals if and only if, its Galois group* $\mathbb{G}(f)$ *is a solvable group;*

(ii) *For* $n \geq 5$*, a generic*[69] *n-th order polynomial* $f(x)$ *over* $\mathbb{Q}$ *has the Galois group* $\mathbb{G}(f) = S_n$*, i.e the full* n*-object permutation group.*

From the above definition of group solvability, it is a mere mechanical task to check whether any given finite group is solvable. The only catch is that it is often quite quite difficult to actually determine the group $\mathbb{G}(f)$ *for a given polynomial* $f(x)$*. Once this is done, however, theorem* (i) *enables a straightforward determination as to whether* $f(x)$ *is solvable by radicals or not.*

The correspondence between the solvability of $f(x)$ *by radicals and the solvability of* $\mathbb{G}(f)$ *as a group is, in fact, more intimate than indicated by Theorem* (i)*. If the finite sequence of nested subgroups of a solvable* $\mathbb{G}(f) \equiv G^{(0)}$ *is* $G^{(0)}, \; G^{(1)}, ..., G^{(r)} = \{\mathbf{1}\}$*, then the number of distinct sets* $xG^{(j+1)} = \{xy_1, \; xy_2, ...\}$ *where* $x \in G^{(j)}$*, and* $y_1, y_2, \ldots$ *range over all elements of* $G^{(j+1)}$*, is an integer,* n_{j+1}*; theses sets are called congruences or*

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68 The first theorem exposes the reason for naming the just-described property of some groups "solvability"!

69 By a "generic", or "general", polynomial is meant : any $f(x)$ except some special classes definable via algebraic relations among the coefficients.

cosets, and they form (for each j) an Abelian group.[70]*In Galois theory, it can then be shown that solving $f(x) = 0$ by radicals (if possible at all) can be done by solving successive algebraic equations of orders $n_1, n_2, ..., n_r$. Furthermore, for n_j prime, the j-th of these algebraic equations can be written in the form $y^{n_j} = g$, where g is a function (of the roots of f(x)) whose group of symmetries is $G^{(j-1)}$. The procedure of extracting the n_j-th root of g is then a step in a sequence of field extensions that iteratively build up the field E introduced above.*

The combinations of (i) and (ii), plus the easily demonstrated fact[71] *that S_n is solvable for $n \leq 4$ and insolvable for $n \geq 5$, leads us immediately to Galois' celebrated result – that a general 5^{th} or higher-order polynomial is not solvable by radicals*[72]*.*

We conclude with several examples.

(1) *$n = 1$: any 1^{st}-order (linear) polynomial is, of course, solvable via simple subtraction – even radicals are not needed! Thus if $x + a_0 = 0$, the solution is simply $x = -a_0$. And, indeed, $\mathbb{G}(f)$ is in this case the trivial group $\{\mathbf{1}\}$ (since $S_1 = \{\mathbf{1}\}$ and $\mathbb{G}(f)$ is a subgroup of S_1). This group is (trivially) solvable.*

(2) *$n = 2$: The quadratic equation $x^2 + a_1x + a_0 = 0$ is always solvable by radicals: $d_{1,2} = \frac{1}{2}(-a_1 \pm \sqrt{(a_1)^2 - 4a_0})$. If $(a_1)^2 - 4a_0$ happens to be a perfect square in $\mathbb{Q}$, then both roots are rational numbers and, as explained above, $\mathbb{G}(f) = \{\mathbf{1}\}$ in this case - a solvable group (as in example (1)).*

But if $(a_1)^2 - 4a_0$ is not a perfect square, $\{d_j\}$ are both irrational, and it is readily seen that the only rational functions of (d_1, d_2) over $\mathbb{Q}$ that are rational numbers are symmetric rational functions – which, for $n = 2$, means rational functions of $(d_1 \cdot d_2, \quad d_1 + d_2)$. Thus, by the first definition of the Galois group, $\mathbb{G}(f) = S_2$ in this case. S_2 consists of $2! = 2$ permutations:

$$S_2 = \left\{\mathbf{1}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\right\},$$

[70] The set of cosets $xG^{(j+1)}$, $x \in G^j$, is a group by virtue of the fact that $G^{(j+1)}$ is a *normal* subgroup of $G^{(j)}$, i.e. $x^{-1}yx \in G^{(j+1)}$ for any $x \in G^{(j)}$, $y \in G^{(j+1)}$.

[71] In the examples below it will be shown that S_1, S_2, S_3 and S_4 are *solvable* groups, and that S_5 is insolvable

[72] For rational coefficients. However, this also holds for a general $n \geq 5$ polynomial over $\mathbb{R}$. This can be seen by either noting that $\mathbb{Q} \subset \mathbb{R}$, or by applying Galois' theory but replacing $\mathbb{Q}$ with the field of rational functions of the coefficients $\{a_0, a_1, \ldots, a_{n-1}\}$ (of $f(x)$) over $\mathbb{Q}$.

where $\mathbf{1}$ *is again the trivial permutation:*

$$\mathbf{1} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

S_2 *is clearly an abelian group, so its commutators are all* $\mathbf{1}$*;* $\mathbb{G}(f) = S_2$ *is, once again, a solvable group. Thus* $\mathbb{G}(f)$ *is always solvable for* $n = 2$ *– confirming Galois' result.*

(3) $n = 3$*: Theorem (ii) states that for a general* 3^{rd} *order polynomial,* $\mathbb{G}(f) = S_3$*. In particular, this holds for* $f(x) = x^3 - 2$*. This polynomial is easily solved by radicals, with three complex roots* $d_1 = \sqrt[3]{2}$*,* $d_2 = \omega(\sqrt[3]{2})$*,* $d_3 = \omega^2(\sqrt[3]{2})$*, where* $\omega = \frac{-1 \pm i\sqrt{3}}{2}$*. We now use the second (abstract algebra) definition of the Galois group: any permutation of these roots is an automorphism of the extension field* E*. For instance, the permutation*

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

maps ω *into its complex-conjugate* $\omega^2 = \omega^*$*, and vice versa, leaving* $\sqrt[3]{2}$ *invariant; it thus amounts to redefining* $i = \sqrt{-1}$ *as* $-i$*, which clearly preserves addition and multiplication in the field* $E = \mathbb{Q}(\omega, \omega^*, \sqrt[3]{2})$*. Likewise, the cyclic permutation* $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ *merely multiplies each root by* $e^{2\pi i/3}$*, and is equivalent to redefining the "canonical" root,* $d_1 = \sqrt[3]{2}$*, to be* $\omega\sqrt[3]{2}$*; and similarly with the other* $3! - 3 = 3$ *nontrivial permutations. Thus indeed* $\mathbb{G}(f) = S_3$ *for the particular polynomial* $x^3 - 2$*. By working out all two-element commutators in*

$$S_3 = \left\{\mathbf{1} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\right\}$$

we easily find:

$$S_3^{(1)} = \left\{\mathbf{1}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\right\}$$

This is the subgroup of cyclic permutations, and is abelian; hence $S_3^{(2)} = \{\mathbf{1}\}$*, and* $S_3 = \mathbb{G}(f)$ *is therefore solvable – in agreement with Galois result.*

In this example – and in fact for the general cubic $f(x) \equiv 0$, except special cases – the sequence of nested normal subgroups is:

$$\begin{aligned} G^{(0)} &= S_3 && \text{(a group of order 6, i.e. having 6 elements)} \\ G^{(1)} &= A_3 && = \text{“alternating group”} = \\ &&& \left\{\mathbf{1}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\right\} \\ &&& = \text{subgroup of order 3;} \\ G^{(2)} &= \{\mathbf{1}\}. \end{aligned}$$

The order ratios are nothing but the indices $\{n_j\}$ mentioned above: $n_1 = 6/3 = 2, \quad n_2 = 3/1 = 3$. And as these indices are both prime, Galois theory tells us that the general cubic may be solved by taking a square root and then a cubic root (with rational operations before, after and between these radical operations corresponding to the standard Cardano-Tartaglia solution.

A more cumbersome algorithm having the same radical sequence – an algorithm, in fact, directly related to Galois theory – is as follows.

Let $\alpha,\ \beta,\ \gamma$ denote the roots of a general cubic. The three fundamental symmetric functions:

$$\sigma_1 = \alpha + \beta + \gamma$$

$$\sigma_2 = \alpha\beta + \beta\gamma + \alpha\gamma$$

$$\sigma_3 = \alpha\beta\gamma$$

are just the (rational) non-leading coefficients of $f(x)$ (up to signs). Each σ_j has as its group the full S_3, and any *function with this invariance group can be expressed as a rational function of $\sigma_1,\ \sigma_2,\ \sigma_3$ over $\mathbb{Q}$.*

An example of a function of $\alpha,\ \beta,\ \gamma$ which corresponds to the subgroup $G^{(1)} = A_3, \quad$ is:

$$\tau \equiv \alpha^2\beta + \beta^2\gamma + \gamma^2\alpha$$

Galois theory predicts that τ satisfies a quadratic equation over the rationals – and indeed, some algebra shows that

$$\tau = \frac{1}{2}\left(A \pm \sqrt{B}\right), \quad \text{where}$$

$$A = \alpha^2\beta + \beta^2\gamma + \gamma^2\alpha + \alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2 =$$

$$= \frac{1}{3}(\alpha+\beta+\gamma)^3 - \frac{1}{3}(\alpha^3+\beta^3+\gamma^3) - 2\alpha\beta\gamma,$$
$$B = (\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma)^2.$$

A and B are fully symmetric, i.e. their group is S_3, and thus are rational functions of $\sigma_1,\ \sigma_2,\ \sigma_3$; therefore A, B are rational numbers and thus τ solves a rational quadratic equation – as predicted by Galois theory. It can likewise be shown that $\alpha,\ \beta,\ \gamma$ can be obtained from τ by extracting a third root of a rational function of τ over the extension field $\mathbb{Q}(\sqrt{B})$.

(4) *$n = 4$: The generic 4^{th}-order polynomial has $\mathbb{G}(f) = S_4$, and it has been known for centuries that a general quartic is solvable by radicals. This is in agreement with Galois theory, because S_4 is a solvable group: this is proven by constructing the sequence of groups $S_4^{(1)},\ S_4^{(2)}, \ldots$ (as done above for S_3).*

For a general quartic $f(x)$, the sequence of commutator subgroups over the extension field $\mathbb{Q}(\sqrt{B})$ is as follows:

$G^{(0)} = S_4$, of order $4! = 24$; $G^{(1)} = A_4$ = set of even *permutations of order (2).*

$$G^{(2)} = \left\{\mathbf{1},\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix},\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}\right\},$$

of order 4; and then,

$$G^{(3)} = \left\{\mathbf{1},\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}\right\},$$

is of order 2.

The indices are:

$$n_1 = 24/12 = 2; \quad n_2 = 12/4 = 3; \quad n_3 = 4/2 = 2.$$

Thus the general quartic could *be solved by solving a* quadratic, *then a cubic, and finally another quadratic.*

But the well-known Ferrari Solution of quartic proceeds by first solving an auxiliary cubic and then *two successive quadratics. This corresponds to a different sequence of groups, not invoking successive commutator; one of the groups is not a normal subgroup of the preceding one. This illustrates the fact*

the solutions by radicals suggested by Galois theory, are quite cumbersome – one can usually do better!

(5) *$n = 5$: S_5 is not a solvable group. Indeed, by computing all commutators one can verify that $S_5^{(1)} = A_5$, the subgroup of all even permutations; and that $S_5^{(2)} = A_5^{(1)} = A_5$, so that $S_5^{(r)} \neq \{\mathbf{1}\}$ for any natural number r.*

Hence S_5 is insolvable, and the general quintic equation is not solvable by radicals.

A concrete example of a quintic not solvable by radicals is

$$f(x) = x^5 - 4x + 2,$$

for which the abstract "machinery" of Galois theory can be used to prove that, indeed, $\mathbb{G}(f) = S_5$. But for

$$f(x) = x^5 - 2, \qquad d_j = \sqrt[5]{2}e^{2\pi ij/5} \qquad (0 \leq j \leq 4)$$

In this case $f(x)$ is solvable by radicals, and indeed $\mathbb{G}(f)$ is a 20 - element, solvable subgroup of the 120-element S_5 permutation group.

This order-20 subgroup M_{20} of S_5 is the so-called metacyclic group for the case $n = 5$; i.e. it is the subgroup of substitutions

$$j \to rj + g \mod 5, \qquad r \in \{1\ ,2\ ,3\ ,4\},\ g \in \{0, 1, 2, 3, 4\}.$$

This is not a normal subgroup, so unless $\mathbb{G}(f)$ is this metacyclic group – it cannot be used as part of a Galois group-sequence. However, even for an $f(x)$ for which $\mathbb{G}(f) = S_5$, it can be proven that a function of the roots having M_{20} as its group satisfies an algebraic equation (with rational coefficients) of 6^{th} order (since $120/20 = 6$ is the index of M_{20} in S_5).

In some special cases this sextic is solvable by radicals – in which case, so is the original quintic. The $\mathbb{G}(f)$ is again M_{20}, as occurred in the case $f(x) = x^5 - 2$.

1829–1832 CE **Nicolai Ivanovitch Lobachevsky** (1793–1856, Russia). A geometer of great originality. Pioneer of modern geometries which deal with spaces other than Euclidean.[73] Published (1829) the first account of non-euclidean geometry to appear in print.

This revolutionary development marked the liberation of geometry from its traditional mold established by the Greeks. A deep-rooted and centuries-old conviction that there could be only one possible geometry was shattered, and the way opened for the creation of many different systems of geometry. Moreover, it became apparent that geometry is not necessarily tied to actual physical space as long as its postulates are self-consistent. And as in other instances, it turned out, less than a century later, that these "artificial geometries" are not less physical than the Euclidean geometry. **Clifford** (1845–1879) called Lobachevsky "the Copernicus of geometry".

Lobachevsky was born in Makariev, Nizhniy Novgorod. His father died around 1800 and his mother, who was left in poor circumstances, removed to Kazan with her three sons. In 1807 Nicolai entered the University of Kazan, then recently established. In 1823 he rose to a rank of a full professor of mathematics and retained the chair until 1846. His first contribution to non-Euclidean geometry is believed to have been given in a lecture at Kazan in 1826, but the subject is also treated in many of his memoirs.

Gauss (1777–1855) and **Janos Bolyai** (1802–1860, Hungary) share with Lobachevsky the credit for the discovery of non-Euclidean geometry. Although Gauss failed to publish anything on the matter throughout his life, there is ample evidence to show that he was first to reach penetrating conclusions concerning the parallel postulate.

Bolyai published his findings in 1832 in an appendix to a mathematical work of his father. Because of language barriers and the slowness with which information on new discoveries traveled in those days, Lobachevsky's work did not become known in Western Europe for some years[74].

1829–1841 CE **Jacques Charles Francois Sturm** (1803–1855, Switzerland and France). Mathematician and physicist. Made major contributions to

[73] For further reading, see:

- Brannan, D.A. et.al., *Geometry*, Cambridge University Press, 1998, 497 pp.

[74] In 1824, **F.A. Taurinius** (1794–1874, Germany) communicated to Gauss two monographs on non-Euclidean geometry. Earlier, in 1817, **F.K. Schweikert** (1780–1859, Germany), discussed his ideas with Gauss and is also known to have developed a non-Euclidean geometry.

the theory of algebraic and differential equations [Sturm's theorem[75], Sturm-Liouville equation]. Made the first accurate determination of the velocity of sound in water (1826).

Sturm was born in Geneva. After completing his studies at the Geneva Academy, he became (1823) a tutor to the youngest son of Mme de Staël at the Château of Coppet near Geneva. There he met the Duke Victor de Broglie[76]. He then accompanied the Duke to Paris and through him was able to enter the capital's scientific circles. He became a French citizen (1833). In Paris he met Arago, Ampère, Gay-Lussac, Dulong and Fourier. Upon the death of Ampère, he was elected to the vacant seat in the Académie des Sciences (1836), and in 1838 he became a professor of analysis and mechanics at the École Polytechnique, and succeeded Poisson to the chair of mechanics there (1840).

Around 1851 Sturm's deteriorating health obliged him to arrange for a substitute at the Sorbonne and at the École Polytechnique. Four years later he died in Paris. Sturm also made contributions to experimental and mathematical physics in the fields of analytical mechanics, optics, heat conduction and the study of vision.

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[75] *Sturm's Theorem* shows how to find for any equation, by rational methods, the exact number of real roots which lie within a given range of values. (**Descartes**, **Newton**, **Lagrange**, **Fourier** and **Cauchy** had tried to find suitable criteria to decide whether a root of a polynomial lies in a given interval of the domain of definition.) Given a polynomial of degree n, Sturm defined a chain of $n+1$ functions $f(x), f'(x), f_2(x), \dots, f_n(x)$ where $f'(x)$ is the derivative, $f_2(x)$ is the remainder of the division of $f(x)$ by $f'(x)$, $f_3(x)$ is the remainder of the division of $f'(x)$ by $f_2(x)$ etc. Substituting for x a particular value a in the polynomials of Sturm's chain gives a sequence of real numbers: $f(a), f'(a), f_2(a), \dots, f_n(a)$. If two consecutive numbers $f_i(a)$ and $f_{i+1}(a)$ in this sequence have different signs, one speaks of a *sign change*. Let $W(a)$ denote the number of sign changes in the Sturm's chain for a value $x = a$. Sturm's theorem then states: "Let $f(x)$ be a polynomial with only simple zeros, where $a < b$ and $f(a) \neq 0$, $f(b) \neq 0$; then $\{W(a) - W(b)\}$ is equal to the number of zeros of the polynomial in the closed interval $[a, b]$".

[76] Victor Claude, Prince de Broglie was executed at Paris in June 1794. His son, the Duke Achille Charles Léonce Victor de Broglie (1785–1870) escaped with his mother to Switzerland, where they remained until the fall of Robespierre. In 1816 he was married to the daughter of **Madame de Staël**. In 1832, he took office as minister for foreign affairs. His son, Jacques Victor Albert (1821–1901), was a prime minister of France in 1877. Jacques' grandson was the physicist **Louis Victor de Broglie** (1892–1987).

1829–1855 CE **Thomas Graham** (1805–1869, Scotland). Chemist. A founder of *physical chemistry*. Conducted research on gases and solutions. In 1829 he formulated *Graham's law of diffusion*[77], which explains how two gases mix with each other. He also did pioneering work with colloids[78], founding the science of *colloidal chemistry* (1850).

Graham was born in Glasgow. In 1819 he entered the University of Glasgow with the intention of becoming a minister of the Church, but 'converted' to experimental science and concentrated his studies on molecular physics, a subject which formed the main preoccupation throughout his life. He graduated in 1824, and in 1837 was appointed to the chair of chemistry in University College, London. In 1855 he became Master of the Mint.

1829–1858 CE **Isambard Kingdom Brunel** (1806–1859, England). Railway and bridge engineer, and naval architect. One of the greatest English engineers of the 19^{th} century. Son of **Marc Isambard Brunel** (1769–1849). Took a leading part in the systematic development of ocean steam navigation. Designer and builder of railroads, bridges, tunnels, steamships and docks.

Brunel studied in Paris (1820–1823) and during 1823–1828 assisted his father in the Thames-tunnel project. First designed the Clifton suspension bridge over the Avon (1829; completed 1864). In 1833 he became chief engineer of Great Western Railway and constructed all its viaducts, bridges and tunnels, including the Royal Albert bridge across the River Tamar into Cornwall. During 1838–1845 he designed two highly successful steamships for regular transatlantic service: the *Great Western* (1838) was a wooden steamship, measuring 72 m long and 11 m wide with two huge side wheels that drove it at a speed of 9 knots. The *Great Britain* (1845) was the first large iron-hulled screw-driven steamship. Then in 1853 he began the construction of the *Great Eastern*, the largest steamship of its time[79] (1858).

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[77] The ratio of speeds at which two different gases diffuse is inverse to the ratio of the square roots of the gas densities. The same law applies to the flow of gas through a small aperture (*effusion*).

[78] Colloids: tiny particles of one material evenly distributed in another.

[79] It measured 211 m long, 26 m wide with a total tonnage of 18,918 tons, accommodating 4000 passengers. The *Great Eastern* was intended to show the full potentialities of the iron steamship by carrying enough fuel for a voyage to Australia and back, out of the Cape and home via the Horn. This was a bold attempt to overcome the great obstacle to the development of steamships, namely the fact that coal took up so much space and there was no room for the other commodities less profitable than passengers and mail. As an advertisement of the structural possibilities of iron, the ship was a great success; as a demonstration of the economic use of coal it was a dismal failure – it did not attract enough

1830 CE **George Peacock** (1791–1858, England). Mathematician. Was first to study the fundamental principles of algebra and its structure and pass from 'symbolized arithmetic' to 'symbolic algebra'. In 1830 he published *Treatise on Algebra* which attempted to give algebra a logical treatment comparable to Euclid's *Elements*. First to define and introduce *symbolic algebra* as the science which treats the combinations of arbitrary signs and symbols by means defined through arbitrary and consistent laws. As an undergraduate at Cambridge he made friends with John Herschel and Charles Babbage and together they formed the Analytical Society whose aims were to bring the advanced continental methods to Cambridge. In 1836 he was appointed professor of geometry and astronomy at Cambridge.

Peacock was followed by **Duncan Farquharson Gregory** (1813–1844, England, 1840), **Augustus de Morgan** (1806–1871, England, 1860), **George Boole** (1815–1864) and finally **Hermann Hankel** (1839–1873, Germany, 1867). These studies led to the liberation of algebra (as in geometry) and opened the floodgates of modern abstract algebra. Thus, it seemed inconceivable in the early 19^{th} century that there could exist *non-commutative* algebras. However, **Hamilton** and **A. Cayley** applied it soon enough to quaternions and matrices.

The founders of *quantum mechanics* showed in the 1920's that the atoms and electrons must live by the rules of a non-commutative algebra. Later on, non-associative algebras, such as Jordan algebras and the Lie algebras, were introduced.

Abstract Algebraic Structures

The modern abstract point of view requires a pure science to be founded on postulates (assumptions) about undefined elements, which are not necessarily numbers or points but abstractions *(elements), potentially capable of varied*

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passengers to pay the enormous operation costs; it was used (1866) to lay the first transatlantic telegraph cable. In 1881 it was sold for scraps.

During the late 1800's, steel began to replace iron for ships. Steel ships were stronger and lighter than iron ones. In 1881, the *Servia*, a British vessel, became the first all-steel passenger liner to cross the Atlantic.

interpretations consistent with the basic assumptions. With this in mind, one might begin with systems in which a set of undefined elements is given, as well as two operators between these elements, $\oplus$ and $\otimes$. These symbols are used to suggest some kinship with ordinary addition and multiplication, although in different concrete realizations and interpretations might be considerably different from the usual ones. Then our postulate set can include those 'laws' or 'properties' such as closure, commutativity, and associativity for both $\oplus$ and $\otimes$, and also distributivity of $\otimes$ w.r.t. $\oplus$.

One of the simplest sets of abstract elements is a modulus*: a set S of numbers such that the sum and difference of any two members of S are themselves members of S, i.e. $m \in S$, $n \in S \Rightarrow (m \pm n) \in S$. The elements of a modulus need not necessarily be integers or even rational: they may be complex numbers or quaternions.*

The single number 0 *forms a modulus (the* null modulus*). For any modulus S and $a \in S$ we have $a - a = 0 \in S$, and also: $a + a = 2a \in S$. Repeating this argument we see that $na \in S$ for any integer n. More generally, $a \in S$, $b \in S$ implies $xa + yb \in S$ for any integer x, y. Thus the set of values of $xa + yb$ also forms a modulus.*

It can be shown that $xa + yb$ is the set of multiples of $d = (a, b)$, the greatest common divisor *of a and b. But a number representable as $ax + by$ is, per definition,* linearly dependent on a and b. *Clearly, the property of linear dependence on a and b is preserved by addition, subtraction and multiplication by a number and is not affected by interchanging a and b. Indeed,*

$$(ax_1 + by_1) \pm (ax_2 + by_2) = a(x_1 \pm x_2) + b(y_1 \pm y_2)$$

and

$$\lambda(ax + by) = (\lambda x)a + (\lambda y)b.$$

We shall next exhibit five fundamental algebraic structures which have a wide application in the physical world.

Before we discuss algebraic structures, let us consider first the subject of complex numbers.

*When operating with ordinary real numbers, it is noticed that the square root of negative numbers has no meaning, because the square of every real number is positive or zero. However, the solution of quadratic and cubic equations compelled mathematicians to regard expressions of the form $a + b\sqrt{-1}$. If it is assumed that these '*imaginary*' numbers are subject to the same laws (axioms) of common arithmetical operations as the ordinary numbers, then*

all square roots of negative numbers can be expressed in terms of the quantity $i = \sqrt{-1}$*, and the result of arithmetical operations performed any finite number of times on real or imaginary numbers can always be expressed in the form* $a + bi$*, where* a *and* b *are real numbers.*

Clearly, this definition of imaginary numbers runs counter to common sense: First it was stated that expression $\sqrt{-1}$*,* $\sqrt{-2}$*, and so forth, have no meaning, and then it was proposed that these meaningless expressions be called imaginary numbers. This circumstance caused many mathematicians of the 17*th *and 18*th *century to doubt the validity of the use of complex numbers. However, these doubts were dispelled at the beginning of the 19*th *century, when a geometrical interpretation was found for the complex numbers by points in a plane.*

Another purely arithmetical foundation of the theory of complex numbers was discovered by ***Hamilton*** *(1833) who noted that the complex number* $a+bi$ *can be viewed simply as an ordered pair of real numbers, subject to the addition and multiplication rules*

$$(a, b) + (c, d) = (a + c, b + d);$$

$$(a, b)(c, d) = (ac - bd, ad + bc).$$

For example, we have

$$(2, 3) + (1, -2) = (3, 1) \qquad (2, 3)(1, -2) = (8, -1)$$

$$(3, 0) + (2, 0) = (5, 0) \qquad (3, 0)(2, 0) = (6, 0)$$

These examples show, in particular, that the arithmetical operations on pairs with a zero in the second place reduce to the same operations on their first terms, so that the arithmetic of real numbers is just a special case of the arithmetics of complex numbers. Indeed, if we introduce the notation i *for the pair* $(0, 1)$ *then we have*

$$(a, b) = a(1, 0) + b(0, 1) = a + bi$$

$$i^2 = (0, 1)(0, 1) = (-1, 0) = -1$$

i.e., we have the usual notation for complex numbers.

One then defines the conjugate *of a complex number by* $(a, b)^* = (a, -b)$*, the square of the* norm *of* (a, b)*:*

$$(a, b)(a, -b) = (a^2 + b^2, 0) = ||a, b||^2$$

and the multiplicative inverse *of* (a, b) *as*

$$\frac{(a, b)^*}{||a, b||^2},$$

whenever (a, b) is nonzero.

Sylvester (1852) noted that complex numbers could alternatively be represented by matrices[80]

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

plays the 'role' of $i = \sqrt{-1}$.

Thus it would be improper to state that complex numbers were invented so that negative numbers would have square roots, or, equivalently, so that all quadratic equations would have solutions. It is certainly true that they do provide these, as well as many other interesting and useful properties.

Before negative numbers were invented, mathematicians would say that the equation $x + 1 = 0$ has no solution. Similarly, before complex numbers were introduced, mathematicians could state that $x^2 + 1 = 0$ has no solution.

The real reason that complex numbers gained acceptance in mathematical circles, has to do with cubic equations. It was recognized that all cubic equations have at least one real root. However, when the cubic formula was discovered, it was found that sometimes complex numbers were needed as an intermediate step in finding that one real root. One could not just dismiss a cubic as having no solutions; but at the same time maintain that real numbers were insufficient to solve it.

Operations on the complex numbers can be used to describe geometrical operations on the plane. For instance, multiplication by a real number corresponds to scaling of the plane. Multiplication by complex numbers with

[80] The matrices

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

form a group w.r.t. matrix multiplication. This group is *isomorphic* to the multiplicative group of non-zero complex numbers

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \leftrightarrow a + bi.$$

a modulus ("length") of unity corresponds to rotation *of the plane. Adding complex numbers corresponds to* translation *of the plane. Thus,* transformation *of the plane is easily modeled with complex numbers.*

By 1830, it was well established that complex numbers behave algebraically like vectors in a plane.[81]

Let us investigate this interesting observation in more detailed: consider the complex numbers

$$z = x + iy = re^{i\theta}$$
$$iz = -y + ix = re^{i(\theta+\pi/2)}.$$

Multiplication by i then rotates the 'vector' z by 90° counterclockwise. Two consecutive operations of this kind rotate the vector by 180°, yielding a vector that is anti-parallel to the original vector z.

Now suppose that we start from the plane vector $\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y$ and cross it from the left by $\boldsymbol{e}_z$

$$\boldsymbol{e}_z \times \boldsymbol{r} = -y\boldsymbol{e}_x + x\boldsymbol{e}_y.$$

This operation is again rotating the vector by 90° clockwise. Both cases can be represented by the coordinate transformation

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}.$$

We may then write the symbolic equation in the xy plane

$$i \Leftrightarrow \boldsymbol{e}_z \times .$$

Moreover, the product of two complex numbers $\overline{f} = a - ib, \quad g = c + id$ can be written as

$$\overline{f}g = (ac + bd) + i(ad - cb) \Leftrightarrow (\boldsymbol{f} \cdot \boldsymbol{g}) + i\{\boldsymbol{f} \times \boldsymbol{g}\},$$

where

$$\boldsymbol{f} = a\boldsymbol{e}_x + b\boldsymbol{e}_y,$$
$$\boldsymbol{g} = c\boldsymbol{e}_x + d\boldsymbol{e}_y,$$
$$\boldsymbol{f} \times \boldsymbol{g} = \boldsymbol{e}_z\{\boldsymbol{f} \times \boldsymbol{g}\}.$$

When the two vectors are perpendicular, the real part of their product vanishes. If, on the other hand they are parallel, the imaginary part of their product vanishes.

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[81] **Aristotle** knew that forces can be represented as vectors and that the combined action of two forces can be obtained by the 'parallelogram law'. **Simon Stevin** employed this law in problems of statics, and **Galileo** stated the law explicitly.

This notion can be further extended into the realm of the calculus: Let

$$\nabla = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$$

operate on $S = u + iv$. *Then* $\overline{\nabla}$ *gives the divergence and the rotation of a vector* $\boldsymbol{S} = u\boldsymbol{e}_x + v\boldsymbol{e}_y$

$$\begin{aligned}\overline{\nabla}S = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u+iv) &= \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\ &= div\ \boldsymbol{S} + i\{curl\ \boldsymbol{S}\}_z\end{aligned}$$

If S is analytic, the Cauchy-Riemann relations

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

enable us to write

$$\overline{\nabla S} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u - iv) = 0$$

$$\nabla S = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) = 0$$

The complex representation of plane vectors can be extended to plane tensors. Consider the symmetric plane dyadic

$$\overleftrightarrow{A} = a_{11}\boldsymbol{e}_x\boldsymbol{e}_x + a_{12}(\boldsymbol{e}_x\boldsymbol{e}_y + \boldsymbol{e}_y\boldsymbol{e}_x) + a_{22}\boldsymbol{e}_y\boldsymbol{e}_y,$$

having the scalar invariant $A_1 = a_{11} + a_{22}$. *Let us set our former correspondence* $\boldsymbol{e}_y = (\boldsymbol{e}_z\times)\boldsymbol{e}_x = i\boldsymbol{e}_x$. *We may then recast* $\overleftrightarrow{A}$ *in the symbolic form*

$$\overleftrightarrow{A} = (a_{11} - a_{22})\boldsymbol{e}_x\boldsymbol{e}_x + 2ia_{12}\boldsymbol{e}_x\boldsymbol{e}_x = A_i\boldsymbol{e}_x\boldsymbol{e}_x,$$

where $A_i = (a_{11} - a_{22}) + 2ia_{12}$ *is called the complex invariant of* $\overleftrightarrow{A}$. *We can then use it as a complex representation of the dyadic* $\overleftrightarrow{A}$.

1. Group

A group G is a set of elements a, b, c, etc. (objects, symbols, quantities) for which a composition law $*$ between any two elements (ordered pair) has been uniquely defined and for which the following four conditions are fulfilled.

(i) *Closure: If a belongs to G and b belongs to G, then $a * b$ also belongs to G.*

(ii) *Associativity:* For any three elements a, b, c in G

$$(a * b) * c = a * (b * c).$$

(iii) *Existence of a unit element (the identity):* There exists an element e in G such that operating with e has no effect on a, namely $e * a = a * e = a$ for every a of G. [Actually the slightly weaker condition of the *right identity* $a * e = a$ would also suffice.]

(iv) *Existence of the inverse element:* Corresponding to each a of G there exists an element denoted by a^{-1} such that $a^{-1} * a = a * a^{-1} = e$ for every a in G. [Again, the existence of a *right inverse* only could suffice.]

Of course, $a * b \neq b * a$ in general. A group is said to be *Abelian* or *commutative* if in addition to the group axioms (i)–(iv) we also have $a * b = b * a$ for any pair of elements in G. It is usual in this case to call the composition law *addition* and write $a + b$ for $a * b$ which is then called the *sum* of the element a and b. The identity is then denoted by 0 and is called the *zero* element while the inverse of a is called the *negative* of a and denoted by $-a$. Thus, for an Abelian group, the axioms (i)–(iv) take the form:

(i) $a \in G, \quad b \in G \Rightarrow a + b = c \in G; \quad a + b = b + a$

(ii) $a + (b + c) = (a + b) + c$

(iii) $a + 0 = a \quad$ for every $a \in G$

(iv) $a + (-a) = 0 \quad$ for every $a \in G$

Examples of groups:

(1) The set of all integers with ordinary addition as the composition law.

(2) *The set of all $n \times m$ matrices A with complex elements a_{ij} under the addition law*

$$(A+B)_{ij} = a_{ij} + b_{ij} = (A)_{ij} + (B)_{ij}.$$

The negative of A is defined by $(-A)_{ij} = -a_{ij} = -(A)_{ij}$, so that $A + (-A) = (a_{ij} + (-a_{ij})) = (0) = 0$, where $0 = (0)$, the null matrix, all of whose elements are zero, is the zero element of the group.

(3) *The set of permutation of n objects. Such a permutation may be written*

$$S = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ s_1 & s_2 & s_3 & \dots & s_n \end{pmatrix} = \begin{pmatrix} k \\ s_k \end{pmatrix},$$

$k = 1, 2, \dots n$: this means that the object in cell 1 was sent to cell s_1, the object in cell 2 was sent to cell s_2 etc. under the permutation S. Observe that S remains the same in permuting its columns in any way. Consider the permutation

$$T = \begin{pmatrix} k \\ t_k \end{pmatrix} \equiv \begin{pmatrix} s_k \\ t_{s_k} \end{pmatrix}.$$

The product

$$TS = \begin{pmatrix} k \\ t_k \end{pmatrix} \begin{pmatrix} k \\ s_k \end{pmatrix}$$

means that we first carry out the permutation S that sends the object in cell k to cell s_k, and then from s_k to t_{s_k}, namely

$$TS = \begin{pmatrix} k \\ t_{s_k} \end{pmatrix}, \quad k = 1, 2, \dots n.$$

In general, $TS \neq ST$. The permutation

$$E = \begin{pmatrix} k \\ k \end{pmatrix}$$

which leaves the objects where they are is the unit element, and

$$S^{-1} = \begin{pmatrix} s_1 & s_2 & s_3 & \dots & s_n \\ 1 & 2 & 3 & \dots & n \end{pmatrix}.$$

The algebra of permutations can alternatively be executed with the aid of matrices proper: A permutation on n objects is represented by

an $n \times n$ orthogonal matrix (of determinant ± 1) in which the column (k, a_k) in

$$\begin{pmatrix} 1 & 2 & 3 & \dots & k & \dots & n \\ a_1 & a_2 & a_3 & \dots & a_k & \dots & a_n \end{pmatrix}$$

is represented by placing unity in the k^{th} row and the a_k^{th} column of the matrix, and zero elsewhere in that row. Thus, for example

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

(4) *The set of all non-singular complex square matrices of order n under matrix multiplication.*

(5) *The set of all non-singular complex quaternions under quaternion multiplication.*

2. Ring

A ring R is a set of elements a, b, c etc. which is closed under two distinct composition laws between any two elements and for which the following four conditions are fulfilled:

(i) *R is an additive abelian (commutative) group*

(ii) *$a \in R$; $\quad b \in R \Rightarrow ab \in R$ $\quad$ for any a, b (Closure under 'multiplication')*

(iii) *$a(bc) = (ab)c$ $\quad$ for any a, b, c of R (Associativity for 'multiplication')*

(iv) *$a(b + c) = ab + ac$ $\quad$ and $(b + c)a = ba + ca$ (Distributivity)*

Note again that the commutative operation $(+)$ and the operation of multiplication may be only abstractly connected with the usual connotations of these expressions. In general ab need not be equal to ba and the corresponding ring is then called a non-commutative ring.

An immediate consequence of (iv) is $a \cdot 0 = 0$ for any $a \in R$, where 0 is the zero element of R. In some rings the product of two non-zero elements

may itself be zero, i.e. $ab = 0$ *while* $a \neq 0$, $b \neq 0$. *In that case, these elements are called zero divisors. Furthermore, a ring need not contain either a unit element* e *in the sense that* $ae = e$, *or a multiplicative inverse* a^{-1} *of a given* a. *A commutative ring which contains no zero divisors is called an integral domain. In this domain* $ab = ac$ *implies* $b = c$ *if* $a \neq$ *zero (cancellation law).*

Examples of rings:

(1) *The set of all integers,* $0, \pm 1, \pm 2, \dots$ *under ordinary addition and multiplication. The integer* 1 *serves as a unit element for multiplication but there does not exist an integral inverse* n^{-1} *for any* n *except* $n = \pm 1$. *Note that the subset of all even integers* $0, \pm 2, \pm 4, \dots$ *is itself a ring. This ring does not contain even a unit element.*

(2) *The set of all polynomials* $A(x) = a_0 + a_1 x + \cdots + a_n x^n$ *of arbitrary degrees under addition and multiplication*[82]*: we define a zero polynomial as the one with all its coefficients zero and it is simply denoted by* 0. *The negative* $-A(x)$ *of* $A(x)$ *is evidently the polynomial whose coefficient of* x^k *is* $-a_k$ $(k = 0, 1, \dots, n)$.

In this manner, the set of all polynomials becomes an additive abelian group. We make it into a ring by defining the product polynomial of $A(x)$ *and* $B(x)$ *(of orders* m, n *respectively) as one of degree* $n + m$,

$$A(x)B(x) = c_0 + c_1 x + \dots c_k x^k \cdots + c_{n+m} x^{n+m}$$

$$c_k = \sum_{r+s=k} a_r b_s; \qquad (k = 0, 1, \dots n + m)$$

[82] The exact significance of the symbol x varies subtly with the context. In the early stages of algebra x denoted some *unknown number*, to be discovered eventually at the outcome of the analysis. Later, with the introduction of the concept of a function the unknown x is replaced by the *variable* x, which can range over all numbers of a given set. But unknown and variables obey exactly the same algebraic laws and it is not always necessary to have a clear idea which one is being used. Sometimes one passes from unknown to variable without noticing. Thus it may be required to solve the equation $x^3 + 3x + 2 = 0$ for the unknown x. But a convenient method of solving this equation is by finding the intersection of the curve $y = x^3$ with the line $y + 3x + 2 = 0$. Here, the symbol x ceased to be an unknown and has become a variable.

The laws of algebra are not concerned, however, with whether x represents an unknown, a variable or a constant. A further important generalization is introduced here: the letter x can denote an *indefinable*, i.e. just a symbol about which nothing is assumed, except that it obeys a certain arbitrary system of formal algebraic operations, that may or may not be commutative.

One can similarly define rings of polynomials in several unknowns.

(3) *The set of all square matrices under the composition laws of addition and multiplication of matrices. The ring is non-commutative, contains zero divisors* [e.g. $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ -a & -b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$] *and contains both singular and non-singular matrices* [i.e., *some elements have no inverse*].

(4) *The set of complex quaternions with the product definition*

$$AB = (a_0 b_0 - \boldsymbol{a} \cdot \boldsymbol{b}, \qquad a_0 \boldsymbol{b} + b_0 \boldsymbol{a} + \boldsymbol{a} \times \boldsymbol{b}),$$

where $A = (a_0, \boldsymbol{a})$; $B = (b_0, \boldsymbol{b})$.

Here again, the ring consists of all quaternions, both singular $[|A| = a_0^2 + a_1^2 + a_2^2 + a_3^2 = 0]$ *and non-singular, and must be distinguished from the group of nonsingular complex quaternions introduced earlier. This ring too, contains zero divisors* [e.g. $(1 + ie_1)(1 - ie_1) = 0$ *where* e_1 *is a quaternion unit* $e_1^2 = -1$] *and is non-commutative.*

3. Division ring (skew-field)

We observe that a general ring is a group w.r.t. addition and only a semigroup w.r.t. the multiplication since only closure and associativity are assumed to hold relative to this second composition law. If on the other hand, the elements of the ring (with exception of its zero element), form a multiplicative group as well, it is called a division ring.

More explicitly, a division ring is a ring that contains a unit element e *and also the inverse* a^{-1} *of every* $a \neq 0$ *such that* $ae = a$ *and* $aa^{-1} = e$. *Here in addition to the unique solution* $x = -a + b$ *of an equation* $a + x = b$, *an equation* $ay = b$ *also possesses the unique solution* $y = a^{-1}b$ *and hence, the name division ring for this structure.*

In general, a division ring is not commutative w.r.t. multiplication[83].

[83] However **J.H.M. Wedderburn** (1882–1948) proved (1905) that any finite division ring must be commutative (and thus a *field*).

Example of a non-commutative division ring:

The set of all real quaternions under the definition of the inverse to $A(a_0, \boldsymbol{a})$

$$A^{-1} = \frac{\bar{A}}{|A|^2}$$

where $\bar{A} = (a_0, -\boldsymbol{a})$; $\quad |A|^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2$. *We also have a unit element* $(1, \mathbf{0})$.

4. Field

A commutative division ring is a field. In other words: a field is a set of elements forming a ring w.r.t. two binary operations, addition and multiplication, for which the set of all elements except the unit element w.r.t. addition forms an abelian group w.r.t. multiplication. In yet simpler language one could say that a field is a set of entities which is closed w.r.t. the four rational arithmetical operations of addition, subtraction, multiplication and division by any non-zero member of the set.

Obvious instances of fields are the set of all rational numbers, the set of all real numbers and the set of all complex numbers. We notice that each of these examples is a subfield of the one that succeeds it. Note that $x^2 = 2$ *cannot be solved over the field of rationals while* $x^2 + 1 = 0$ *cannot be solved over the field of real numbers.*

Consider the field of rational numbers $\mathbb{Q}$. *Each rational number can be represented uniquely as a point on a straight line, the 'number axis'. Each such point is a rational point. The rational number* A *is said to be less than the rational number* B $(A < B)$ *if* A *lies left of* B *on the axis. Equivalent statements are that* B *is greater than* A $(B > A)$ *if, or that* $B - A$ *is positive.*

It then follows that, if $A < B$, *the points (numbers) between* A *and* B *are those which are both* $> A$ *and* $< B$. *Any such pair of distinct points, together with the points in between, is called a segment, or interval,* (A, B). *The distance of a point* A *from the origin, considered as positive, is called the absolute value of* A *and is indicated by the symbol* $|A|$. *By definition*

$$\begin{aligned} |A| &= A, \text{ if } A > 0 \\ &= -A, \text{ if } A < 0 \end{aligned}$$

It is clear that if A and B *have the same sign, the equality* $|A + B| = |A| + |B|$ *holds, while if* A and B *have different signs, we have* $|A + B| < |A| + |B|$. *Hence, combining these two statements we have the general inequality*

$$|A + B| \leq |A| + |B|,$$

which is valid irrespective of the signs of A and B.

The absolute value $|x|$ *of* $x \in \mathbb{Q}$ *therefore satisfies the three properties*

(i) $|x| \geq 0, \qquad |x| = 0 \Longleftrightarrow x = 0$

(ii) $|xy| = |x||y|$

(iii) $|x + y| \leq |x| + |y|$

Any function on $\mathbb{Q}$ *with properties (i)–(iii) is called a norm. [The absolute value* $|x|$ *is not the only norm possible over the rationals]*

Once the norm is defined, one can go further and define a metric D *over the rationals*

$$d(x, y) = |x - y|$$

which renders the distance between any two rational points on the number axis. It has the following properties

(i) $d(x, y) = 0$ *iff* $x = y$

(ii) $d(x, y) = d(y, x)$

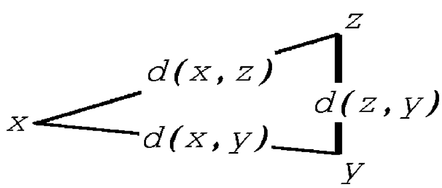
(iii) $d(x, y) \leq d(x, z) + d(z, y)$

for all $z \in \mathbb{Q}, \quad x \in \mathbb{Q} \quad$ and $\quad y \in \mathbb{Q}$.

In the field of complex numbers $\mathbb{C}$, *with the metric*

$$d(a + bi, c + di) = \sqrt{(a - c)^2 + (b - d)^2}$$

the above condition (iii) is known as the triangle inequality. Indeed, in the complex plane, with the above metric, (iii) states that the sum of two sides of a triangle is greater than the third side:



The integral domain of polynomials over a field can be extended to form a field by the use of infinite series. If a_0 differs from zero, the inverse of the polynomial $(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)$ can formally be generated by assuming the existence of an infinite series of ascending powers of x such that

$$(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)(b_0 + b_1x + \cdots + b_nx^n + \dots) = 1.$$

Solving successively for the coefficients b_k we obtain

$$b_0 = \frac{1}{a_0}; \quad b_1 = -\frac{a_1}{a_0^2}; \quad b_2 = \frac{a_1^2 - a_0a_2}{a_0^3}; \quad b_3 = -\frac{a_1^3 - 2a_0a_1a_2 + a_3a_0^2}{a_0^4}$$

etc. No question of convergence arises; the system of polynomials is merely extended to include infinite series. The system is not yet a field, as the element x has no reciprocal. But the system of polynomials and series of the form $x^p(a_0 + a_1x + a_2x^2 + \dots)$ where p is a positive or negative integer does form a field.

5. Linear Vector Space[84]

One considers an additive abelian group V with elements $\mathbf{0}, \boldsymbol{x}, \boldsymbol{y} \dots$ whose general element is denoted by $\boldsymbol{v}$. One then considers a field F with elements

[84] For further reading, see:

- Deskins, W.E., *Abstract Algebra*, Dover: New York, 1995, 624 pp.
- Littlewood, D.E., *University Algebra*, Dover, 1970, 324 pp.
- Childs, L.M., *A Concrete Introduction to Higher Algebra*, Springer-Verlag, 2000, 522 pp.

$0, \lambda, \mu, a, b \ldots$ *We call* V *a linear Vector Space over the ground field* F *if we can define an operation called scalar multiplication which associates with each* $\lambda \in F$ *and each* $\boldsymbol{v} \in V$ *one unique element of* V *denoted* $\lambda\boldsymbol{v}$ *and satisfying the following axioms in addition to those of the abelian group for* $\boldsymbol{v}$ *and the field* F*:*

$$\begin{aligned}
&\lambda(\boldsymbol{x}+\boldsymbol{y}) = \lambda\boldsymbol{x} + \lambda\boldsymbol{y} \\
&(\lambda+\mu)\boldsymbol{x} = \lambda\boldsymbol{x} + \mu\boldsymbol{x} \\
&\lambda(\mu\boldsymbol{x}) = (\lambda\mu)\boldsymbol{x} \equiv \lambda\mu\boldsymbol{x} \\
&1\boldsymbol{x} = \boldsymbol{x}; \qquad 1 \textit{ is the unit element of } F.
\end{aligned}$$

The elements of V *are called vectors while the elements of* F *are called scalars. If the ground field* F *is the real number field,* V *is a real linear vector space; if* F *is the complex number field,* V *is called a complex vector space. An obvious consequence of the above axioms is*

$$\boldsymbol{x} \in V, \boldsymbol{y} \in V \Rightarrow \lambda\boldsymbol{x} + \mu\boldsymbol{y} \in V \qquad \textit{for arbitrary } \lambda, \mu \in F.$$

If each element $\boldsymbol{v}$ *of* V *is equivalent to a finite sum of* n *scalar-multiplied fixed vectors of* V

$$\boldsymbol{v} = \lambda_1\boldsymbol{v}_1 + \lambda_2\boldsymbol{v}_2 + \ldots \lambda_n\boldsymbol{v}_n,$$

the vector space is said to be finite-dimensional over F*. If, in addition,* $\boldsymbol{v}_1, \ldots \boldsymbol{v}_n$ *are linearly independent*[85]*,* V *is called an* n*-dimensional vector space over* F*, usually designated* V_n*.*

Let the ground field F *possess a norm as defined above. Then a vector space* V *over* F *is called normed if to every element* $\boldsymbol{v}$ *of* V *there corresponds a real, non-negative number* $\|\boldsymbol{v}\|$ *called the norm of* $\boldsymbol{v}$ *such that*

(i) $\|\lambda\boldsymbol{v}\| = |\lambda|\|\boldsymbol{v}\|, \quad \lambda \in F, \quad \boldsymbol{v} \in V$

(ii) $\|\boldsymbol{v}_1+\boldsymbol{v}_2\| \le \|\boldsymbol{v}_1\| + \|\boldsymbol{v}_2\| \qquad$ *for* $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$

(iii) $\|\boldsymbol{v}\| > 0 \qquad$ *for* $\boldsymbol{v} \ne 0$

It is easily proven that $\|\mathbf{0}\| = 0$*.*

An example of a norm is the magnitude of the real vector $\boldsymbol{v}$*, namely* $|\boldsymbol{v}|$*, its euclidean distance from a fiducial origin.*

A vector space is called metric if for each pair of elements $\boldsymbol{v}_1, \boldsymbol{v}_2$ *in the space there is a real, non-negative number* $d(\boldsymbol{v}_1, \boldsymbol{v}_2)$ *such that*

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[85] A set of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2 \ldots \boldsymbol{v}_n$ is said to be *linearly dependent* if scalars $\lambda_1, \lambda_2 \ldots \lambda_n$, not all of them zero, exist such that $\lambda_1\boldsymbol{v}_1 + \lambda\boldsymbol{v}_2 + \ldots \lambda_n\boldsymbol{v}_n = \mathbf{0}$.

(i) $d(\boldsymbol{v}_1, \boldsymbol{v}_2) = 0 \qquad$ *iff* $\qquad \boldsymbol{v}_1 = \boldsymbol{v}_2$

(ii) $d(\boldsymbol{v}_1, \boldsymbol{v}_2) \le d(\boldsymbol{v}_3, \boldsymbol{v}_1) + d(\boldsymbol{v}_3, \boldsymbol{v}_2) \qquad$ *for all* $\boldsymbol{v}_3$ *in* V

These properties imply that $d(\boldsymbol{v}_1, \boldsymbol{v}_2) = d(\boldsymbol{v}_2, \boldsymbol{v}_1)$. *Clearly a normed vector space can be made metric by taking*

$$d(\boldsymbol{v}_1, \boldsymbol{v}_2) = \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|,$$

which is the 'distance' between the vectors $\boldsymbol{v}_1$ *and* $\boldsymbol{v}_2$.

Examples of linear vector spaces:

(1) The set of all 3-dimensional real vectors as oriented line segments with the triangle law of vector addition and with the real number field as the ground field.

(2) The set of all n*-tuples of the form*

$$x = (x_k) = (x_1, x_2, \ldots x_n)$$

where x_k *are elements of a ground field, and*

$$x + y = (x_k) + (y_k) = (x_k + y_k) = (x_1 + y_1, x_2 + y_2, \ldots x_k + y_k)$$

The zero element is clearly $0 \equiv (0, 0, \ldots 0)$ *and the negative of* x *is* $-x = (-x_k)$; *also* $\lambda x = (\lambda x_k) = (\lambda x_1, \ldots, \lambda x_n)$. *The* n*-tuple* x *when written as a row is called a* row vector *and when written as a column is called a* column vector.

(3) The set of all matrices A *of order* n*, with complex elements, on defining* $(\lambda A)_{ij} = \lambda A_{ij}$*, where* λ *is a complex number.*

(4) The set of all polynomials $A(x)$ *of degree less than or equal to some number* n *over a field* F*, on defining*

$$\lambda A(x) = (\lambda a_0) + (\lambda a_1)x + \ldots (\lambda a_n)x^n, \quad \lambda \in F.$$

It may be noted that in the above abstract structures no mention has been made of a vector product; *this is because that structure is in general not a vector at all from the viewpoint of linear transformation theory; i.e. its components do not transform to different coordinate systems as do those of an ordinary vector, unless* $n = 3$.

Table 4.1 overviews the development of abstract algebra during the 19^{th} *century.*

Table 4.1: Major events and turning points in the evolution of Abstract Algebra, 1771–1880

Mathematician	Year(s)	Achievement
J.L. Lagrange	1771–1774	Employment of symmetric and similar functions in the solution of algebraic equations by radicals.
A.T. Vandermonde	1771	First group-theoretic theorem. In his memories "*Memoire sur la resolution des équatious*" he approached the general problem of solvability of algebraic equations through the study of functions invariant under permutations of the roots of the equations. **Kronecker** (1888) claimed that the study of modern algebra began with this paper by Vandermonde.
R. Ruffini	1799–1813	Introduction of subgroup of substitutions; notions of transitivity and primitivity of groups. First to prove (with groups) the *Abel-Ruffini Theorem*, using permutation groups. May have come up first with some of the ideas of Galois.
C.F. Gauss	1801	Showed that equation $x^p - 1 = 0$ can be reduced to solving a series of quadratic equations, whenever p is *Fermat prime**.
	1815	Algebraic proof of the *fundamental theorem of algebra*. Pioneered early concepts of: group; special (abelian) case of Galois group; *field*; *splitting field*; *quotient ring*; Extensions of cyclotomic fields; primitive roots.

Table 4.1: (Cont.)

Mathematician	Year(s)	Achievement
N.H. Abel	1824–6	Complete independent proof of *Abel-Ruffini theorem* of the impossibility of the algebraic solution of general algebraic equation of degree higher than the forth.
E. Galois	1823–1829	Complete theory of *finite fields*. Theory of *field extensions*; Solvability conditions of algebraic equations by radicals. Completion of the theory of equations. *Group* concept; *Normal subgroups*. A turning point in the rise of group theory. Advent of abstract algebra. Decisive paper published by J. Liouville only in 1846.
G. Peackock **D.E. Gregory** **A. de Morgan** **G. Boole**	1834–1841	*Symbolic algebra* and *logic*
C.G. Jacobi	1834	Theories of *determinants*, *quadratic forms* and *invariants*.
W.R. Hamilton	1843	Advent of *hypercomplex numbers*; *Quaternions*.
H.G. Grassmann	1844	*Polyadic algebra* (n-dimensional 'exterior algebra'); harbinger of tensor algebra.
A. Cauchy	1844–1846	*Permutation subgroups*, *splitting fields* (1815); *Cauchy theorem* in group theory: ('every group whose order is divisible by a prime number p must contain one or more subgroups of order p').

Table 4.1: (Cont.)

MATHEMATICIAN	YEAR(S)	ACHIEVEMENT
A. Cayley	1849–1859	Theories of *matrices* (1858); abstract *finite groups* [*Cayley theorem: every finite group is isomorphic to a subgroup of* S_n]. Theory of algebraic invariants. With **Hamilton** and **Grassmann** opened abstract algebra to a variety of structures.
S.H. Aronhold **J. Silvester** **R.F.A. Clebsch** **L.O. Hesse** **P. Gordan**	1850–1872	Theory of invariants (later to become essential in *tensor algebra*)
J.A. Serret	1866	Gave the first exposition of Galois' ideas in his book '*Cours d'algebre superiere*'.
C. Jordan	1870	Consolidation of group theory (*normal subgroups*, *simple groups*, *homomorphism*, *matrix groups*). First full and clear presentation of Galois theory.
L. Kronecker	1870	Generalized Gauss' work (1815) and solved the general problem of polynomial splitting field.
L. Sylov	1872	Extended Cauchy's theorem
J.W Dedekind **B. Peirce**	1872–1880	Creation of structural theory of semisimple algebras.
W.K. Clifford	1878–80	'Clifford algebras'; biquaternions.

Table 4.1: (Cont.)

(*) *Example*, since $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$, the equation to be solved is $x^4 + x^3 + x^2 + x + 1 = 0$. We put $z = x + \frac{1}{x}$. Since $x^5 = 1$, we have

$$\frac{1}{x} = x^4, \quad z = x^4 + x, \quad z^2 = x^2 + 2 + \frac{1}{x^2} = x^2 + 2 + x^3.$$

This yields the equation $z^2 + z - 1 = 0$ for z and the equation $z^2 - zx + 1 = 0$ for x. Solving these equations we obtain

$$z_{1,2} = \frac{-1 \pm \sqrt{5}}{2}, \quad x_{1,2} = \frac{z \pm i\sqrt{z+3}}{2}.$$

Thus,

$$x_{1,2} = \frac{-1 \pm \sqrt{5}}{2} + i\sqrt{\frac{5 \pm \sqrt{5}}{8}}; \quad x_{3,4} = \frac{-1 \pm \sqrt{5}}{4} - i\sqrt{\frac{5 \pm \sqrt{5}}{8}}.$$

Vandermonde (1771) produced an *algebraic solution* in radicals for the binomial equation $x^{11} - 1 = 0$

1830 CE **Joseph Jackson Lister**[86] (1786–1869, England). Optician. Showed how microscope lenses could be made to correct for *chromatic and spherical aberrations*[87] (seventy years after the first achromatic *telescope* lenses were made). His invention improved the resolving power of compound microscopes. This, in turn, enabled physiologists to advance cell theory on a solid basis[88].

1830–1833 CE **Marshall Hall** (1790–1857, England). Physician and physiologist. Described the mechanism by which a stimulus can produce a response independently of sensation or volition and coined the term "*reflex*". Maintained his theory in face of denunciation by colleagues. Denounced *bloodletting*[89] as a treatment for disease (1830).

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[86] Not to be confused with the English surgeon **Joseph Lister** (1827–1912), founder of *antiseptic surgery* (1865–1877) who first demonstrated that the use of carbolic acid as an antiseptic reduced the danger of surgery.

[87] When a bundle of rays originates from an axial point, the image distances are not the same for all rays but depend on the original slope angle at the object point. This means that the *rays do not converge to a single focus*. This common feature of spherical reflecting and refracting surfaces (such as lenses, mirrors, prisms etc.) is known as *spherical aberration.* Due to *dispersion*, the focal length of a simple lens also varies with the wavelength. This variation, called *chromatic aberration*, can be reduced substantially by means of a lens combination in which the component lenses are made of glasses having different dispersions. An *achromatic* combination of focal length f for two thin lenses in contact is obtained if the focal lengths of the component lenses are

$$f_1 = f\left(1 - \frac{\delta_1}{\delta_2}\right), \qquad f_2 = f\left(1 - \frac{\delta_2}{\delta_1}\right),$$

where

$$\delta_1 = \frac{1}{n_1 - 1}\frac{dn_1}{d\lambda}, \qquad \delta_2 = \frac{1}{n_2 - 1}\frac{dn_2}{d\lambda},$$

$n_1,\ n_2$ being the indices of refraction and λ the wavelength. Since $\frac{dn}{d\lambda}$ also varies with wavelength, a lens can be achromatized over a limited wavelength interval only. Spherical aberration, on the other hand, can be minimized by appropriate choice of *lens curvatures* and *separations*.

[88] **Pierre Turpin** (1826) reported his observations of *cell division* in algae; **Franz Meyen** observed (1830) that each cell is an independent unit which nourishes itself, build itself up and incorporates raw nutrients that are taken up into different substances and structures. **Robert Brown** (1831) discovered the widespread occurrence of nuclei in cells. **Hugo von Moll** (1805–1872, Germany) showed that plant-cells alone possessed walls (1835–1839).

[89] An ancient practice, found in virtually all periods and cultures, based on magic

1830–1833 CE **Charles Lyell** (1797–1875, England). Founder of modern historical geology. Building on Hutton's concept of *gradual* change through existing physical causes, Lyell marshaled all the observations he could collect in support of the doctrine that the earth has changed slowly and gradually through the ages by means of processes that are still going on.

Between 1830 and 1833 he published the three volumes of his *Principles of Geology*, which organized existing information about that science with lucidity and clarity. Until then many persons believed that changes in the earth occurred in sudden *worldwide* upheavals. Almost singlehandedly, Lyell established *uniformitarianism* at the expense of *catastrophism*, as the accepted philosophy for interpreting the history of the earth. In so doing he introduced, with profound impact, the concept of *unlimited time*. Geological problems now could be solved by reference to natural laws *still active* and available for study in the real world about us instead of by reference to former, shadowy, mythical, or supernatural events; the present became a key to the past.

Lyell's wide influence prepared the ground for the succeeding accomplishments of the 19^{th} century, including those of **Charles Darwin**, whose ideas on the gradual development of living things could not have flourished without the intellectual framework of vast time. Hence, the uniformitarian doctrine was eminently successful in nourishing scientific progress.

In retrospect, however, it appears that the pendulum swung a bit too far[90]. Not only did Lyell strictly reject any process that could not be shown to accord with constant and presently verifiable laws of nature, but he would not even entertain the thought that rates of change, or the *relative importance* of geological agents, ever differed from what they have been within human experience. In short, strict uniformitarianism possessed its own rigid and stifling aspects, brought on by allowing, for all the geological past, only the present rates of natural processes.

Lyell was born in Kinnordy, Scotland. In 1816 he entered Exeter College, Oxford, studying law and geology. During 1821–1826 he was simultaneously practicing law and active in geological research, becoming a member of the Royal Society in 1826. In 1827 he finally abandoned the legal profession and

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and other supernatural explanations. Although various exuberant Renaissance *phlebotomists* were attacked by Paracelsus (1493–1541) and J.B. van Helmont (1579–1644) as upholders of outmoded traditions, bloodletting was still widely used into the 19^{th} century and died out only gradually toward 1900.

[90] e.g. Lyell's doctrine would preclude catastrophic episodes in the earth's history such as the one posited in the contemporary theory of the extinction of the dinosaurs, or nonlinear processes such as the reversal of the polarity of the geomagnetic field.

devoted himself to geology. In 1841 he spent a year in traveling through the United States and Canada; he returned the America in 1845. During these journeys he studied the annual average accumulation of alluvial matter in the Mississippi delta and the coal-formations in Nova Scotia.

Among his characteristics were great thirst for knowledge, perfect fairness and sound judgment. He was buried in Westminster Abbey.

1830–1842 CE **Auguste Comte** (1798–1857, France). Philosopher and social thinker. Founder of the history of science (1832). Founded a philosophical system concerned with the impact of modern science on society, known as *positivism*. In it he tried to arrange the entire field of scientific study in a comprehensive and logical order. Each science in the hierarchy contributes to the entries that follow it, but not to those who precede it; the list is headed by mathematics and followed by astronomy, physics, chemistry, biology and sociology[91].

Comte adopted the view there are three phases in the historical development of human society: (1) initial *theological* phase, where man's speculations were dominated by superstition and prejudices; (2) *metaphysical* phase, where man's search for reality took the form of rational speculation unsupportable by facts; (3) final *positive* phase, where dogmatic assumptions began to be replaced by factual and rational scientific knowledge. This phase brings the historical process to its ultimate state of perfection[92].

One may consider positivism as a type of social physics describing a society rigidly governed by natural laws, with reason playing a key role in social evolution. This evolution goes through three stages: a military-theological stage, a critical-metaphysical stage, and a scientific-industrial stage. In this last stage man no longer concerns himself with ultimate causes, as he did during the metaphysical stage, but is satisfied with the material world and with whatever he might learn from observing it. Here was a philosophy that

[91] The idea of such an order is extremely old, going back as far as Aristotle, and was later adopted by the philosophical movement of the *encyclopedists*.
Comte himself coined sociology for the *science of man*, the last and most complex study in the hierarchy. He considered himself as its founder.

[92] Such a view reflects the "rampant optimism" of the 19^{th} century, shared by Hegel, Spencer and Marx. **Herbert Spencer** (1820–1903, England) sought to interpret society in terms of principles derived from *mechanics* (within 50 years of Comte and Spencer a positivist account of mechanics came to be given by **Mach**). In his *First Principles* he amassed an enormous amount of data, systematically arranged and accompanied by consistent body of theory. He was the chief exponent of the philosophy of evolution.

accepted science as its only guide and authority and which for that reason was eminently suited to the late 19^{th} century.

Comte maintained that it is necessary to study the evolution of the different sciences to understand the development of the human mind and the history of mankind. It is not sufficient to study the history of one or of many particular sciences; one must study the history of *all* sciences, taken together. Indeed, as early as 1832, Auguste Comte made an application to the minister Guizot for the creation of a chair, devoted to the general history of the sciences[93] (*histoire générale des sciences*).

Comte further maintained that Positive humanity will be ruled by the moral authority of a scientific élite, while the executive power will be entrusted to technical experts, an arrangement that is similar to the ideal state of Plato's *Republic*.

Comte was born in the ancient university town of *Montpelier*, at a time when social and political conditions were highly unstable. He was the son of a respectable and conventional family of government clerks. His father was a monarchist and a rigid Catholic. When at the École Polytechnique in Paris (1816), he was expelled for taking part in a student rebellion against one of their professors. This later prevented him from gaining university employment.

Throughout his life Comte was frail in health, and suffered from recurrent mental depression which drove him to the verge of suicide. He made a living by giving private lessons in mathematics and by gifts from friends and admirers. He was twice committed to an insane asylum; the first time, as a result of his unhappy marriage (1825–1842); the second, after the death of his friend Clotilde de Vaux (the wife of a man imprisoned for life) in 1846.

1831–1843 CE **James Clark Ross** (1800–1862, England). Polar explorer. On June 1, 1831, Ross located the *north magnetic pole*[94] in Boothia Peninsula. Commanded Antarctic expedition (1839–1843), discovering Ross Sea,

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93 It was sixty years before this wish of his was granted; the course entrusted to Pierre Laffitte was inaugurated at the Collège de France in 1892, thirty-five years after Comte's death. The real heir to Comte's thought was **Paul Tannery** (1887).

94 At the *magnetic poles*, on the earth's surface, the horizontal component of the magnetic field vanishes and a completely free magnetic needle sets itself vertically. The line joining the poles is the magnetic axis of the earth. In 1963, the poles were approximately at $75\,^\circ$N, $101\,^\circ$W and $67\,^\circ$S, $143\,^\circ$E. In contradistinction, the *geomagnetic poles* (the best dipole approximation to the earth's true field) are at $78\frac{1}{2}\,^\circ$N, $69\,^\circ$W and $78\frac{1}{2}\,^\circ$S, $111\,^\circ$E.

the Ross Ice Shelf, Victoria Land, and Mount Erebus, an active volcano. His uncle **John Ross** and **William Edward Parry** trained him during six arctic voyages in search of the *Northwest Passage* (1818–1834).

Tracking the North Magnetic Pole (1831–2000)

The discovery of the directive property of a magnetic needle in the earth's field and the invention of the mariners compass is obscure. The earliest mention in European literature is ascribed to the monk ***Alexander Neckham*** *(1157–1217). Using a model of the earth made from loadstone (a naturally occurring magnetic rock),* ***William Gilbert*** *came to the conclusion (1600 AD) that the earth behaved substantially as a* uniform magnetized sphere*, its magnetic field being due to causes within the earth, and not from any external agency, as was supposed at that time. The field of a uniformly magnetized sphere can be represented by a dipole at its center. Gilbert showed that there should be two points on the earth where a magnetized needle should stand vertically: at the North and South magnetic poles.*

This is basically the same definition used today. At the magnetic poles, the earth's magnetic field is perpendicular to the earth's surface. Consequently, the magnetic dip, or inclination (the angle between the horizontal and the direction to the earth's magnetic field), is 90°*. And since the magnetic field is vertical, there is no force in a horizontal direction. Therefore, the* magnetic declination*, the angle between true geographic north and magnetic north, cannot be determined at the magnetic poles.*

Gilbert believed that the North Magnetic Pole coincided with the north geographic pole. Magnetic observations made by explorers in subsequent decades showed that this was not true, and by the early nineteenth century, the accumulated observations proved that the pole must be somewhere in Arctic Canada.

In 1829, ***John Ross*** *set out on a voyage to discover the Northwest Passage. His ship became trapped in ice off the northwest coast of Boothia Peninsula, where it was to remain for the next four years. John's nephew,* ***James Clark Ross****, used the time to make magnetic observations along the Boothia coast. These convinced him that the pole was not far away, and in*

the spring of 1831 he set out to reach it. On June 1, 1831, at Cape Adelaide on the west coast of Boothia Peninsula, he measured a dip of $89°59'$. *For all practical purposes, he had reached the North Magnetic Pole.*

In 1839, ***Gauss****, by spherical harmonic analysis, showed that the field of uniformly magnetized sphere was an excellent first approximation to the earth's magnetic field. Gauss further analyzed the irregular part of the earth's field, i.e. the difference between the actual observed field and that due to a uniformly magnetized sphere. With the data then available, he showed that both the regular and irregular components of the earth's field were of internal origin.*

The next attempt to reach the North Magnetic Pole was made some 70 *years later by the Norwegian explorer* ***Roald Amundsen****. In 1903 he left Norway on his famous voyage through the Northwest Passage, which, in fact, was his secondary objective. His primary goal was to set up a temporary magnetic observatory in the Arctic and to re-locate the North Magnetic Pole.*

A pole position was next determined by scientists shortly after World War II. **Paul Serson** *and* **Jack Clark** *measured (1947) a dip of* $89°56'$ *at Allen Lake on Prince of Wales Island* ($73.9°$ *N*, $100.9°$ *W*). *This, in conjunction with other observations made in the vicinity, showed that the pole had moved some* 250 *km northwest since the time of Amundsen's observations. Subsequent observations by scientists in 1962, 1973, 1984, and in 1994, showed that the general northwesterly motion of the pole is continuing, and that during the* 20^{th} *century it has moved an average of 10 km per year*[95].

If, as Gilbert believed, the earth acts as a large magnet, the pole would not move, at least not so rapidly as it does. We now know that the cause of the earth's magnetic field is much more complex. We believe that it is produced by electrical currents that originate in the hot, liquid, outer core of the earth.

In nature, processes are seldom simple. The flow of electric currents in the core is continually changing, so the magnetic field produced by those currents also changes. This means that at the surface of the earth, both the strength and direction of the magnetic field will vary over the years. This gradual change is called the secular variation of the magnetic field.

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[95] These measurements were:

1962	Loomer and Dawson	75.1° N	100.8° W
1973	Niblett and Chairboneau	76.0° N	100.6° W
1984	Newitt and Niblett	77.0° N	102.3° W
1994	Newitt and Barton	78.3° N	104.0° W
1999, 2000	Newitt, McKee, Mandea and Orgeval	81.3° N	110.8° W

The position of the North Magnetic Pole is strongly influenced by the secular variation in its vicinity. For example, if the dip is $90°$ *at a given point this year, that point will be the North Magnetic Pole, by definition. However, because of secular variation, the dip at that point will change to* $89°58'$ *in about two years, so it will no longer be the pole. However, at some nearby point, the dip will have increased to* $90°$*, and that point will have become the pole. In this manner, the pole slowly moves across the Arctic.*

The more rapid daily motion of the pole around its average position has an entirely different cause. If we measure the earth's magnetic field continually, such as is done at a magnetic observatory*, we will see that it changes during the course of a day, sometimes slowly, sometimes rapidly. The ultimate cause of these fluctuations is the sun. The sun constantly emits charged particles that, on encountering the earth's magnetic field, cause electric currents to be produced in the upper atmosphere. These electric currents disturb the magnetic field, resulting in a temporary shift in the pole's position. The distance and speed of these displacements will, of course, depend on the nature of the disturbances in the magnetic field, but they are occurring constantly. When scientists try to determine the current average position of the pole, they must average out all of these transient wanderings.*

In April and May of 1994, Larry Newitt, of the Geological Survey of Canada, and Charles Barton, of the Australian Geological Survey Organization*, conducted a survey to determine the average position of the North Magnetic Pole at that time. Working out of Resolute Bay, N.W.T., they established a temporary magnetic observatory on Longheed Island, close to the predicted position of the pole. This allowed them to monitor the short-term fluctuations of the magnetic field that result in the daily motion of the pole.*

The strength and direction of the magnetic field were measured at this site, and at seven additional sites in the region. From these observations, the point at which the average dip was $90°$ *could be determined.*

They determined that the average position of the North Magnetic Pole in 1994 was located on the Noice Peninsula, southwest Ellef Ringnes Island, at $78.3°N$, $104.0°W$.

Note that the magnetic North and South Poles are not *diametrically opposite, each being about 2300 km from the point antipodal to the other. The magnetic poles must not be confused with the geomagnetic poles, which are the points where the axis of the* Gaussian geocentric dipole *(which best approximates the earth's field) meets the surface of the earth.*

The geomagnetic poles are situated approximately at $78\frac{1}{2}^{\circ}N$, $69°W$ *and* $78\frac{1}{2}^{\circ}S$, $111°E$ *and the geomagnetic axis is thus inclined at* $11\frac{1}{2}^{\circ}$ *to the earth's geographical axis. If the* geocentric dipole field *were the total field, the dipoles*

and geomagnetic poles would of course coincide. A better approximation to the earth's field can be obtained by displacing the center of the equivalent dipole by about 300 km towards Indonesia. ***Vestine*** *(1953) has determined the position of the eccentric dipole from 1830 to 1945 and found a change in longitude of about* $0 \cdot 30°$ *per year.*

1831–1839 CE In 1831 the *cell nucleus*[96] was discovered by **Robert Brown** (1773–1858).

During 1838–1839 botanist **Jacob Matthias Schleiden** (1804–1881, Germany) and physiologist **Theodor Ambrose Hubert Schwann** (1810–1882, Germany) originated what we now call *cell biology.*

1831–1846 CE **Michael Faraday** (1791–1867, England). Distinguished experimental physicist and chemist. Discovered (1831) *electromagnetic induction* and introduced the concepts of *lines of force* and a physical *field* (1845). As a chemist he discovered and isolated *benzene* (C_6H_6) in 1825.

His experiments started in 1821, when he showed that a current carrying wire is surrounded by circular lines of magnetic field which he called 'lines of force'. Electromagnetic induction was independently discovered by **Joseph Henry** in 1832. On the other hand, **Oersted** preceded Faraday in discovering the magnetic field of a current (1820).

In 1831 Faraday discovered that electric current is generated by changes in the magnetic field — a phenomenon complementary to Oersted's discovery of the magnetic effects of currents. In 1845 he discovered *diamagnetism* and *paramagnetism.* He also showed that a magnetic field affects the polarization of light in a medium. In the same year he conjectured that light is essentially electromagnetic waves. In 1846 he suggested that electromagnetic energy is transmitted by a transverse vibrations of the lines of force, and no fluid agent, such as the 'ether', is needed for the transmission of light.[97]

1831–1848 CE **Macedonio Melloni** (1798–1854, Italy). Physicist. First to claim that radiant heat and light were different modes of the same process.

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[96] *Nucleus* = little nut (Latin: Diminutive of nux, nuc).

[97] For further reading, see:

- Williams, L.P., *Michael Faraday, A Biography*, Basic Books: New York, 1965, 531 pp.

Measured the heating effect (infrared radiation) from the sun's light scattered from the moon and reaching the earth during the night.

Melloni was born in Parma, and became a professor of Physics there (1824–1831). Had to flee to France on account of political activities. Returning to Naples (1839), he became the director of the Vesuvius Observatory. He died of cholera.

From Thales to Faraday and beyond, or — What is Electricity?

*As early as 600 BCE, **Thales of Miletos**, is supposed to have made the first observation on this mysterious entity, by noting that amber rubbed with another substance attracted certain light objects. Since then its exact nature has been a matter of dispute. The ancients considered it a kind of soul or spirit inhabiting otherwise lifeless objects. **Cardano** (1557) described it as a material substance, a fluid that flows from object to object. **Galvani** (1791) held that it was a "vital force", the element essential to life, for which philosophers had searched for centuries.*

***Du Fay** (1733) offered evidence that there were not one but two different types electricity, vitreous and resinous. **Benjamin Franklin**, in a dangerous experiment, showed (1752) that lightning and electricity were akin. **Oersted** (1820) proved that there was a relationship between electricity and magnetism.*

***Faraday**, a consummately skillful experimentalist, went even further. He knew of Oersted's observation; he also knew that heat and chemical reactions could generate electricity and vice versa. With an insight which characterizes a great scientist, he stated: "I believe that the various forms under which the forces of matter are made manifest, have one common origin; or, in other words, are so directly related and mutually dependent, that they are convertible, as it were, into one another".*

Before Faraday, electricity was a plaything of natural philosophers, a source of entertainment for the fashionable audiences who attended their lectures. No one had the slightest intimation of its practical possibilities.

Faraday had little interest in such possibilities. He was concerned with fundamental research — with the establishment of principles linking seemingly diverse phenomena. He was, however, more farsighted than his predecessors, and in response to a query by W. Gladstone, he is said to have replied, "Some day you will tax it".

Faraday's discoveries should have led immediately to a major electrical industry, but it took a very long time to develop. The reason was mainly economic: although it would have been possible to have produced electricity in a big way in 1830, electricity could not be sold because there were no buyers!

The first call for electricity was through fashion: people were becoming moderately rich, not rich enough to afford silver spoons, but what about having electroplated spoons? For that a good source of current was required and the magneto machine of Faraday, slightly improved, worked very well for this purpose. Then it was used for where really bright lights were needed, arc lights, and for lighthouses. Gradually the uses increased and as the demand for it increased, so did production.

1832 CE Following the *Reform Act*, elementary education in England became the concern of the state rather then that of the Church. The government stepped in with grants, and toward 1850, elementary education became universal in *public schools*, for the first time in history.

1832 CE **Joseph Henry** (1797–1878, U.S.A.). Physicist and inventor. Discovered the principles of electromagnetic self-inductance. Proposed a single wire telegraph (1816).

1832 CE A Chicago carpenter, **George W. Snow(e)**, reinvented[98] the *balloon frame* which revolutionized home construction. This simple method,

[98] The idea was not original. Carpenters in 17^{th} century Virginia employed a similar method when confronted with pressures to build rapidly. But no matter the type of frame, carpenters could not reduce substantially the handwork necessary for building a house until the 1880's. Then, Chicago carpenters replaced mortized-and-tenoned sills with box sills that used only dimensional lumber joined by nails. By this time, factories produced most windows, doors, and trim, as well as kiln-dried dimensional lumber with tighter tolerances. Carpenters on the site merely fit and installed these products.

The balloon frame evolved slowly over the course of the 19^{th} century. Companies

utilizing standard size boards and machine-cut nails allowed even unskilled workers to build houses, quickly, cheaply and easily.

Traditional building methods used heavy, hand-hewn timbers and hand carved mortise-and-tenon joints held by hand-cut dowels or hand-made nails. An entire wall was assembled on the ground by skilled craftsmen and then lifted into place by a crew of about twenty laborers.

The balloon frame was built with much lighter pre-cut 2×4 studs and held together by factory-made nails. This reduced the cost and made affordable building materials available to middle-income and low-income families. These houses could be built quickly and easily by only two workers using basic carpentry techniques.

The method was first used by Augustine Taylor on St. Mary's Church in Fort Dearborn, near Chicago. His crew framed the church with 2×4s and 2×6s — using studs, joists, and cross members, all nailed together. There were no mortises, no tenons, no dovetail joints to be carved. It was built with just boards and nails.

Professional carpenters said the church would blow away in a stiff wind and labeled the technique "balloon construction." It was a derisive term which stuck. The style also stuck as most homes today still used this method with some modifications. In early construction, the studs ran from the foundation to the roof. In case of fire, this long open space created a chimney effect and allowed fires to spread rapidly. Today, the studs are broken by the floors and the spaces between the studs are filled with insulation reducing the chimney effect.

Two architectural practices, the balloon frame and Chicago Construction, made Chicago the world's first vertical city. Builders using the balloon frame method created a skeleton of two-by-four covered by wooden siding. First widely used in 19^{th}-century Chicago and still employed today, the balloon frame not only sped up the building process; It also made construction less costly.

The new balloon frame helps to explain the astonishingly rapid expansion of Chicago. By 1848, it became an important port equipped with facilities for handling the biggest inland ships in the world with 100 trains a day arriving on eleven different railroads. From a small village (1830), with a population less than 200, it grew (1887) into a city of 800,000 people.

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in Chicago then produced ready-made houses with balloon frames that were sold to various Western cities attempting to meet the needs of rapidly expanding populations.

The balloon frame was a precursor to a great Chicago innovation: the practice of attaching a facade onto a strong yet light steel frame. Though skyscrapers were born in New York, the method called Chicago Construction, developed by Chicago architects and engineers between 1880 and 1883, provided the basic structural system for building modern steel-and-glass office towers.

Balloon frame construction helped to make possible the incredible growth of Western U.S., where trees were scarce. Wood from the Midwest, cut into standard-size boards, was shipped by rail to the West. Most wooden buildings erected today still use the method of construction derived from this system.

1832–1846 CE **Joseph Liouville** (1809–1892, France). A remarkable mathematician of the 19^{th} century.

He was born into a distinguished family in St. Omer, France. He studied at the École Polytechnique and was appointed professor there in 1833. In 1836 he founded the *Journal des Mathématique Pures et Appliquées*, which upheld the high standard of French mathematics throughout the 19^{th} century. He was appointed professor at the Sorbonne and the Collège de France in 1839.

Liouville contributed significantly to many fields of mathematics, especially to boundary-value problems for 2^{nd} order linear differential equations (*Stürm-Liouville theory*). He was also interested in number theory, differential geometry and Hamiltonian dynamics. Today he is also remembered for having published the works of **Galois**, after the latter's untimely death.

In 1838 he discovered an important theorem which found applications both in classical and quantum mechanics. It states that the volume of any region in phase-space remains invariant under any Hamiltonian time evolution. Otherwise stated it means that the 'phase-fluid' moves like an incompressible fluid[99]. Of historical importance is his general method of solution of integral equations by successive substitution (1837).

A number of important theorems bear his name: *Liouville theorem* in the theory of functions states that if $f(z)$ is analytic for all values of z and $|f(z)| < k$, where k is constant, then $f(z)$ is constant. Another *Liouville theorem* states that an elliptic function $E(z)$ with no poles in a cell is merely a constant. Then there are a number of theorems that concern the

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[99] **Boltzmann** (1867) applied the theorem in the context of his statistical-mechanics theory. An explicit *equation* was first derived by **Gibbs** (1884), who recognized its potential usefulness in astronomy.

solvability of second order differential equations[100], and in particular of the Riccati equation.

Liouville proved the existence of *transcendental numbers* (1844) and constructed an infinite class of such numbers. He wrote over 400 papers in total, many of them of major importance in mathematics.

1832–1863 CE **Jacob Steiner** (1796–1863, Switzerland). An outstanding geometer. Laid the foundations of modern synthetic geometry. His mathematical works are confined to geometry, which he treated synthetically, to the total exclusion of analysis. In his own field he surpassed all his contemporaries. His investigations are distinguished by their great generality, rigor and profound intuition.

Steiner clarified many of the concepts of projective geometry and stressed the fundamental importance of the *principle of duality*. Using exclusively synthetic methods he was able to prove theorems that belong to the realm of analysis. Among his contributions: *the Steiner-Poncelet theorem*, which states that second order problems (Euclidean constructions) can be solved with the aid of a straight-edge and a circle with a given center.

Steiner was born in the village of Utzendorf (canton Bern). He learned to read and write at the age of 14. At 18 he became a pupil of **Heinrich Pestalozzi** and afterward studied mathematics at Heidelberg and Berlin, where he made his living by giving private lessons. He was helped by **A.L. Crelle**, and due to the influence of **G.C.J. Jacobi** and the brothers **Alexander** and **Wilhelm von Humboldt** he was appointed professor of geometry at the University of Berlin (1834), where he taught for the rest of his life.

1832–1873 CE **Joseph Antoine Ferdinand Plateau** (1801–1883, Belgium). Physicist. Devised an experimental method of visualizing minimal surfaces, and described it in his 1873 treatise on molecular forces in liquids. The essence of the matter is that if a piece of wire is bent into a closed curve and dipped in a soap solution, then the resulting soap film spanning the wire will assume the shape of a minimal surface in order to minimize the potential energy due to surface tension. During 1830–1869 Plateau performed many striking experiments on surface tension and capillary phenomena, and

[100] There is a partial nonlinear differential equation, which bears his name: $u_{xy} = e^u$. It has the *exact* solution $e^u = 2\frac{\alpha'(x)\beta'(y)}{[\alpha(x)+\beta(y)]^2}$, where $\alpha(x)$, $\beta(y)$ are arbitrary functions of x and y respectively.

since his time the problem of minimal surfaces has been known as *Plateau's problem*.[101]

Plateau was born in Brussels. From 1835 and on he was a professor of physics at Ghent. He did most of his work in the fields of physiological optics and molecular forces. We owe to him the *stroboscopic*[102]method of studying the motion of a vibrating body, by looking at it through equidistant radial slits in a revolving disc.

In 1829 he imprudently gazed at the midday sun for 20 seconds, with the view of studying after effects. It caused him to lose his eyesight in 1843. But this calamity did not interrupt his scientific activity. Aided by his wife, son and son-in-law, he continued to the end of his life his researches on vision, although he did not see many of his own experiments.

In 1832 he developed the *phenakistoscope*[103], the first device that gave pictures the illusion of movement: Plateau placed two discs on a rod. He painted pictures of an object or a person along the edge of one disc. Each picture slightly advanced the subject's position. Slots were cut in the other

[101] For further reading, see:

- Hildebrandt, S. and A. Tromba, *The Parsimonious Universe*, Springer-Verlag, 1995, 330 pp.
- Courant, R., *Dirichlet's Principle, Conformal Mapping and Minimal Surfaces*, Interscience-Wiley: New York, 1950, 330 pp.

[102] Plateau and **Simon von Stampfer** (1792–1864, Austria) invented the *stroboscope* independently around 1823. Stampfer was a professor at the Technical College of Vienna. The stroboscope and the principle of persistence of vision were at the base of all early attempts to produce moving pictures, culminating with the *cinematograph* of the Lumiére brothers (1885).

[103] The *phenakistoscope* was constructed with **Simon Ritter**. It was then developed in a number of directions in an attempt to produce moving pictures. **Franz von Uchatius** (1811–1881, Germany) was the first person to project visible moving images on the screen: he scanned a series of painted slides (places around a disc) through slits cut in a second disc. As the discs were rotated, apparently moving images were projected by light onto a screen (1853). **Ottomar Anshutz** (1846–1907, Prussia) made the first noteworthy attempt (1892, two years before Edison's peepshow *kinetoscope*) to project moving sequences of *photographs* using his *projecting Electrotachyscope* which was in principle just an elaborated stroboscope.
The *commercial history* of the moving pictures began with **Edison's** kinetoscope (1854) and the invention of the Kodak celluloid film by **George Eastman** (1888).

disc. When both discs were rotated at the correct speed, the pictures seemed to move as they appeared in the slots.

1833 CE **Charles Babbage** (1792–1871, England). Mathematician. The great ancestral figure in the history of computers. The first man to put forward detailed proposals for an all purpose automatic calculating machine. He designed and tried to build a complicated machine, dubbed the *Analytical Engine*. The design for his vast mechanical calculators rank among the most startling intellectual achievements of the 19^{th} century. Yet Babbage failed in his efforts to realize those plans in physical form, because the demands of his devices lay beyond the capabilities of Victorian mechanical engineering.

Contemporary computers are based on many of the principles used in Babbage's machine: It was designed so that it would perform mathematical operations from a set of instructions ('program'). The machine would 'read' the program from 'punched cards', an idea derived partly from the punched cards of the *Jacquard loom* (1805). The computer was equipped with a memory and a central processor. A long sequence of different operations could be performed with no human intervention after the punched cards were fed in.

The first machine conceived by Babbage, already in 1812, was the *Difference Engine* which he intended as a device for computing and printing tables of mathematical functions. He noticed that tables of polynomials can be easily constructed to any desired accuracy if one employs their first, second etc. *differences*, using the addition operation only. Since most functions can be represented to sufficient accuracy (at least over a limited range) by means of polynomials, their values can be constructed in a similar way. It was this process that Babbage proposed to mechanize with his *Difference Engine*.

Babbage constructed a small machine with 3 registers which would tabulate quadratic functions. This he demonstrated in 1822, to such effect that he secured the support of the Royal Society for the construction of a full size machine to compute and check tables of 6^{th} degree polynomials to no less than 20 decimal places. The machine was never constructed. A part of it is now in the London Science Museum. [In 1853, **Pehr Georg Scheutz** (1785–1873, Sweden), stimulated by some published accounts of Babbage's ideas and funds from the Swedish Academy, completed an improved version of the Difference Engine that would tabulate 4^{th} degree polynomials to 14 decimal places.] In 1832 Babbage lost interest in the *Difference Engine*.

Babbage was born in Teignmouth, Devonshire. He was educated at a private school, and afterwards entered St. Peter's College, Cambridge, where he graduated in 1814. Though he did not compete in the mathematical tripos, he acquired a great reputation at the university. In the years 1815–1817, he contributed three papers to the *Philosophical Transactions* and in 1816 was

made a fellow of the Royal Society. Babbage's attention seems to have been drawn at an early stage to the number and importance of errors introduced into astronomical and other tables.

From 1828 to 1839 Babbage held the post of Lucasian Professor of Mathematics at Cambridge — but without delivering a single lecture at the university. He was busy enough in other directions, however. Not only did he attempt to reform the Royal Society, Greenwich Observatory, and the teaching of mathematics at Cambridge, but he also found time to analyze the operation and economics of the Post Office, the pin-making industry and the printing trade, to publish one of the first reliable actuarial 'life tables', and to make some of the earliest dynamometer measurements on the railway, running a special train on Sundays for the purpose.

The essential constituents of the 'Analytical Engine' are:

- a *store* (sometimes called a *memory*) for holding numbers — both those forming the data of the problem and those generated in the course of the calculation;
- an *arithmetic unit* — a device for performing arithmetic operations on those numbers (Babbage called it the *mill*);
- a *control unit* — a device for causing the machine to perform the desired operation in the correct sequence;
- *input devices* whereby numbers and operating instructions can be supplied to the machine;
- *output devices* for displaying the results of the calculation.

For storage Babbage proposed to use columns of wheels, each wheel being capable of resting in any one of ten positions and so of storing one decimal digit. Transfer of numbers between the store and the mill was to be accomplished by means of elaborate mechanisms of gears, rods, and linkages. The store itself was to accommodate 1000 numbers, each number being represented by no less than 50 decimal places. It seems that Babbage intended that numbers would normally be set on the storage wheels or on the mill by hand, but he also envisaged supplying mathematical tables to the machine in punched-card form. Several alternative kinds of output were envisaged: direct printing, the production of moulds from which printer's blocks could be cast, and punched cards.

Babbage's ideas were sound, but there was no technology in the mid 19^{th} century to implement them. For that reason, the Analytical Engine was never completed, although Babbage continued to work on it until his death,

spending much of his own money. His schemes were simply too ambitious (consider, for instance, his idea of working with numbers to 50 decimal places!) at a time when simple desk calculators were far from reliable mechanically and could not be made in any quantity.

As time went on the frustrated inventor became increasingly embittered by a sense of failure[104]. He quarreled with many of his contemporaries, from his own craftsmen to the astronomer royal; he became ever more intolerant of criticism, more caustic in his judgments, more out of sync with his time. He suffered the unhappy fate of a misunderstood genius who is too far ahead of his time.

It took the world a century to catch up with him. Today, when his germinal ideas are bearing so rich a fruit, we can appreciate the magnitude of his achievements and the depth of his prophetic insight.

In 1831, **Babbage**, **John Frederick William Herschel** (1792–1871) and **David Brewster** (1781–1868) created the 'British Association for the Advancement of Science' to promote British science, which was on the decline. The reason was that the mathematics taught in Britain during the early years of the 19^{th} century did not go much beyond the level of Newton's time. In the calculus, Newton's somewhat clumsy notation[105] was adhered to, whilst the more elegant symbolism introduced by Leibniz, and the advances made by the French, were largely ignored. Moreover, scientific research in England was still largely an amateur activity, and a reform in the universities was needed.

The idea of the association had originated from a national congress of German scientists, which Babbage had attended in Berlin in 1828. Other founders appear to have been stimulated by the writings of **Francis Bacon**, who in 1626 had suggested the formation of a national academy for the advancement of the sciences and crafts.

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[104] Babbage's vain attempts to make a universal digital adding machine, are partly the result of the obtuse avarice and shortsightedness of Prime Minister **Robert Peel** (1788–1850); just when the 'analytical engine' was near completion and the inventor desired to adopt a new principle – Peel declined government grants, unwilling to accept further risk.

[105] **Herschel** introduced the new mathematical notation:

- $\sin^{-1}$, $\cos^{-1}$ etc., to indicate the *inverse* trigonometric functions (1813);
- single-line representation of a continued fraction (1820); e.g.,

$$\sqrt{13} = 3 + \frac{1}{\underset{*}{1+}}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{\underset{*}{6+}}\frac{1}{1+}\cdots$$

1833 CE **Jean Marie Constant Duhamel** (1797–1872, France). Applied mathematician. Known for his resolution of boundary value problems for the diffusion equation (*Duhamel's Theorem*[106]), and the *Duhamel Superposition Integral*[107] in the theory of linear systems. In acoustics, Duhamel studied the vibration of strings and suggested, independently of *Ohm* (1843), that one perceives a complex sound signal as a linear superposition of elementary sinusoidal components.

Duhamel entered the École Polytechnique in Paris in 1814. The political events of 1816, which caused reorganization of the school, obliged him to return to Rennes, where he studied law. He taught at the École Polytechnique from 1830 to 1869, becoming a professor of analysis and mechanics in 1836. The commission of 1850 demanded his removal because he resisted a program for change, but he returned as professor of analysis in 1851, replacing Liouville.

1833–1845 CE **Urbain Jean Joseph LeVerrier** (1811–1877, France). Astronomer. His main work was in *celestial mechanics*. His discovery of a discrepancy in the motion of the perihelion of Mercury was important as early evidence for Einstein's GTR.

During 1833–1843 he developed formulas for calculating past changes in the earth's orbit, and reconstructed the orbital history of the past 100,000 years. Published in 1843, these calculations were based on the orbits and masses of the seven planets known at the time. He was then led in 1845 to postulate the existence of planet *Neptune* [co-predicted by **John Couch**

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106 *Duhamel's Theorem*: If $T = F(\boldsymbol{r}, t_0, t)$ is the temperature at point $\boldsymbol{r}$ in a heat conducting solid at time t due to zero initial temperature, a *fixed* heat source $s(\boldsymbol{r}, t_0)$ and fixed boundary temperature $\phi(\boldsymbol{r}, t_0)$ for some t_0 such that $0 < t_0 < t$, then the temperature in the same body due to zero initial temperature and *variable* heat source $s(\boldsymbol{r}, t)$ and boundary temperature $\phi(\boldsymbol{r}, t)$, will be given by the *Duhamel Integral*

$$T(\boldsymbol{r}, t) = \frac{\partial}{\partial t}\int_0^t F(\boldsymbol{r}, \tau, t-\tau)d\tau.$$

Note that the theorem does *not* tell us how to find F, but only how to reduce a problem with time-dependent source and boundary conditions to time-independent source and boundary conditions.

107 *Duhamel's Superposition Integral*: Given the step-response $h(t)$ of a causal linear system, its response to arbitrary excitation $f(t)$ is given by

$$g(t) = f(0^+)h(t) + \int_0^t f'(\tau)h(t-\tau)d\tau.$$

Adams[108] (1819–1892, England)] on the basis of calculations of its perturbation of the orbit of Uranus. He encouraged astronomer **Johann Gottfried Galle** (1812–1910, Germany) to search for it. The latter indeed found it on Sept. 23, 1846, only 52 seconds of arc from LeVerrier's predicted position.

In 1859 LeVerrier postulated the existence of a planet ('Vulcan') between Mercury and the sun, as the cause of an *anomalous* precession of Mercury's perihelion (that part of the precession which remains after the perturbations due to known planets are subtracted). However, no such planet was ever found. In 1916, Einstein explained the anomalous precession of Mercury's orbit as a consequence of small non-Newtonian spacetime curvature effects in his theory of General Relativity. LeVerrier worked at the Paris observatory for the most of his life.

1833–1855 CE **Wilhelm Eduard Weber** (1804–1891, Germany). Physicist. With Gauss, investigated terrestrial magnetism, and devised the *electromagnetic telegraph* (1833). Introduced the absolute system of electrical units after Gauss' system of magnetic units. His ratio between the electrodynamic and electrostatic units of charge (1855) was crucial to Maxwell in his electromagnetic theory of light. Weber found the ratio was 3.1074×10^8 m/sec but failed to take any notice of the fact that this was close to the speed of light. The Weber, a magnetic flux unit, is named in his honor.

Weber entered the University of Halle (1822) and wrote his doctoral dissertation there (1826). He became a professor at Göttingen from 1831. When Victoria became Queen of Britain (1837) her uncle became ruler of Hanover and revoked the liberal constitution. Weber was one of seven professors at Göttingen to sign a protest and all were dismissed. He remained in Göttingen without a position until 1843, when he became a professor of physics at Leipzig. In 1848 he returned to his old position at Göttingen. He and Dirichlet became temporary directors of the astronomical observatory there.

1833–1861 CE **William Whewell** (1794–1866, England). Philosopher, historian of science and mathematician. Suggested to the British Association for the Advancement of Science in Cambridge that their members be called

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[108] The idea that an unknown planet is causing the observed perturbation of the orbit of Uranus occurred to Adams already in 1841, when he was still an undergraduate at Cambridge University, following an earlier conjecture made by **Mary Fairfax Sommerville** (1780–1872), a writer on mathematics and physical science. After 4 years of work Adams obtained a solution, calculated the position of the unknown planet and sent his results to **G.B. Airy**, then the Astronomer Royal of England. Unfortunately, Airy was not interested in the prediction and made no search for the perturbing body.

scientists. The word gradually caught on and began to displace *natural philosopher*. (Today, we would call few of those BAAS members scientists, since most were amateurs or supporters of science.)

In his book *The Philosophy of Inductive Sciences* (1849) he analyzed the method exemplified in the formation of ideas, in the new induction of science, and in the applications and systematizations of these inductions. Whewell articulated that the success of western science is due to the broad theoretical consistency of *physics*, that with its astonishing congruity with mathematics, came to undergird *chemistry*, which in turn proved foundational for *biology*[109].

Whewell was born at Lancaster. His father, a carpenter, wished him to follow his trade, but his success in mathematics in local grammar-school enabled him to proceed to Cambridge (1812). He was a professor of mineralogy (1828–1832) and philosophy (1838–1855) at Cambridge. He died from the effects of a fall from his horse.

1834 CE **Emile** (**Benoit Pierre**) **Clapeyron** (1799–1864, France). Engineer and physicist. Born in Paris and educated at the École Polytechnique. He went to Russia in 1820 with **G. Lamé** at the invitation of the czar Alexander I to supervise a program of public works. Upon his return in 1830 he was employed in the Paris-Versailles railroad project. In 1834 he revived the forgotten theory of Carnot by applying it to practical steam engine problems. By considering a *Carnot engine* operating between two reservoirs differing infinitesimally in temperature, and by letting the working substance undergo a change in phase, he derived an important relation, giving the slope of the equilibrium lines in a pressure-temperature diagram. This was later generalized by Clausius, and is known today as the *Clausius-Clapeyron equation*[110].

[109] He introduced the concept of *consilience* as literally a "jumping together" of facts and theory to form a common network of explanation across the scientific disciplines. He said: "The Consilience of Inductions takes place when an Induction, obtained from one class of facts, coincides with an Induction, obtained from another different class. This Consilience is a test of the truth of the Theory in which it occurs."

Western scientists succeeded because they believed that the abstract laws of the various disciplines in some manner interlock and interconnect. *Consilience* proved to be the way of the natural sciences.

[110] For a pure crystalline solid the change of state from solid to liquid (the process of *melting*) takes place at a single definite temperature under fixed pressure. This temperature is called the melting point for that pressure. The melting point at a given pressure is the temperature at which the solid and the liquid are in equilibrium under that pressure. Melting is accompanied by a change of volume, which may be either an *increase* or a *decrease* (there is a decrease in

1834 CE Foundation of the Statistical Society of London. Though it has contributed little to the theory of statistics, it has had a considerable influence on the practical work of carrying out statistical investigations in the United Kingdom and elsewhere.

1834–1837 CE **Charles Wheatstone** (1802–1875, England). Experimental physicist and practical founder of modern telegraphy. In 1834 he measured the velocity of current electricity by examining sparks produced at the ends of a lengthy electric circuit with a revolving mirror. He estimated that electricity traveled at a speed which was one half the speed of light. The great velocity of electrical transmission suggested to him the possibility of utilizing it for sending messages, and after many experiments and business collaboration with **William Fothergill Cooke** (1806–1879), a patent for an electric telegraph was taken out in their joint name in 1837. Wheatstone is also known for his "bridge", a circuit for comparison of resistors.

Wheatstone was born in Gloucester. He was educated at several private schools. In 1823 he and his brother inherited their father's business. In 1829 he retired to devote himself to experimental research in sound physics. By 1834 he was appointed professor of experimental philosophy at Kings College, London. In 1868, after completion of his automatic telegraph, he was knighted. Wheatstone was the uncle of **Oliver Heaviside**.

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volume when ice melts, but increase in volume when wax melts).

The effect of a change of pressure on the melting point is such that dp/dT has always the same sign as $V_2 - V_1$ [$V_{2,1}$ = volume of a unit mass of liquid (solid)]. The thermodynamic equation governing this phenomenon is known as the *Clausius-Clapeyron* equation:

$$\frac{dT}{dp} = \frac{T(V_2 - V_1)}{LJ}.$$

Here dT is the change in the absolute temperature T of the melting point caused by the change in pressure dp, L is the latent heat of melting in cal. per gram, and the conversion factor J is the mechanical equivalent of heat (1 calorie $\equiv$ 4.2 joules).

If $V_2 > V_1$, the substance expands on melting and dT/dp is positive, whence increasing the pressure *raises* the melting point. If $V_2 < V_1$, the substance contracts on melting, dT/dp is negative, whence increasing the pressure *lowers* the melting point.

The *Clausius-Clapeyron equation* applies to changes of state in general, e.g. change of vapor-pressure with temperature, and enables dp to be calculated if dT and other quantities are provided. As it stands, the equation cannot be integrated unless $L(T)$, $V_1(T)$ and $V_2(T)$ are explicitly known.

1834–1840 CE **Jean Charles Athanase Peltier** (1785–1845, France). Experimental physicist and meteorologist. Discovered experimentally that a junction between two dissimilar metals tends to absorb heat when an electric current is passed across it in one direction, but tends to lose heat when the current is passed in the opposite direction. The thermoelectric cooling or heating of the junction was later termed the *Peltier effect*, and is now commonly used e.g. to cool semiconductor chips. Introduced the concept of *electrostatic induction* (1840).

Peltier was born at Ham (Somme). He was originally a watchmaker, but retired from business about the age of 30 and devoted himself to experimental and observational science.

1834–1856 CE **James Nasmyth** (1808–1890, England). Engineer and inventor. Developed the *self-acting principle* in machine tool design, by which a mechanical hand moving along a slide holds a tool. Using this principle, Nasmyth invented the *planning mill* and a *nut-shaped machine*.

Nasmyth was born in Edinburgh, the son of a noted artist. He became assistant to **Henry Maudslay**, tool designer and manufacturer. In 1834, Nasmyth started the Bridgewater Foundation at Manchester, which became famous for machine tool and steam-engine construction. He then invented the *shaper* (1834) and the *steam-hammer* (1839).

When **James Watt** began his experiments with the steam engine (1763) he could not find anyone who could drill a perfect hole! As a result, his engines leaked steam until the Englishman **John Wilkinson** (1728–1808) invented the *boring machine*. The *planner* was developed (1800–1825) by **Matthew Murray**, **Joseph Clements**, and **Richard Murray**. The principle of the *lathe* had been known since ancient times, and probably originated with the *potter's wheel*. Until 1800, lathes were crude machines that could not be used to cut screw threads accurately. In that year, **Henry Maudslay** (1771–1831, England) invented the first good screw-cutting lathe.

Nasmyth did much for the improvements of machine-tools, and his inventive talent devised many new appliances.

1834–1884 CE **James Joseph Sylvester** (1814–1897, England). One of the foremost mathematicians of his time. Developed the theories of matrices, algebraic invariants and quadratic forms, partition of numbers[111], algebraic

[111] Sylvester addressed the problem of "*denumeration*", i.e. the number of partitions of a number N into m parts $n_1, n_2, \dots, n_m$, repeated or not. This is the same thing as finding the number of solutions in integers of $n_1x_1 + n_2x_2 + \cdots + n_mx_m = N$. Sylvester (1855) introduced the name "*denumerant*" for this number of partitions and denoted it by the symbol

elimination and substitution, determinants, theory of equations, mechanics, optics and astronomy. Sylvester coined the terms *matrix*, *latent roots* (*eigenvalues*) and *Jacobian*.

Sylvester was born of orthodox Jewish parents in London as James Joseph. His eldest brother emigrated to the United States, where he took the name of Sylvester, an example followed by the rest of the family. In 1831 Sylvester entered St. Johns College, Cambridge. Being a Jew he was ineligible for fellowship and could not even take a degree. This last, however, he obtained at Trinity College, Dublin, were religious restrictions were no longer in force. After leaving Cambridge he was appointed to the chair of natural philosophy at University College, London, where his friend **A. de Morgan** was one of his colleagues, but he resigned in 1840 in order to become professor of mathematics in the University of Virginia, U.S.A. There, however, he remained only a few months, for a certain event entailed unpleasant consequences and caused his return to England[112].

$D(N; n_1, n_2, \dots, n_m)$. He then proved that the denumerants are the coefficients in the expansion of the *generating function*

$$\frac{1}{(1-t^{n_1})(1-t^{n_2})\cdots(1-t^{n_m})} = \sum_n D(n; n_1, n_2, \dots, n_m)t^n.$$

Multiplying this equation by $(1-t^{n_m})$ and equating coefficients of t^n of two equivalent sums, we get the relation

$$\begin{aligned} D(N; n_1, n_2, \dots, n_m) =& D(N-n_m; n_1, n_2, \dots, n_m) \\ &+ D(N; n_1, n_2; \dots, n_{m-1}) \end{aligned}$$

which upon repetition, enables one to evaluate the denumerant.

112 Sylvester was America's first Jewish professor. He arrived in Charlottesville late in November 1841 and left suddenly in March 1842. Being both a Jew and an Englishman he attracted the hatred of the local protestant community even before his arrival. The *watchman of the South*, organ of the Presbyterian Church, the most influential denomination in Virginia, led a venomous racist attack on his appointment, driven by the fear that "his powerful ascendancy over the young minds may contaminate their pure Christian morality". This crusade provoked some of his students to abuse him to such a point that he had no choice but leave the hornet's nest.

The virtual ouster of Sylvester did great damage not only to his career, but especially to the advancement of science itself. His most creative years were lost to humanity. Disgraced, outcast from the mathematical community, unable to secure any teaching post, unemployed for more than a year in New York City, Sylvester sought his livelihood for 10 years as an actuary and at the legal bar.

He then proceeded to spend almost ten years in business and then turned to the study of law, in connection with which, in 1850, he first met A. Cayley. The two men were to remain lifelong friends, and ultimately both left the law. In 1855 Sylvester took a position at the Royal Military Academy at Woolwick. In 1877 he was appointed professor of mathematics in the Johns Hopkins University, Baltimore, where he stayed until 1883. His stay there gave an enormous impetus to the study of higher mathematics in America, and during that time he contributed to the *American Journal of Mathematics*, of which he was the first editor. In 1883 he was appointed to the Savillian chair of geometry at Oxford[113], from which he retired in 1893 due to failing health.

Sylvester was a good linguist and a diligent composer of verse, in English, Latin and Greek.

1835 CE *The Roman Catholic Church* finally takes the books of Copernicus, Galileo and Kepler off its *Index of Prohibited Books* (the decision to lift the ban was made in 1822). Thus, heliocentricity is officially restored 13 centuries after Justinian and 21 centuries after Aristarchos of Samos.

1835 CE **Samuel Colt** (1814–1863, US). Designer and manufacturer of the first successful repeating pistol – the *'colt revolver'*. It had a cylinder of several chambers that could be discharged in succession by the same locking and firing mechanism. The idea for a revolver dates back to the early 1500's, but Colt was the first person to make it simple and rugged enough for long use.

Samuel Colt was born in Hartford, Conn. He established a factory there, where he also produced arms used during the Mexican War and the Civil War.

1835 CE **Augostino Maria Bassi** (1773–1856, Italy). Bacteriologist. Anticipated **Pasteur** and **Koch** in formulating *germ* theory.

Demonstrated that a disease of silkworms was caused by a parasitic *fungus* (1835). Theorized that many diseases are caused by parasites. This discovery gave impetus to the germ theory of infectious diseases.

At the beginning of the 19th century Bassi studied the silkworm disease (*muscardine*). He discovered (1807) that it was caused by a minute parasitic

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He also took a few private pupils. One of them was **Florence Nightingale**, then six years younger than her teacher.

[113] After the abolition of the *religious tests* (1871), this appointment could go through.

fungus (the fungus was later named *Botrytis bassiana* after its first discoverer) that was transmitted by infected food and from animal to animal by contact. He went on to describe methods for treating fungally infected worms, which was of considerable interest at the time, as *muscardine* was causing financial losses to those working in the European silk trade.

Bassi was born in a village near Lodi in what was then part of the Austrian Empire but is now a part of Italy. He graduated in law and worked as a civil servant in Italy while devoting much of his spare time to the study of living organisms using an early version of the microscope.

Although **Anton van Leuwenhoek** first discovered and described such minute microorganisms as bacteria (1676), the link between these tiny organisms and the induction of infectious diseases was not recognized for another two hundred years. Bassi was the first to understand this link.

1835 CE **Gaspard Gustave de Coriolis** (1792–1843, France). Physicist. Presented an analysis of a body's motion in a rotating frame in his paper: "*Memoire sur les équations due mouvement relatif des systèmes de corps*". He applied his study to fluid motions on a rotating earth[114].

[114] Nevertheless, problems associated with the dynamics of the earth's rotation continue to challenge scientists even today. Consider, for example, the phenomenon of the "*bathtub vortex*", i.e. the rotation developed when water drains out through a hole at the bottom of a vertical tank. In a carefully controlled experiment, water in the tank is allowed to settle for some 24 hours before opening the drain to begin the experiment, so that the residual vorticity is reduced to a value less than that corresponding to the earth's angular velocity; a perceptible *counterclockwise* (looking down on the tank) rotation appears in the Northern Hemisphere after 10–15 min of drain, indicating that vorticity can be developed from the earth's rotation. Under similar controlled conditions, a *clockwise* rotation (looking down on the tank) is developed in the Southern Hemisphere.

The equation of motion for a homogeneous, inviscid and incompressible fluid of density ρ relative to a reference frame having angular velocity $\boldsymbol{\omega}$ (relative to an inertial frame), is

$$\frac{D\boldsymbol{v}}{Dt} \equiv \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \boldsymbol{g} - 2(\boldsymbol{\omega} \times \boldsymbol{v}) - \operatorname{grad} \frac{P}{\rho},$$

where P, $\boldsymbol{v}$ are the pressure and velocity fields and $\boldsymbol{g}$ is the body force per unit mass, assumed to be conservative. This equation assumes that centripetal acceleration terms involving the square of the earth's angular velocity are negligible. Taking the curl of the above equation $(\boldsymbol{\Omega} = \operatorname{curl} \boldsymbol{v})$, one obtains the

evolution equation for the vorticity $\boldsymbol{\Omega}$:

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{\Omega} = (\boldsymbol{\Omega} + 2\boldsymbol{\omega}) \cdot \nabla \boldsymbol{v}.$$

The flow for the situation to be considered (that of vorticity generation as water flows out of an exit in the center of the bottom of a vertical cylindrical tank), is *symmetric*. We accordingly employ the cylindrical coordinate system (r, θ, z), with the z direction being downward. In these coordinates (neglecting the term $\boldsymbol{v} \cdot \nabla \boldsymbol{\Omega}$ for the small velocities under consideration), the equations for the components of the vorticity vector become

$$\frac{\partial \Omega_z}{\partial t} = \Omega_r \frac{\partial v_z}{\partial r} + (\Omega_z + 2\omega_z) \frac{\partial v_z}{\partial z};$$

$$\Omega_r = -\frac{\partial v_\theta}{\partial z}; \qquad \Omega_z = \frac{1}{r} \frac{\partial}{\partial r}(r v_\theta).$$

They give the growth of the vertical vorticity component Ω_z in a frame having angular velocity ω_z in terms of the prescribed velocity gradients $\frac{\partial v_z}{\partial r}$, $\frac{\partial v_z}{\partial z}$ and $\frac{\partial v_\theta}{\partial z}$.

Additional relations are obtained from the equation of continuity

$$\operatorname{div} \boldsymbol{v} = \frac{1}{r} \frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z} = 0.$$

We take v_θ to be independent of θ on account of axial symmetry. The temporal growth of the circulatory velocity v_θ associated with the growth of the vertical vorticity Ω_z is taken as the primary effect, and we assume that only the vertical component ω_z of the angular velocity of the frame of reference is operative in generating vorticity. Then, a solution satisfying $\operatorname{div} \boldsymbol{v} = 0$ is obtained with $\Omega_r = 0$, $\Omega_\theta = 0$, $v_z = az$, $v_r = -\frac{1}{2}ar$, $\omega_z = -\omega_E \sin\lambda$, where ω_E is the angular velocity of the earth's rotation and λ is the latitude of the location of the experiment (assumed to be in the Northern Hemisphere). The explicit form of the solution for Ω_z is

$$\Omega_z = -2\omega_E \sin\lambda(e^{at} - 1) + (\Omega_z)_0 e^{at},$$

where $(\Omega_z)_0$ is the *residual vorticity* of the water in the tank on beginning the discharge at time $t = 0$. If this residual vorticity is absolutely removed by stilling, then vorticity will be generated and grow (initially) exponentially with time, according to

$$\Omega_z = -2\omega_E \sin\lambda(e^{at} - 1),$$

being negative in the Northern Hemisphere (counterclockwise when viewed from above). Taking $\omega_E = 7.3 \times 10^{-5}$ radians per sec, we find that at latitude 50° North, a counterclockwise vorticity of about one revolution in 6 sec would be generated after 15 min with a value of $a = 0.01\ \text{sec}^{-1}$. The actual value of a depends upon the dimensions of the tank. Note, though, that the assumed velocity profile in the no-rotation limit $(v_z = az,\ v_r = -\frac{1}{2}ar)$ does not hold for any actual draining container; furthermore, eventually Ω_z is large enough to invalidate the approximates made, so Ω_z does not *continue* to grow exponentially.

Coriolis was assistant professor of mathematics at the Ecole Polytechnique, Paris (1816–1838). He introduced the terms *'work'* and *'kinetic energy'* with their present scientific meaning. In 1835 he wrote a mathematical theory of *billiards.*

Let $\boldsymbol{\omega}$ be the angular velocity vector of the earth, with the vector pointing in the direction of advancement of a right-hand screw (that is, northward along the earth's axis). Since the earth turns once in 24 hours, the magnitude of $\boldsymbol{\omega}$ is $\frac{2\pi}{24\times 3600} = 0.729 \times 10^{-4}$ rad/sec.[115] If $\boldsymbol{v}$ is a body's velocity relative to the earth's frame, then the Coriolis acceleration $= -2\boldsymbol{\omega} \times \boldsymbol{v}$. If u and v are the east and north components of the velocity $\boldsymbol{v}$, the corresponding components of the Coriolis acceleration are: $C_u = 1.46 \times 10^{-4}(v \sin\lambda)$; $C_v = -1.46 \times 10^{-4}(u \sin\lambda)$, where λ is the latitude.

If a car is driven at 90 mph (40 m/sec) in any direction, the Coriolis acceleration at latitude $\lambda = 45°$ will tend to push it to the right with an acceleration of 0.4 cm/sec^2. Meanwhile, gravity will be acting downward with an acceleration of 980 cm/sec^2. As a result, suspended objects in the car will tend to lean to the right by 4 parts in 10,000, or 1.4 minutes of arc. There is no need to bank the freeways for the Coriolis effect. The effect is tiny, but sufficient to cause lateral wear on railway tracks, except near the equator. For example, suppose a train of mass 500 tons $= 5 \times 10^5$ kg moves with a speed of $v = 40$ m/sec toward the north at latitude $\lambda = 30\,°$, so that the component of its speed perpendicular to the earth's axis is $v_V = v \sin\lambda = 20$ m/sec. The Coriolis acceleration it experiences is about 3×10^{-3} m/sec^2, and the Coriolis force it experiences on the track through the flanges of its wheels is thus about 1.5×10^3 Newton.

In the Northern Hemisphere winds will circle around a low-pressure area in a counterclockwise direction, as recorded on a weather map. As a low-pressure area is developing, air will be drawn into its center, and as the wind gathers speed it will be deflected, by the Coriolis effect, toward the right. The net result is a circulation of air around the low-pressure area in a counterclockwise direction. In the southern hemisphere this sense of circulation is reversed, as borne out by meteorological observation.

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[115] Other measures of rotation-rate are:

- RPM – rotations per minute;
- deg/s – degrees per second.

The relations between them are:

1 RPM = 360 deg / 60 sec = 6 deg/sec
1 rad/sec = 180 deg / π sec = 57.3 deg/sec

Objects dropped from a tower will be deflected, except at the poles, toward the east by the amount $d = \frac{2}{3}\omega_0 \cos\lambda\sqrt{\frac{2h^3}{g}}$, where h is the height of the tower, g the local acceleration of gravity, λ the local latitude angle and ω_0 the angular speed of the earth's spin. For $\lambda = 40^\circ$ and $h = 100$ m, $d = 1.6$ cm. This effect can be made plausible by the following line of reasoning: consider a tower of height h located at the equator. The velocity of the tower's base is $v_{_R} = 2\pi R/T$ and the velocity of the tower's top is $v_{_{R+h}} = 2\pi(R+h)/T$, directed from west to east because of the earth's rotation. Accordingly, an object at rest at the top will have an eastward horizontal component of its velocity with respect to ground of amount $v_{_E} = v_{_{R+h}} - v_{_R} = \frac{2\pi h}{T} = \omega_0 h$. Neglecting the fact that the local vertical and horizontal are slowly rotating with the earth, one estimates the deflection toward the east, if the falling time is t, to be of amount $d \cong v_{_E}t$ with $t = \sqrt{\frac{2h}{g}}$. Combining these result, we obtain the above formula except for the factor $\frac{2}{3}$.

In 1735, exactly 100 years before Coriolis published his theory, **George Hadley** (1685–1768, England) proposed a theory based on the *conservation of angular momentum* to explain the existence of *trade winds*.

In 1775 **Laplace** included the horizontal components of the 'Coriolis acceleration' in his hydrodynamic tidal equations, antedating the work of Coriolis.

1835–1837 AD **Edward Blyth** (1810–1873, England and India). Chemist and naturalist. Presented a precursor of Darwin's work on evolution. In a number of papers, published in the *Magazine of Natural History*, heralded elements of the theory of evolution by natural selection, some twenty years ahead of **Charles Darwin** (1859)[116]. He stated therein:

> 'A variety of important considerations here crowd upon the mind, foremost of which is the inquiry that, as *man*, by removing species from

[116] In his book *Darwin and the Mysterious Mr. X* (1979), **Loren Eiseley** vigorously promoted the thesis that Darwin read Blyth's papers and quite likely had derived a major inspiration from it without ever mentioning this in his writings. Eiseley argues that Darwin was to use many of Blyth's ideas years later when writing his "Origin", yet he had given Blyth little or no acknowledgment. Darwin, however, having been influenced by Blyth's ideas, changed natural selection around to mean evolutionary descent of all beings from a common ancestor. Loren Eisely wrote: "But let the world not forget that Edward Blyth, a man of poverty and bad fortune, shaped a key that dropped half-used from his hands when he set forth hastily on his own ill-fated voyage. That key, which was picked up and forged by a far greater and more cunning hand, was no less then *natural selection*."

> their appropriate haunts, superinduces changes on their physical constitution and adaptations, to which extent may not the same take place in wild *nature*, so that, in a few generations, distinctive characters may be acquired, such as are recognized as indicative of *specific diversity*. May not then, a large proportion of what are considered species have descended from a common *heritage*?'

Blyth was an ardent *creationist*, and his papers flowed with his sense of awe and reverence for the God of creation who had so wonderfully and wisely made all of his creatures.

Unlike Darwin, Blyth was not born into wealth. His father died when he was ten, leaving his widowed mother to raise four children. She managed to send Edward, her eldest son, to school where he excelled in chemistry and natural history. He went to India (1841) and was eventually appointed a curator of the Museum of the Royal Society of Bengal. He lived there for many years on a meager stipend. Plagued by continuing poor health, and afflicted by a personal tragedy, he returned to England (1862), living on a small pension.

1835–1839 CE **Theodor Ambrose Hubert Schwann** (1810–1882, Germany). Physician and physiologist. Laid the foundation of *cell-biology*. Discovered and isolated *pepsin* (1835), a digestive catalyst *enzyme*[117] (which he called *ferment*), the first known animal enzyme. Discovered (1837) that yeast is made of small living organisms[118].

The theory of fermentation was immediately attacked by the leading chemists of the time: **Berzelius** (1839) concluded that microscopic evidence was of no value and that nothing was living in yeast! In the same year **Justus von Liebig** and **Friedrich Wöhler** added sarcasm to scorn by ridiculing the Schwann-de la Tour discovery. It took a man of the caliber of **Pasteur** to settle the problem once and for all.

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[117] The latter term was coined (1876) by **Wilhelm Künhe** (1837–1900, Germany) from Greek words meaning *in yeast*, because they acted outside cells as ferments did inside cells such as yeast. Künhe, a physiologist, discovered the enzyme *trypsin* in the pancreatic juice. Born in Hamburg, died at Heidelberg.
Schwann coined (1839) the word *metabolism*, taken from the Greek and meaning literally "throw into different position" (and therefore implying 'change').

[118] This was independently discovered in the same year by the French inventor **Charles Cagniard de la Tour** (1777–1859).

Schwann was born at Neuss in Prussia. Educated at Bonn and Würzburg, where he graduated M.D. in 1834. During 1838–1847 he lectured at the Catholic University of Louvain, and in 1847 he was appointed professor at Liége, where he remained.

1835–1840 CE **Nachman (Kohen) Krochmal**[119] (1785–1840, Ukraine). Philosopher of history. The first thinker to view Jewish history not as a distinct and independent entity, but as a part of the whole of civilization in the framework of world history. In his *Moreh Nevuchei ha-Zman* (*Guide to the Perplexed of the Age*) he set forth his ideas on reconciling essential Judaism with modern thought; he showed that while the history of every nation undergoes the inevitable stages of growth, blossoming and decay, that of Israel is cyclic, i.e., always rises again to begin a new cycle.

Drawing from **Maimonides**, **Avraham Ibn Ezra**, **Yehuda Halevi**, **Maharal**, **Kant**, **Hegel** and **Schelling**, Krochmal's philosophy of Jewish history is based on the concept of '*national spirit*' that consists of its religious greatness and spiritual gifts; this spirituality permeates all the people's intellectual achievements, explains the ability of the Jews to overcome the forces of decline and is the secret of their self-rejuvenation and national revival.

Krochmal was born in Brody (Poland), lived most of his life in Zalkieve and died in Tarnopol. He was a merchant, and later a bookkeeper at Nestrov, near Lvov. He ordered his disciples to send the manuscript of his book to **Yom-Tov Lippmann Zunz** (1794–1886, Germany) in Berlin, who published it posthumously (1851).

1835–1865 CE **Panfuty Lvovich Chebyshev** (1821–1894, Russia). An outstanding, versatile mathematician with rare talent for solving difficult problems by elementary methods.

Conjectured at the age of 14 that $\left\{\frac{x}{\log x}\right\}$ is a good approximation to the number of primes less or equal to x. In probability theory, Chebyshev introduced the concepts of *variance* and *arithmetic mean* of random variables. Known also for his *inequality*[120], *set of polynomials* and *problem*[121]. He was the

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[119] Known by his acronym: RANAK.

[120] If $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and $b_1 \geq b_2 \geq b_3 \geq \cdots \geq b_n \geq 0$ then

$$\frac{1}{n}\sum_1^n a_k b_k \geq \left(\frac{1}{n}\sum_1^n a_k\right)\left(\frac{1}{n}\sum_1^n b_k\right),$$

$n = 1, 2, \ldots$.

[121] To find the probability that two integers, chosen at random, are prime to one another (the answer is $\frac{6}{\pi^2}$).

principal founder of the theory of approximations. Proved (1850) Bertrand's conjecture (1845) that there is always at least one prime between n and $2n$ for $n > 3$.

Chebyshev was born at Borovsk. He was educated at the University of Moscow, and in 1859 became a professor of mathematics in the University of St. Petersburg, a position from which he retired at 1880.

Approximations — Minimax vs. Least-Squares

"After having spent years trying to be accurate, we must spend as many more in discovering when and how to be inaccurate".

Ambrose Gwinett Bierce (1842–1914)

Mathematical models of natural processes inevitably contain some inherent *errors. These errors result from incomplete understanding of natural phenomena, the stochastic or random nature of many processes, and uncertainties in experimental measurements. Often, a model includes only the most pertinent features of the physical process and is deliberately stripped of superfluous detail related to second-level effects. Therefore, we approximate because we must!*

Even if an error-free mathematical model could be developed, it could not, in general, be solved exactly *on a digital computer. A digital computer can only perform a limited number of simple* arithmetico-logical *operations on finite, rational numbers. Fundamentally important* mathematical *operations such as differentiation, integration, and evaluation of infinite series cannot, in general, be implemented directly on a digital computer. All computers have finite memories and computational registers; only a discrete subset of the real, irrational numbers may be generated, manipulated, and stored. Thus, it is impossible to represent infinitesimally small or infinitely large quantities, or even a continuum of real numbers on a finite interval.*

Algorithms that use only finitistic arithmetic operations and certain logical operations (such as a branching based upon algebraic comparison or logical) are called numerical methods. *The error introduced in approximating the solution of a mathematical problem by a numerical method is usually termed the* truncation error *of the method. When a numerical method is actually run on a digital computer after transcription to computer program form, another kind of error, termed* round-off error, *is introduced. These are caused by the rounding of results from individual arithmetic operations because only a finite number of digits can be retained after each operation, and will differ from computer to computer, even when the same numerical method is being used.*

The art of approximations is as old as mathematics itself; the greatest mathematicians since **Archimedes**, *including* **Newton**, **Euler**, **Lagrange**, **Gauss**, **Legendre** *and* **Ramanujan** *devised ingenious and even beautiful ad-hoc techniques to approximate* π, e, *arclengths, areas, sums of series, roots of equations, solutions of differential equations and other entities.*

However, the rapid growth of applied mathematics in the wake of the Industrial Revolution called for the establishment of a discipline of approximations, through which algorithms could be systematized and developed methodically to answer the growing needs of the exact sciences. The banner of this new trend in mathematics was carried by **Panfuty L. Chebyshev**.

To understand the ideas of Chebyshev, a brief survey of polynomial approximation is needed: It is sometimes useful to approximate one function $f(x)$, *by a sum of 'suitable', simpler function. Such simpler functions are: monomials* $\{x^k\}$, $k = 0, 1, \ldots, n$, *trigonometric functions* $\{\sin kx, \cos kx\}$ *or exponential functions* $\{e^{\lambda_k x}\}$. *A linear combination of monomials leads to an algebraic* polynomial *of degree* n, $p_n(x) = \sum_{k=0}^{n} a_k x^k$. *Polynomials are easy to evaluate, and their sums, products, differences, derivatives and integrals — are also polynomials. In addition, they remain polynomials under the transformations of scaling and of origin translation*[122]. *These favorable properties are possessed by the trigonometric functions as well. The Weierstrass*

[122] A natural generalization of polynomial approximation consists in approximation by *ratios* of polynomials, that is, by rational functions. Such approximations are expressed conveniently in terms of *continued fractions*. As an example, consider the continued-fraction expansion of **Thorvald Nicolai Thiele** (1909)

$$f(x) = a_0 + \cfrac{x - x_0}{a_1 + \cfrac{x - x_0}{a_2 + \cfrac{x - x_0}{a_3 + \cdots}}}$$

approximation theorem[123] *(1885) then provides the analytical justification for*

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with coefficients

$$a_0 = f(x_0), \qquad a_k = \frac{k}{\left[\dfrac{d\rho_{k-1}(x)}{dx}\right]_{x_0}}$$

for $k = 1, 2, \dots$ where the function $\rho_k(x)$ follows from the recursion relation

$$\rho_k = \rho_{k-2}(x) + \frac{k}{\rho'_{k-1}(x)}$$

for $k = 1, 2, \dots$ with $\rho_{-1}(x) = 0$, $\rho_0(x) = f(x)$. Also

$$a_k = \rho_k(x_0) - \rho_{k-2}(x_0).$$

For small values of $x - x_0$, the Thiele expansion can be seen as an alternative to the Taylor expansion of $f(x)$ about $x = x_0$.
As an example, take $f(x) = e^x$ at $x_0 = 0$, to obtain

$$a_0 = 1, \quad a_{2n} = 2(-1)^n, \quad a_{2n+1} = (-)^n(2n+1).$$

A generalization of the above expansion leads to an analog of the *Bürmann-series* expansion in the form:

$$F(x) = A_0 + \cfrac{G(x) - G(x_0)}{A_1 + \cfrac{G(x) - G(x_0)}{A_2 + \cfrac{G(x) - G(x_0)}{A_3 + \cdots}}}$$

where

$$A_k = \Phi_k(x_0), \qquad \Phi_k(x) = P_k(x) - P_{k-2}(x), \qquad \Phi_{k+1}(x) = (k+1)\frac{G'(x)}{P'_k(x)},$$

$$P_{-2}(x) = P_{-1}(x) = 0, \qquad \Phi_0(x) = F(x).$$

The first few Φ's are readily found to be governed by the equations $\Phi_0 = F$, $\Phi_1 = \frac{G'}{F'}$, $\Phi_2 = 2\frac{G'}{\Phi'_1}$, $\Phi_3 = 3\frac{G'}{F' + \Phi'_2}$. Thus, for example, if we take $F(x) = e^x$, $G(x) = \sin x$, $x_0 = 0$, we obtain, near $x = 0$,

$$e^x = 1 + \cfrac{\sin x}{1 + \cfrac{\sin x}{-2 + \cdots}}.$$

[123] If $f(x)$ is continuous in the closed interval $[a, b]$ then, given any $\varepsilon > 0$, there is some polynomial $p_n(x)$ of degree $n(\varepsilon)$ such that $|f(x) - p_n(x)| < \varepsilon$, $a \le x \le b$.

believing that polynomials can yield good approximations for a given function. Weierstrass' theorem is of little value in cases where $f(x)$ is unknown, except for a few sampled values. But even if $f(x)$ is known, the theorem does not tell us how *the polynomial $p_n(x)$ can be produced.*

If the data is given in the form of $n+1$ paired values $\{x_i, f(x_i)\}$, $i = 0, 1, \ldots, n$, the determination of the approximating polynomial

$$p_n(x) = \sum_{i=0}^{n} a_i x^i$$

boils down to the determination of the coefficients a_i from the set of $n+1$ equations $p_n(x_i) = f(x_i)$, $i = 0, 1, \ldots, n$. The result is known as the interpolating polynomial *of the n^{th} degree. It does not guarantee accurate approximation of $f(x)$ for $x \neq x_i$, unless $f(x)$ itself is a polynomial of degree n or less.*

There are situations which render the above procedure inefficient. This is especially true when the degree of reliability of the discrete data is not well established. There is no sense then in attempting to determine a polynomial of high degree which fits the vagaries of such data exactly and hence, in all probability, is represented by a curve which oscillates violently about the true function. In this case it is preferable to apply a postulate that is often known as the principle of least squares *(**Gauss**, 1795; **Legendre**, 1806).*

The basic idea behind this principle is the requirement that $f(x)$ and its approximant $p_n(x)$ (or some other function) agree as closely as possible *in a specific sense. Of the many meanings which might be ascribed to "as closely as possible", the principle assumes that the* best approximation *is that for which the integral (or sum) of the* squared error *is least.*

More generally, if $W(x_i)$ is a measure of the relative precision of the value assigned to $f(x)$ when $x = x_i$, the criterion is modified by requiring that the squared error at x_i be multiplied by the weight *$W(x_i)$ before the sum is calculated.*

For a given $f(x)$ and basis functions $\phi_k(x)$, one minimizes the integral

$$I = \int_a^b W(x) \left[f(x) - \sum_{k=0}^{\infty} A_k \phi_k \right]^2 dx.$$

Setting $\frac{\partial I}{\partial A_k} = 0$ yields at once

$$A_k = \int_a^b W(x) f(x) \phi_k dx,$$

provided ϕ_k are orthonormal with weight $W(x)$ in $[a, b]$. One can show that the truncated Fourier series

$$T_M(x) = \frac{1}{2}A_0 + \sum_{k=1}^{M}(A_k \cos kx + B_k \sin kx)$$

minimizes the integral

$$I = \int_0^{2\pi} [f(t) - T_M(t)]^2 \, dt.$$

In other words, to minimize I we should choose

$$A_k = \frac{1}{\pi}\int_0^{2\pi} f(t) \cos kt \, dt, \qquad B_k = \frac{1}{\pi}\int_0^{2\pi} f(t) \sin kt \, dt.$$

The minimum value of I then assumes the form

$$I_{\min} = \pi \sum_{k=M+1}^{\infty} (A_k^2 + B_k^2).$$

If $f(x)$ is discontinuous, the truncated Fourier-series does not give a good approximation to $f(x)$ in the vicinity of the discontinuity points, no matter how large M is chosen (Gibbs' phenomenon)[124].

[124] In 1904, **Lipót Fejer** (1880–1959, Hungary) has shown that a better approximation to $f(x)$ is obtained if one replaces $T_M(x)$ by the arithmetic mean of the partial sums

$$s_N(x) = \frac{1}{N}\{T_1(x) + T_2(x) + \cdots + T_{N-1}(x)\}.$$

As N increases, this series tends to a limit that is equal to the mean discontinuity, i.e.

$$\frac{1}{2}[f(x+0) + f(x-0)].$$

Since trigonometric series can in turn be represented by power series, $s_N(x)$ can be approximated by a polynomial in x.

Fejer Theorem: Let $f(x)$ be a function of the real variable x, $-\pi \leq x \leq \pi$, and defined by the equation $f(x + 2\pi) = f(x)$ for all real values of x; and let $\int_{-\pi}^{\pi} f(x)dx$ exist and (if it is an improper integral) let it be absolutely convergent. Then the Fourier series associated with the two limits $f(x \pm 0)$ exist and its average is

$$s = \lim_{N\to\infty} s_N(x) = \frac{1}{2}[f(x+0) + f(x-0)].$$

Another popular criterion for how close is "as closely as possible", termed the minimax principle, requires that the coefficients of the approximating polynomial $p_m(x)$ *be chosen so that the maximum magnitude of the difference* $f(x_i) - p_m(x_i)$, $i = 0, 1, \ldots, n$ $(m < n)$ *be as small as possible. Then the minimax polynomial of degree* m *must satisfy the condition* $\max_i |f(x_i) - p_m(x_i)| = \text{minimum}$, *that is,* $p_n(x)$ *must minimize the maximum error. In more general form* $\max_{a \le x \le b} |f(x) - p_m(x)| = \text{minimum}$. *The principle was created by Chebyshev (1859), and the minimax polynomials are closely related to the Chebyshev polynomials of the first kind* $T_n(x)$.

These polynomials are defined by $T_n(x) = \cos(n \cos^{-1} x)$. *This may look trigonometric at first glance, but it is indeed algebraic [the symbol* T_n *comes from the French spelling used for his name in French, Tchebychef*[125]*]. To see this, we recall de Moivre's theorem*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta.$$

Expanding the binomial and taking the real part, we get, with $x = \cos\theta$:

$$T_n(x) = \cos n\theta = \frac{n}{2}\sum_{k=0}^{[n/2]} (-)^k \frac{\Gamma(n-k)}{k!(n-2k)!}(2x)^{n-2k}.$$

With the aid of the recurrence formula $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ *one can generate* $T_n(x)$ *for any* n. *In particular* $T_0(x) = 1$; $T_1(x) = x$; $T_2(x) = 2x^2 - 1$; $T_3(x) = 4x^3 - 3x$.

The Chebyshev polynomials have a number of interesting and useful properties:

Fejer has shown that

$$s_N(x) = \int_0^\pi [f(x+t) + f(x-t)] \left\{ \frac{\sin^2 \frac{N}{2} t}{\sin^2 \frac{t}{2}} \right\} \frac{dt}{2\pi N}$$

where the function in the curly braces is the *Fejer kernel*. Since

$$\lim_{N\to\infty} \left\{ \frac{\sin^2(\frac{N}{2}t)}{\frac{1}{2}\pi N t^2} \right\} = \delta(t),$$

the above result is then obvious.

[125] **Abram S. Besicovitch** (1891–1970) once said, in his thick Russian accent: "Zey are called T-polynomials because T is the first letter of ze name Chebyshev".

(1) $T_n(x)$ *is a polynomial of degree* n*. If* n *is even,* $T_n(x)$ *is an even polynomial; if* n *is odd,* $T_n(x)$ *is an odd polynomial. The coefficient of* x^n *in* $T_n(x)$ *is* 2^{n-1}.

(2) $T_n(x)$ *has exactly* n *real zeros on the interval* $[-1,1]$*. These zeros are located at* $x_j = \cos \frac{2j+1}{n}\frac{\pi}{2}$, $j = 0, 1, 2, \ldots, n-1$.

(3) $|T_n(x)| \leq 1$, $-1 \leq x < 1$ *for all* n*. For* $n > 0$, $T_n(x)$ *attains its bounds* ± 1*, alternately at the points* $x_j = \cos \frac{\pi j}{n}$, $j = 0, 1, \ldots, n$; $T_n(x_j) = (-1)^j$.

(4) The Chebyshev polynomials are orthogonal in the interval $[-1,1]$ *over weight* $W(x) = (1-x^2)^{-1/2}$*, namely*

$$\frac{2}{\pi}\int_{-1}^{1}\frac{T_i(x)T_j(x)}{\sqrt{1-x^2}}dx = \left\{\begin{matrix} 0 & i \neq j \\ 1 & i = j \neq 0 \\ 2 & i = j = 0 \end{matrix}\right\}.$$

Given a function $f(x)$*, the least-squares approximation to* $f(x)$ *in terms of the Chebyshev polynomials yield the series* $\sum_{k=0}^{\infty} A_k T_k(x)$ *with* $A_k = \frac{\varepsilon_k}{\pi}\int_{-1}^{1}\frac{f(x)T_n(x)}{\sqrt{1-x^2}}dx$ *where* $\varepsilon_0 = 1$, $\varepsilon_k = 2\ (k \neq 0)$.

The polynomials $T_n(x)$ *also satisfy the discrete orthogonality relation*

$$\frac{2}{n}\sum_{k=1}^{n} T_i(x_k)T_j(x_k) = \left\{\begin{matrix} 0 & i \neq j \\ 1 & i = j \neq 0 \\ 2 & i = j = 0 \end{matrix}\right\}$$

with $0 \leq i < n$, $0 \leq j < n$*. Here* $x_k\ (k = 1,\ ...,\ n)$ *are the* n *zeros of* $T_n(x)$*. If* $f(x)$ *is an arbitrary function in the interval* $[-1,1]$*, and if* N *coefficients* c_j $(j = 0, 1, \ldots, N-1)$ *are defined by*

$$c_j = \frac{2}{N}\sum_{k=1}^{N} f(x_k)T_j(x_k),$$

then the approximation formula

$$f(x) \approx \left[\sum_{k=0}^{N-1} c_k T_k(x)\right] - \frac{1}{2}c_0$$

is exact for x *equal to all* N *zeros of* $T_N(x)$.

The importance of this should be appreciated because it means that we can approximate the continuous case in a natural way by simply doing the discrete

case. This property is not enjoyed by the other classical set of orthogonal polynomials.

(5) *Since* $T_n(x)$ *belongs to a class of functions of the special form* $T_n(x) = f\left[nf^{-1}(x)\right]$, *one derives the unique relation*

$$\begin{aligned} T_m\left[T_n(x)\right] &= f\left[mf^{-1}T_n(x)\right] = f\left[mf^{-1}fnf^{-1}(x)\right] = f\left[mnf^{-1}(x)\right] \\ &= T_{mn}(x) = T_{nm}(x) = T_n\left[T_m(x)\right]. \end{aligned}$$

(6) *Minimax Property. Let* $p_n(x)$ *be any polynomial of degree* n *with leading coefficient unity. Then*

$$\max_{-1\le x\le 1} |2^{1-n}T_n(x)| \le \max_{-1\le x\le 1} |p_n(x)|,$$

i.e. $\left\{T_n(x)/2^{n-1}\right\}$ *has the smallest maximum magnitude on the interval* $[-1, 1]$ *of all polynomials in this class.*

This property is of great interest in numerical computations, since any error that can be expressed as an n^{th} *degree polynomial, can be minimized by equating it with* $T_n(x)/2^{n-1}$, *provided there is freedom of choice in selecting the base points* x_i *[which one chooses as the roots of* $T_n(x)$*]. It has been shown that if the function* $f(x)$ *can be expanded in terms of Chebyshev polynomials* $f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$, *then the partial sum* $p_M(x) = \left\{\sum_{k=0}^{M} a_k T_k(x) - \frac{1}{2}a_0\right\}$ *will usually be a very good approximation to the minimax polynomial, that is,* $p_M(x)$ *will be near-minimax*[126], *but whereas the minimax polynomial is very difficult to find, the Chebyshev approximating polynomial is very easy to compute.*

The polynomial $p_M(x)$ *is known as the minimax polynomial approximation to* $f(x)$*. If* $f(x)$ *is given by a polynomial* $p_n(x)$*, it is possible in many cases to obtain a minimax polynomials with* $M < n-1$*. The procedure for replacing a polynomial of a given degree by one of lower degree is known as economization.*

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[126] *Note that* $p_M(x)$ *is just the Fourier cosine series expansion of the function* $f(\cos\theta)$.

1835 CE **Giusto Bellavitis** (1803–1880, Padua, Italy). Mathematician. Created a two dimensional vectorial system (calculus of 'equipollences'), thereby describing geometrical entities that are in all ways equivalent in behavior to complex numbers. He gave numerous and ingenious applications of his method to mathematical and physical problems. Made significant contributions to algebraic geometry and descriptive geometry.

Bellavitis was born in Bassano. He was an autodidact who did not pursue regular studies. During 1822–1843 he worked for the municipal government of Bassano, occupying his free time with mathematical studies and research. In 1845, he became a professor of descriptive geometry at the University of Padua (through competitive examination). In 1866 he was elected a senator of the Kingdom of Italy.

1835–1846 CE **Jean Léonard (Louis) Marie Poiseulle** (1799–1869, France). Physician and physiologist. Discovered experimental laws for viscous laminar flow in straight circular pipes, known as *Hagen-Poiseulle flow*[127]. He wanted to understand the flow of blood through capillaries and determined the relevant laws in painstaking detail.

If a pressure difference Δp drives the viscous fluid (of *shear viscosity* η) in a cylinder of length L and radius R, the velocity profile is given by the parabola

$$V(r) = \frac{\Delta p}{4\eta L}(R^2 - r^2),$$

where $0 \leq r \leq R$ (*Poiseulle law*).

This *Hagen-Poiseulle flow* is a steady unidirectional axisymmetric flow in a circular cylinder. The law of flow is derivable theoretically through a straightforward integration of the Navier-Stokes equations for steady, axially-homogeneous axially-directed incompressible flow,

$$\eta \nabla^2 \boldsymbol{V} = -\frac{\Delta p}{L}$$

where $\boldsymbol{V}(\boldsymbol{r}, t) = V(r)\boldsymbol{e}_z$, $\boldsymbol{e}_z$ being a unit vector along the axis. We find

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = -\frac{\Delta p}{\eta L}.$$

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[127] **Gotthilf Heinrich Ludwig Hagen** (1797–1884, Germany). Hydraulic engineer in Prussian state service. Known for his studies of laminar and turbulent flow, and for the independent discovery of the law of laminar flow in circular pipes (1839). A *laminar flow* is a an orderly non-turbulent flow in which the fluid particles move in smooth layers without mixing.

Integrating twice w.r.t. r and imposing the boundary condition $V(R) = 0$, we arrive at the Poiseulle law.

The volume velocity (volume flow per unit time) through the tube is given by $Q_c = \frac{\pi R^4}{8\eta}\frac{\Delta p}{L}$ (*Poiseulle equation*) and serves to determine η when all other entities are known. The sensitive dependence of Q_c on R explains why small changes in diameter can cause large changes in flow[128]. The application of this law to flow in blood vessels must be modified by the *elastic* properties of the capillary wall and the presence of *erythrocytes*. (It is convenient to define the resistance to flow via the relation $Q_c = \frac{F}{\Omega}$, where $F = \pi R^2(\Delta p)$ is the driving force and $\Omega = \frac{8\eta L}{R^2}$ is the resistance.)

The above relation is sometimes recast in the form $\Delta p = Q_c \cdot r$ where Δp =mean arterial pressure, Q_c =cardial output and r =total peripheral resistance.

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[128] If the pipe has an elliptic cross section

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the solution of the above problem becomes

$$V(x,y) = \frac{\Delta p}{2\eta L}\left(\frac{a^2b^2}{a^2+b^2}\right)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right),$$

which describes the flow of a fluid of viscosity η through an elliptic pipe. The flux through the pipe is

$$Q = \iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1} V(x,y)dxdy = \frac{\pi \Delta p a^3 b^3}{4\eta L(a^2+b^2)}.$$

Setting $a = b = R$, we deduce that the flux through a circular pipe of radius R is given by $Q_c = \frac{\pi \Delta p R^4}{8\eta L}$. Since the area of an ellipse is πab, a circular pipe with the same cross-sectional area as the ellipse must have radius $\sqrt{ab}$. Hence the fractional flux reduction caused by deforming the circle into an ellipse with the same area is $1 - \frac{Q}{Q_c} = \frac{(a-b)^2}{a^2+b^2}$. This is always non-negative and is clearly minimized by taking $a = b$. Thus for a given cross-sectional area and driving pressure drop, a circular pipe carries a greater quantity of fluid per unit time than any elliptical one, and this is why pipes are circles! This optimality property is intuitively clear: because the circle, being the curve of minimum length for a given enclosed area, minimizes the area of viscous friction between fluid and pipe per unit pipe length.

Blood pressure, the force per unit area exerted by the blood against a vessel wall and normal to it, depends on the volume of blood contained within the vessel and the compliance of the vessel wall (how easily they can be stretched). If the volume of blood entering the arteries were equal to the volume of blood leaving the arteries during the same period, *arterial blood pressure* would remain fixed. This is not the case, however. During ventricular *systole*, a stroke volume of blood enters the arteries from the ventricle while only about 1/3 as much blood leaves the arteries to enter the arterioles. During *diastole*, no blood enters the arteries, while blood continues to leave, driven by elastic recoil (**Hale**, 1733).

The maximum pressure exerted in the arteries when blood is ejected into them during systole, the *systolic pressure*, averages 120 mm Hg. The minimum pressure within the arteries when blood is draining off into the remainder of the vessels during diastole, the *diastolic pressure*, averages 80 mm Hg. The arterial pressure does not fall to 0 mm Hg because the next cardiac contraction occurs and refills the arteries before all the blood drains off.

In computing the average pressure (or *mean arterial pressure*) responsible for driving blood forward into the tissues throughout the cardiac cycle, it must be taken into account that arterial pressure remains closer to diastolic than to systolic pressure for a longer portion of each cardiac cycle; numerically, mean arterial pressure $=$ diastolic pressure $+\frac{1}{3}[$ systolic $-$ diastolic$]$.

Poiseulle's main interest was the flow of blood through the vessels of the circulatory system, but he actually worked with water because of the difficulty at that time of preventing blood from clotting on exposure to air. Poiseulle's law is so well established experimentally that it is often used in order to determine the viscosity coefficient of viscous fluids. When blood is examined in this manner[129], its viscosity coefficient is found to be about 5 times the value for water, if the diameter of the tube is relatively large $(\eta_B = 0.035$ poise[130], where 1 poise $=$ 1 dyn sec/cm^2).

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129 *Estimate of total number of capillaries in the body*: The cardiac output is about $K = 5.5$ liter/minute. The mean blood flux, Q_c, through a typical capillary of radius 3.5 micron (a mean value for the entire body) is calculated from Poiseulle's equation to be 0.13×10^{-6} milliliter/sec. The total number of capillaries in the body, N, is then calculated from $K = fNQ_c$ $(f \simeq 0.7$ is the fraction of capillaries that are open), yielding $N \approx 1.0 \times 10^9$. This estimate agrees with other estimates in order of magnitude. The mean velocity through a capillary is 2.5 mm/sec.

130 A unit of viscosity named after Poiseulle. In 1975 the unit was changed to {Pascal·sec}, where Pascal (Pa) is a unit of pressure ($= 1$ N/m^2) and 1 Pa·sec $= 10$ poise.

1836 CE Introduction of the marine *screw propeller*, developed by **John Stevens** (1749–1838, U.S.A.) and later applied by **John Ericsson** (1803–1889, Sweden, U.S.A.) and **Francis Pettit Smith** (1808–1874, England).

The Rainbow —
From Noah to Airy (1836) and Beyond

"The triumphal arch through which I march,
With hurricane, fire, and snow,
When the powers of air are chained to my chair,
Is the million-colored bow;
The sphere-fire above its soft colors wove,
While the moist earth was laughing below".

Percy Bysshe Shelley, *'The Cloud'*

I. Phenomenology

The rainbow (formerly known as iris*) is best seen in the sky after a rain storm when the sun is low but is shining brightly through a section of the sky that is clear. To see the rainbow, one must turn one's back to the sun and look toward a region that still has rain clouds. Under good conditions one sees a colored arc consisting of concentric circular bands having their common center on the line joining the eye of the observer to the sun. Each band within the bow has its own color, with blue-violet on the inside (lower boundary) and red on the outside (upper boundary). This is known as the* primary rainbow; *it has an angular radius of about 41°, and exhibits a fine display of the colors of the* spectrum.

Sometimes an outer bow, the secondary rainbow, *is observed; this is much fainter than the primary bow, and it exhibits the same play of colors, with the important distinction that the order of colors is reversed — the red being inside and the violet outside. Its angular radius is about 57°. It is also to be noticed that the space between the two bows is considerably* darker *than the rest of the sky. The third or* tertiary *bow, having about the same radius as that of the primary and colors in the same order, lies* between the observer and the sun, *but is so faint that it is rarely seen in nature. In addition to these prominent features there are sometimes to be seen a number of colored* bands, *situated at or near the summits of the bows, close to the inner edge of the primary and the outer edge of the secondary bow; these are known as the* spurious, supernumerary *or complementary rainbows.*

The higher the sun, the lower the bow; if the sun is on the horizon, the observer will see a full 180° of the rainbow (and an observer on a high mountain might see the whole circle of the bow). If the sun is higher in the sky, only a small section of the upper arc can be seen. If the sun is higher than 42° above the horizon, the rainbow disappears completely. However, if one is looking down into a canyon where there is a waterfall or even down the mist produced by a lawn sprinkler, one can see a miniature rainbow though the sun is high in the sky. Under the right conditions, one may be able to see the whole 360° of the bow from an airplane!

(Occasionally, the light from the moon forms a feeble lunar rainbow. *But this phenomenon is rarely seen except about great waterfalls and along certain showery coasts.)*

Seven colors are discernible in each rainbow: violet, indigo, blue, green, yellow, orange, and red. But these colors blend into each other so that the observer really perceives only four or five of them. The angular width of each color band varies, and depends chiefly on the size *of the raindrops in which the rainbow forms.*

Like other optical phenomena, rainbows have been used by people as a means of predicting the weather. A well-known weather proverb says:

Rainbow in the morning, sailors take warning
Rainbow at night, sailors delight.

This bit of weather lore relies on the fact that weather systems in the mid-latitudes usually move from west to east. Remember that an observer must be positioned with his back to the sun and facing the rain in order to see the rainbow. When a rainbow is seen in the morning, the sun is located to the east of the observer and the raindrops that are responsible for its formation must therefore be located to the west. In the early evening, the opposite situation exists — the rain clouds are located to the east of the observer.

Thus, we predict the advance of foul weather when the rainbow is seen in the morning because the rain is located to the west of the observer and is traveling toward him. On the other hand, when the rainbow is seen late in the day, the rain has already passed. Although this proverb does have a scientific basis, a small break in the clouds, which lets the sun shine through, can generate a late-afternoon rainbow. In this situation, a rainbow may certainly be followed shortly by more rainfall.

Rainbows differ among themselves[131]*, as one snowflake from another. Close observations of rainbows show that not even the* colors *are always the*

[131] Furthermore, each observer sees his "own" rainbow, generated by a different set of droplets and different sunlight from that which produces another person's rainbow. In this sense each observer would find "his" rainbow responding to

same; neither is the band of any color of constant angular width; nor is the total breadth of the several colors at all uniform; similarly, the purity and brightness of the different colors are subject to large variations. All these differences depend essentially upon the size of the drops.

II. Optics

The rainbow is produced by the combined effects of refraction, reflection, dispersion, scattering and *diffraction* of *sunlight by drops of rain*[132].

Consider first the formation of the primary bow in terms of geometrical optics[133] *(ray theory). Consider first a monochromatic (fixed wavelength, λ) plane wave of sunlight falling on a spherical drop of water. According to ray*

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his motion in the same sense as his shadow. Rainbow, like beauty, is in the eye of the beholder.

132 If we look at a very bright rainbow through a monochromatic red glass we see a succession of circular arcs, alternately bright and dark, similar to the *diffraction rings* which are formed when the light from a point source (sunlight or a distant arc lamp) falls through a small circular stop on to a white screen.

133 The laws of *geometrical optics* are asymptotic laws of propagation of electromagnetic waves (light), valid in the limit of wavelengths small relative to typical spatial dimensions of the problem. In this regime one assumes that the wave-fronts near any point are sufficiently characterized by their normals and by their local radii of curvature. This approximation breaks down *near* the rainbow, where the *Airy theory* applies. The visible spectrum stretches from $\lambda = 0.7\mu m$ (red) to $\lambda = 0.4\mu m$ (violet), while the diameters of a rain drop range from 200μ to 2000μ. Visible *fog* may have characteristic particle sizes as small as 5–20μ.

The general explanation for rainbows is that of all the parallel rays of light which fall on a drop of water and emerge after one or more internal reflections, those which emerge in appreciably the same direction reinforce each other, and therefore produce a definite sensation on the eye. The order of the colors is explained by the fact that the direction of these "accumulated" rays depends on the index of refraction of the drop, and is therefore different for different colors.

This explanation is not entirely satisfactory, nor are the results it predicts absolutely consistent with the facts. The rays that leave the drop in the same direction take slightly different paths in the drop, and may therefore be in a condition to produce *interference* effects in accordance with the principles of

theory, this wave is represented by its normal (ray). Because of the drop's spherical symmetry, it is sufficient to determine the effects of this interaction in the plane of a great circle containing the ray, rendering the problem 2-dimensional[134].

Let the ray enter with incidence angle i (w.r.t. the local normal of the sphere) and angle of refraction r (w.r.t. the same normal), be internally reflected n times, and finally be refracted into air again. The path of the ray will lie throughout in the initial plane of incidence. Simple geometric considerations show that the angle by which the incident ray will be bent from its original direction (known as total deviation) is given by

$$D = 2(i - r) + n(\pi - 2r).$$

Since $\sin i = \mu \sin r$ *(Snell's law of refraction; $\mu =$ index of refraction of water relative to air), the deviation becomes a function of* $\{i, \mu, n\}$, *having the explicit form*

$$D = \pi n + 2i - 2(n+1)\sin^{-1}\left\{\frac{1}{\mu}\sin i\right\}.$$

Now, according to ray theory, the intensity *of the emergent ray at large distance R from the sphere is*

$$I_n = \frac{a^2}{R^2} I_0 \epsilon_n^2 G,$$

where a is the radius of the drop, I_0 the incident intensity, ϵ_n the fraction that yields the refracted part of the total energy, and

$$G = \frac{\sin i \cos i}{\sin\theta \left|\frac{dD}{di}\right|}$$

is the divergence coefficient *($\theta =$ azimuth angle of emergent ray at the sphere's center relative to incident direction in the plane of incidence).*

Clearly, the intensity is very sensitive to $\left|\frac{dD}{di}\right|$, *and since* $\frac{a^2}{r^2}$ *is usually very small, it is just this amplification due to G that makes the rainbow visible! But, alas, the rays which render* $D(i)$ *extremal [i.e.* $\frac{dD}{di} \equiv 0$*] are precisely*

Physical Optics. In the purely ray-theory calculation, one must determine the general form of the *caustic surface* enveloped by a system of rays, originally parallel and emerging after any number of reflections within a drop of water.

[134] To obtain the 3-dimensional picture, one then *rotates* this plane about a line in the plane that bisects the angle between the incident and emerged paths of any given ray in the same plane.

those which invalidate the geometrical optics approximation. Simple calculus shows that the angle of incidence of such rays is given by $\cos i_c = \sqrt{\frac{\mu^2-1}{n^2+2n}}$, *and that the total deviation there is a minimum.*

If the rays suffer one internal reflection (primary bow), the deviation is a minimum when the ray is incident at an angle $\cos^{-1}\left\{\sqrt{\frac{\mu^2-1}{3}}\right\}$. *If we take* $\mu = 1.3311$ *for red rays* $(\lambda = 6562.9$ Å), *we find that the corresponding angles are* $(n = 1)$ $i_c = 59°31'$, $r_c = 40°21'$, $D = \pi - 42°22'$. *With* $\mu = 1.3435$ *for violet* $(\lambda = 3968.5$ Å) *we obtain:* $i_c = 58°48'$, $r_c = 39°33'$, $D = \pi - 40°36'$.

Now, if a line be drawn through the eye parallel to the direction of sun's rays, all drops which lie on a cone of semi-opening angle $\{\pi - D\}$ *with this line as axis, will be in a position to allow the emergent parallel rays to enter the eye*[135]*. The apparent arc is therefore composed of arcs of different colors; and the angular radii will be* $42°22'$ *for the red arc and* $40°36'$ *for the violet arc. However, as each point of the sun's disc sends rays giving rise to a bow, the apparent breadth of the bow exceeds the difference of these radii by the sun's angular diameter (ca* $32'$*), yielding altogether* $2°18''$.

Thus, geometrical optics allows us only to explain the overall shape and color-geometry of the primary (and secondary) bow. However, intensities at or near the zones of minimum deviation for each color, must be evaluated from the reconstruction of the caustic formed by the physical-optics process of interference of the refracted and reflected wave-fronts inside the drop.

To this end, ***Airy*** *(1836) first derived the equation of the emergent wave-surface, which he showed to be the involute of the caustic surface. His lengthy analysis boils down at the end to the rather simple equation* $y = \frac{h}{3a^2}x^3$ *where*

$$h = \frac{(n^2+2n)^2}{(n+1)^2(\mu^2-1)}\sqrt{\frac{(n+1)^2-\mu^2}{\mu^2-1}}.$$

Here $y(x)$ *is the curve in the plane of incidence which results from the intersection of the involute with the said plane. It represents the distortion of the straight-line segment of the incident plane wave due to its interaction with the drop, i.e. the initial plane wave surface becomes curved on emerging from*

[135] Since the rainbow may be regarded as consisting of coaxial, hollow conical beams of light of different colors seen edgewise from the vertex, they may have great depth, or extent, in the line of sight. The drops that produce the bow may be nearby or far away, and the question "what is the rainbow's distance" is therefore meaningless.

the drop; it is bent in opposite directions on either side of the least deflected ray. Only those rays that lie close to the inflection point $(x = 0,\ y = 0)$ *reach the eye and give rise to the rainbow image on the retina*[136].

Airy then calculated the amplitude at a distant point in the direction θ *from the ray of minimum deviation. Taking the origin of the coordinates at the point of inflection of the emitted wave-front near the drop, the spatial part of the amplitude is the real part of the Airy "rainbow integral"*

$$A = A_0 \int_{-\infty}^{\infty} e^{-ik[y(x)\cos\theta - x\sin\theta]}\, dx$$

where $k = \frac{2\pi}{\lambda}$ *is the light wavenumber,* $y(x) = \frac{h}{3a^2}x^3$, *and* A_0 *is the amplitude per unit length of the front.*

The above integral can be transformed into a form which yields the luminous image intensity produced in the eye by this active part of the wave surface: $A^2 = M^2 f^2(z)$. *Here*

$$f(z) = \int_0^{\infty} \cos\frac{\pi}{2}(u^3 - zu)\, du, \qquad z = 2\sqrt[3]{6}\frac{\sin\theta}{(\cos\theta)^{1/3}}a^{2/3}\lambda^{-2/3}h^{-1/3},$$

$$h = \frac{(n^2+2n)\sin i_c}{(n+1)^2\cos^3 i_c}, \qquad M = 2A_0\left[\frac{3a^2\lambda}{4h\cos\theta}\right]^{1/3}.$$

Airy's theory predicts periodic changes of intensity *of monochromatic light via the function* $f(z)$; *the first maxima does* not *coincide with* $z = 0$, *nor, therefore, with* $\theta = 0$, *the direction of the ray of minimum deviation.*

When the source of light simultaneously emits radiations of various wavelengths, as does the sun, a corresponding sequence of bows, each consisting of a sequence of maxima and minima, are partially superimposed on each other. In this way different colors are mixed, and thus the familiar polychromatic rainbow produced. The mixing of colors is governed by two causes: First, the angular intervals between two successive maxima increase with $\lambda^{2/3}$, *and consequently, coincident distribution of the intensities of any two colors is impossible. Second, since the direction of the ray of minimum deviation varies*

[136] This equation, then, represents a curve *very nearly* coincident with that portion of the wave-front to which the rainbow phenomena are due, and, since the effects computed from it substantially agree with those observed when the drops are not too small, the approximation is sufficient for most practical use. Indeed the approximation that raindrops are perfectly spherical involves, perhaps, a greater error (the *undersides* of a falling drop depart most from a spherical shape — the largest drops look like hamburger buns with concave undersides).

with the refractive index, the direction of the zero ($\theta = 0$) point on the intensity curve, near which the first maximum lies, correspondingly varies. These two causes, together, produce all sorts of colors mixings that in turn give rise to widely varied sensations.

The actual intensity is proportional to $\left\{\frac{a^{7/3}}{\lambda^{1/3}}\right\}$. *The breadth of the line is proportional to* $\left\{\frac{\lambda^{2/3}}{a^{2/3}}\right\}$. *Thus, the rainbow bands produced by very small droplets (fog) are not only broad, but also feeble; as their colors necessarily are faint they frequently are not distinguished — the bow appearing as a mere white band.*

Note that in the above analysis certain contributions to the incident radiation and scattering by the drop were totally neglected, since they contribute little to the rainbow. (Thus, for example, radiation backscattered to the sun through reflection off the back surface was ignored.)

III. History

The rainbow affords a means of bridging the gap between the sciences and the humanities.

Mankind has been thinking, talking, and writing about the rainbow for thousands of years. Virtually every volume on mythology contains legends connected with the rainbow, and practically all modern textbooks of physics include some exposition of the optical principles which account for the bow.

Man's story of the rainbow, like other aspects of the history of science, has no inescapable origin and no discernible end. There is no record, oral or written, of the precise date at which a rainbow was first noticed; and even now, at the dawn of the 21^{th} *century, it is not possible to boast that the formation of the bow is accounted for in all its details. The primeval theories of the rainbow must have arisen from man's sense of wonder; and now, thousands of years later, the theory has become enmeshed with the intricacies of advanced mathematics. To trace the gradual development of this theory from early primitive conjectures to sophisticated contemporary formulations is to tread the pathway of human knowledge.*

The earliest documented mentions of the rainbow are in Homer's Iliad, *a Chaldean story of the flood, an early Sumerian hymn and above all in the beautiful Biblical passage on God's covenant with* **Noah** *(Genesis* ***9****, 13–16):*

> *"I do set my bow in the cloud, and it shall be for a token of a covenant between me and the earth".*
>
> *"And it shall come to pass, when I bring a cloud over the earth, that the bow shall be seen in the cloud".*
>
> *"And I will remember my covenant, which is between me and you and every living creature of all flesh; and the waters shall no more become a flood to destroy all flesh"*[137].
>
> *"And the bow shall be in the cloud; and I will look upon it, that I may remember the everlasting covenant between God and every living creature of all flesh that is upon the earth".*

Most exegetes interpreted the passage broadly as indicating that God here gave to the already familiar beauty of the rainbow a new significance, causing it to be a symbol of divine promise. Rain, sunlight, and cloud formations would appear to have been sufficiently similar (throughout temperate regions and during the period of man's existence) to those familiar today to justify the assumption that rainbows were observed by our most primitive ancestors. The circular-arc form of the rainbow may have been perceived by the earliest forms of life endowed with a sense of color-vision. At any rate, the rainbow probably existed more than a billion years ago, independently of the observer, as soon as suitable atmospheric conditions for its formation came into being.

Primitive peoples viewed the rainbow with fear and misgiving, as is evident from the various myths and legends. There is no precise date at which mythology gave way to science in the theory of the rainbow, nor did the transition take place at the same time or at the same rate in all cultures. With the four potamic[138] *civilizations (Tigris-Euphrates, Nile, Indus, Yangtze) flourishing several thousand years ago, one might expect to find some theory of the rainbow; yet there is no evidence of an attempt at a scientific explanation in those cultures.*

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[137] God indeed kept his promise, for He used the *fire* next time to wipe out Sodom and Gomorrah (*Genesis* **19**, 24–25).

[138] In a very broad and over-simplified sense, one can recognize, in the development of civilizations, three general stages which may be designated respectively as *potamic*, *thalassic*, and *oceanic*, according as the dominant cultures centered about rivers, seas or oceans. The first of the stages left nothing scientific on the rainbow. With the advent of the thalassic civilizations, which thrived throughout the whole Mediterranean area during the first millennium BCE, the situation changed.

Among the peoples pressing down from the north were the Hellenes, who occupied the peninsula between the Adriatic and Aegean Seas and then spread east and west to colonize the shores of Asia Minor and the tip of Italy (Magna Graecia). They acquired with amazing alacrity all the knowledge that the potamic civilizations had accumulated, and then they looked about for new intellectual fields to conquer. Unencumbered by hoary traditions and relatively unhampered by political and cultural authoritarianism, Greek scholars investigated nature with an exhilarating freedom and ingenuity. With them the scientific point of view became a dominant characteristic, for they sought to coordinate observations of natural phenomena into a consistent theoretical structure.

Anaximenes *(ca 575 BCE), a member of the Ionian school led by Thales of Miletos, was first to issue a naturalistic statement on the rainbow. First, he pointed out the obvious relation of the rainbow to the appearance of the sun. Then, he explained the colors as resulting from the admixture sunlight with the blackness of the cloud. Finally, he claimed that the cloud is bending the rays of the sun toward the eye.*

No documents of the period have survived the ravages of time, and the little that is known of the Milesian school is reported by others who lived long afterwards.

Anaxagoras *(ca 460 BCE) declared that the rainbow is but a reflection of the sun from a spherical cloud, as from a mirror. His theory that the rainbow is caused by* reflection *persisted, in variously elaborated forms, for about* 2000 *years. One cannot, however, determine whether Anaxagoras, and later* ***Democritos*** *(to whom Albertus Magnus ascribed the idea that the colors of the rainbow are due to positions from which it is viewed), were aware of the optical law of reflection, or if it had been applied in those days to a geometrical demonstration of the formation of the rainbow.*

The law may have been discovered shortly after the Periclean age, for ***Plato*** *(in the Timaeus, ca 380 BCE) seems to have been aware of some uniformity in the angles in optical reflection.*

Aristotle *(ca 340 BCE) did* not *contribute significantly to the physics of the rainbow, and his "theory" is today untenable. The most serious deficiency is the ascription of the bow to reflection alone, with no role accorded the essential phenomenon of refraction. A characteristic of his explanation which was perhaps even more obstructive was the macroscopic approach — the concentration of attention on the cloud and the meteorological sphere, rather than on the "little mirrors" of the cloud, where the key to the problem was, in the end, to be found. Finally, one misses in his account any mensurational element, although his geometry, even without measurement, was more sophisticated than that of any successor for well over a millennium. Moreover,*

his work includes the idea that the size of the rainbow could be explained geometrically in terms of the relative positions of the sun, the rain cloud, and the eye of the observer.

Whatever may be one's judgment on the place of Aristotle in the history of science, criticism must be in terms of the status of knowledge at that time. The rainbow concerns one of the most elusive portions of science; and when one compares the idiosyncrasy of the atmosphere with the regularity of the heavens, it is easier to appreciate why the rainbow appeared so enigmatical to the ancients. In view of the fact that Aristotle placed the explanation for rainbow not in optics, but in meteorology, along with hydrology, seismology, geology, and other portions of natural philosophy, it is greatly to his credit that he gave a thoroughly mathematical treatment to the bow.

Aristotle was undoubtedly acquainted with the colors formed when sunlight passes through a glass prism, but he seems not to have associated these with the rainbow. He was in fact, primarily a philosopher and biologist; and hence it is all the most surprising that the first mathematical theory of the rainbow *should have come from him. The surprise deepens into admiration when one realizes that no superior explanation was proposed for a period of more than* 1500 *years.* ***Archimedes*** *(ca 250 BCE), the greatest mathematical scientist of antiquity, was especially interested in optical phenomena; yet, so far as one knows, he left the problem of the rainbow quite untouched*[139].

Seneca *added little of permanent value in the theory of the rainbow. His chief contribution is his emphasis upon the role of the* individual raindrops *or "mirrors". The practical Romans were ever poor mathematicians, and one looks to them in vain for any improvement over the Aristotelian geometrical theory of the rainbow.*

Ptolemy*, who left us in his* Optics *the earliest surviving tables of angles of refraction from air to water, could have attributed the bow to refraction, but this is not mentioned in that part of* Optics *which came down to us.*

The first man to refute the old idea that the rainbow is due to reflection of the sun's rays by the surface of a cloud (as from a concave or convex mirror) was ***Robert Grosseteste*** *(ca* 1217 *to 1235 CE) in his book De Iride Seu de Iride et Speculo. He hinted vaguely to the role of refraction in the formation*

[139] Here one sees a sharp difference in approach of the two outstanding scientists of ancient times. **Aristotle** gave answers — often times rough-and-ready, occasionally more sophisticated — to *all* questions that turned up; and hence many of his answers have not stood the test of time. **Archimedes** concentrated his attention upon a few aspects of mechanics and optics, and his treatises are as impeccable today as when they were written.

of the rainbow but gave no specific explanation and made no attempt at quantitative treatment.

Albertus Magnus *(ca 1260 CE) reiterated the part played by the individual drops, and in that sense he was the initiator of the microscopic doctrine. In his* Opus Majus *(1266–1267 CE),* ***Roger Bacon*** *followed the lead of Grosseteste and Albertus and stated that the rainbow must be produced by many reflections in numberless drops of water. Nevertheless, he utterly failed to clear up the problem of the rainbow, and the seed planted by Grosseteste sprouted elsewhere, in far-away Poland:* ***Witelo*** *(b. 1230 CE) was brought up in the neighborhood of Cracow, but he had been educated at Paris, as well as at Padua and Viterbo, and hence may have been acquainted with the work of Grosseteste. He was however mostly influenced by* ***Alhazen****'s Treasury of Optics. He wrote, sometimes between* 1270 *and* 1278, *a treatise on Optics.*

In his theory of the rainbow, some rays were reflected directly from the convex surfaces of drops, others were refracted through drops *before being reflected at the other surfaces of other drops lying further within the medium. Refraction served primarily to condense the light; the drops served as spherical lenses, to enhance the lights impression upon the eye. He mistakenly believed that the reflections, as well as refractions, participated in the formation of the colors.*

Witelo also furnished tables of refraction from water (or glass) to air. In so doing he used some of Ptolemy's values from the reciprocal law, *(i.e. independence of the refracted ray on the sense in which the path is traversed). Witelo tried, unsuccessfully, to find general mathematical relations between angles of incidence and refraction, but on the other hand he anticipated Newton's discovery of dispersion, believing that the refraction of different rays through different angles produced the various colors. He did* not *succeed, however, to render an overall picture of the rainbow.*

The next significant advance in the theory of the rainbow was made by ***Dietrich of Freiberg*** *in his book* De Iride Radialibus Impressionibus *(1304–1310). Possessed of experimental skill and persistence as well as theoretical imagination, and deeply versed in the optical learning available at the time, he was admirably equipped to exploit to the full the accumulated wisdom of the rainbow, and draw from it correct and clear physical conclusions.*

Consequently, his explanation is an essentially correct *(though incomplete)* description of the mechanism producing the rainbow, *and vastly superior to that of any one of the eminent scholars before him who had sought unsuccessfully to explain the bow. The merits of his contribution are summarized as follows:*

- *Provided for the first time a clear-cut and unambiguous qualitative theory of the formation of the primary and secondary bows in terms of total reflections and refractions in a raindrop.*
- *Discovered that each drop is responsible but for one color in the bow.*

His theory was nevertheless wrong, because he did not discard the Aristotelian macroscopic circle of altitude, and in his microscopic raindrop model he did not use the essential geometric angle between the incident and emergent rays (deviation). Consequently he failed to account for the radius of the rainbow and the tertiary bow. In linking the orthodox macroscopic geometric explanation to a new microscopic consideration of the geometry of the raindrop, he proposed the only quasi-quantitative theory of the rainbow to appear in the long interval from Aristotle to Copernicus. His work, with all its faults, represents one of the greatest scientific triumphs of the Middle Ages.

Similar ideas were presented simultaneously and independently by the Persian scholar ***Kamal al-Din al-Farisi***[140] *between the years* 1302 *and* 1311. *This amazing case of simultaneous discovery can be understood as due to the common intellectual heritage available to them.*

The first clear-cut break with the Aristotelian tradition is due to the Sicilian mathematician **Francisciscus Maurolycus** *(1494–1575) of Messina, Abbot of Castronuovo. In his book Diaphaneon (written* 1553–1567*; published 1611) he abandoned the meteorological sphere and focused attention for the first time on the basic question to which Aristotelian writers had given only fleeting consideration: How can one account for the apparent size of the rainbow? Why is the angle between the incident and reflected ray close to* $45\,^\circ$*? Of course, Dietrich knew from experience that there was a particular path through the drop designated by nature as appropriate for the production of the primary bow, and he successfully traced this path even though he could not explain it in terms of number or measure.*

Maurolycus seems to have felt that the geometrical basis was discoverable without recourse to experimental observation. However, his suggested scheme through which rays are sufficiently reinforced to reach the eye was physically impossible, inasmuch as it was based on reflection without refraction[141].

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[140] Kamal says that he was greatly assisted by his teacher **Qutb al-Din al-Shirazi** (1236–1311), a distinguished Persian scientist. Hence the discovery of the theory presumably belongs to al-Shirazi, its elaboration to al-Farisi. Both Dietrich and al-Shirazi derived their inspiration from the *Meteorologica* of **Aristotle** and *Kitab al-Manazir* (Treasury of Optics) of **Alhazen**.

[141] In another book, *Theoremata de Lumine Umbra* (1521), Maurolycus investigated the optical problems connected with the passage of rays of light through

***Kepler** came close to solving the problem of the rainbow (1608) through the study of refraction in a spherical globe of water. He recognized the fact that colors arise only at places where the refraction is maximum, but lacking the mathematical expression for the law of refraction, he could not make the final step and became discouraged. Yet to Kepler one owes the clear recognition that "to measure is to know"; and to him physics is indebted for the earliest* quantitative *theory of the rainbow based upon refraction in raindrops. Had he but measured more accurately, he might have anticipated the theory that Descartes gave seven years after Kepler's depth.*

*In 1611, **Marco Antonio de Dominis**[142] (1560–1624, Italy), theologician, natural philosopher and mathematician, issued the publication De Radiis Visus et Lucis in Vitris Perspectivis et Iride Tractatus. His explanation of the rainbow, with all its faults, is superior to any other published in the interval of three centuries from* 1311 *to* 1611.

His theory was not derived from any one source, but was rather a mosaic of notions borrowed from the philosophical and optical traditions, *verified or modified perhaps by direct experimental evidence. Dietrich's work was clearly of a higher order in precision and correctness of thought as far as what takes place within the raindrop; but at least Dominis correctly followed Maurolycus in abandoning the old incubus, the Aristotelian meteorological sphere.*

Nevertheless, in several respects Dominis' views are quite inferior to the unpublished opinions of Kepler. In the first place, Kepler's explanation was consistent with the elementary principles of geometrical optics, for he recognized the inevitability of the second refraction. Then, too, Dominis made no

small apertures with and without lenses (*Camera Obscura*). He applied it to solar observations in a darkened room (1535).

[142] Born of a noble Venetian family in the island of Arbe, off the coast of Dalmatia. He was educated by the Jesuits in their colleges at Loreto and Padua. For some time he was employed as professor of mathematics at Padua, and professor of philosophy at Brescia. He was appointed bishop of Segnia (1596), archbishop of Spalato (1598), and primate of Croatia and Dalmatia (1600). His endeavors to reform the church involved him in a quarrel between the papacy and Venice, and made his position intolerable. He crossed to England (1616), where he became convert to Anglicanism and dean of Windsor (1619). He attacked the papacy in a number of publications (1616 to 1619). He was enticed back to Rome by the promise of pardon and the prospect of a cardinal's hat, only to be doomed to bitter disappointment. Upon his return (1623) he was thrown in prison and died soon thereafter in a dungeon of the Inquisition in St. Angelo. Later the Inquisition tried him posthumously and found him guilty. His corpse was exhumed, dragged through the streets of Rome and publicly burnt in the Campo di Fiore.

*attempt to account for the size of the bow, a problem which Kepler essayed, albeit unsuccessfully. Yet, the explanation of Kepler has been universally overlooked and in many an authoritative treatises on physics one can read that "the elementary theory of the rainbow was first given by de Dominis". (**Newton**, **Leibniz**, **Goethe** and others virtually accused Descartes of plagiarism from de Dominis!)*

The abortive efforts to solve the problem of the rainbow came to an end 326 *years after the first scientific theory was propounded by Dietrich. The man who reaped what others have sowed over more than three centuries was non other than **René Descartes**.*

In the third appendix to the Discours de la Méthode*, one to which he gave the title* Les Météores*, Descartes solved for the first time the fundamental problem of the size of the* rainbow*. Following many experiments and calculations he concluded (1637): "I took my pen and made an accurate calculation of the paths of the rays which fall on the different points of a globe of water to determine at what angles, after two refractions and one or two reflections they will come to the eye, and then I found that after one reflection and two refractions there are many more rays which can be seen at an angle of from* 41 *to* 42 *degrees than at any smaller angle; and that there are none which can be seen at a larger angle. I found also that, after two reflections and two refractions there are many more rays which come to the eye at an angle from* 51 *to* 52 *degrees than at any larger angle, and none which come at a smaller angle".*

Thus Descartes gave the 14^{th} *century theory true scientific status by showing the quantitative agreement of theoretical calculations with the results of observation. He discovered the key to the rainbow problem — the reason for the clustering of rays about the angle* 42° *in the primary bow. This he achieved through patient observations and laborious calculations (the* calculus *arrived only in 1671). Yet, he had not really answered* all *the problems connected with the rainbow, as future generations were to find out.*

Descartes' work was not exempt from the rule that new ideas do not meet with immediate acceptance; in fact, it was not integrated into scientific thought for several decades, and consequently, the Aristotelian theory of the rainbow had not suddenly been overthrown. Philosophical disagreement was not the only impediment to the spread of the Cartesian explanation. In 1637 *there were no scientific periodicals, and news traveled slowly. Thus, his work, even though published again in Latin (1656), was slow to achieve the recognition it deserved.*

*The next character in the rainbow drama is **Huygens**, who played a role as a transition figure between the age of Descartes and that of Newton. He held Descartes' explanation of the rainbow in high regard. The chief contribution*

of Huygens to the theory of the rainbow was indirect, and its influence was not felt until well over a century later. In the Cartesian geometrical theory it matters little what light is, or how it is transmitted, so long as propagation is rectilinear and the laws of reflection and refraction are satisfied. But rainbow developments of the 19^{th} *century were to hinge closely on the nature of light, and here Huygens introduced a major change (1679) — the wave theory of light and a new derivation of the law of refraction by means of the "Huygens Principle".*

This led him to conclude that light travels faster in air than in water, contrary to the conclusions of ***Descartes*** *and* ***Hooke****. But Huygens was unable to verify this inference, nor was he able to make use of his principle to explain the colors of the rainbow. The reason for this lay in the fact that he disregarded the oscillatory and dispersive characteristic of waves. Huygens never really accepted the challenge which the problem of the colors presented and felt that, except for the question of color formation, the work of Descartes was definitive. He probably never dreamed that his theory of light some day would revolutionize the explanation of the rainbow.*

It is of interest to note that whereas Descartes had laboriously calculated the paths of innumerable rays, one by one, Huygens expressed the deviation of the emergent ray as a function of the angle of incidence and then calculated, by the method of ***Fermat****, the values for which this deviation is a maximum or a minimum (a procedure equivalent to the use of the calculus).*

Huygens may have been the original inspiration for a little-known treatise on the rainbow Stelkonstige Reeckening Van Den Reegenboog, composed by ***Baruch Spinoza*** *and published posthumously (1687), the year of Newton's Principia. In this manuscript the author combined the use of a variant of the method of Fermat and Cartesian analytic geometry to arrive at the radii* $40°57'$ *and* $54°25'$ *for the two bows.*

Then came ***Newton****. For thousands of years men had looked at colored spectra produced by light passing through spheres and prisms of water and glass; but Newton looked at the spectrum more carefully than had any one of his predecessors. He saw that rays of differing color were refracted by differing amounts. Ever since antiquity it had been realized that the amount by which light was refracted depended on the angle of incidence, as well as upon the media in question; but Newton first showed (1666 to 1672) that it depends also on the color of the light involved, each color having its own characteristic index of refraction. Thus, for the first time color was reduced to an orderly quantitative basis, and, also, for the first time, an adequate explanation was possible for the width of the rainbow.*

Traditionally (since Aristotle) it had been understood that white light was pure and homogeneous, and that color, such as that of the rainbow, was

the result of a loss in strength or purity. Newton's experiments indicated, however, that the reverse is true — only colored light is pure and homogeneous, and it is not a result of weakening. Newton realized that such a drastic departure from previously accepted views would not be accepted by his contemporaries without strong supporting evidence. Consequently, he went out of his way to get an audience for his ideas. In 1672 *he presented to the Royal Society a paper describing his discovery of the composite nature of white light. But the reception accorded Newton's great discovery was a great disillusionment to the young author. Half a dozen scientists, including Hooke and Huygens,* criticized his work. *From that time on Newton was most prudent indeed. He withheld from publication anything further on optics until* 1704, *the year after Hooke, his sharpest critic, had died. Meanwhile his ideas went pretty much unnoticed, with credit sometimes ascribed to others who did similar work [e.g.* ***Edme Mariotte*** *(1679)]. Newton made two other contributions to the theory of the rainbow; he was first to render calculations concerning rainbows of order higher than two (1669–1671, published 1704), and he derived for the first time an explicit formula from which the radii of bows of all orders (and for any index of refraction) can be deduced*[143].

The theory of the rainbow during the Newtonian age had reached the point where no one untrained in advanced mathematics could hope to follow it. In the field of poetry the change in attitude toward the rainbow was variously received. In England, some times later, on December 28, 1817, *in a dinner gathering,* ***Charles Lamb*** *(1775–1834) and* ***John Keats*** *(1795–1821) agreed that Newton had destroyed all the poetry of the rainbow by reducing it to its prismatic colors, and all the guests drank a toast: "Newton's health, and confusion to mathematics". Not long afterwards Keats composed the familiar lines of Lamia:*

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[143] [$\cos i_c = \sqrt{\frac{\mu^2-1}{(n+1)^2-1}}$; $\mu =$ water index of refraction; $n = 1$ for the primary rainbow, etc.; $i_c =$ critical angle of incidence which makes the deviation D extremal;

$$D(i_c) = \pi n + 2i_c - 2(n+1)\sin^{-1}\left\{\frac{1}{\mu}\sqrt{\frac{(n+1)^2-\mu^2}{(n+1)^2-1}}\right\}].$$

> *"Do not all charms fly*
> *At the mere touch of cold philosophy?*
> *There was an aweful rainbow once in heaven:*
> *We know her woof, her texture; she is given*
> *In the dull catalogue of common things.*
> *Philosophy will clip an Angel's wings,*
> *Conquer all mysteries by rule and line,*
> *Empty the haunted air, and gnomed mine –*
> *Unweave a rainbow".*

For 99 *years after the* Opticks *appeared (1704) there was nothing of comparable significance in the story of the rainbow. It was generally assumed that the last word has been written; the theory appeared to be in such satisfactory shape that little refinement seemed to be necessary.*

The first substantial studies in the physiology of color, as well as the first credible explanation of the supernumerary rainbows, *came in the work of* ***Thomas Young*** *(1803). His discovery of optical interference unlocked one of natures best-kept secrets — the cause of supernumerary rainbows: Young saw that for each angle of incidence upon a raindrop greater than that of the Cartesian effective ray, i_c, there is another of smaller angle such that the two rays emerge from the drop in parallel, or nearly parallel, paths.*

It can be shown that these two rays are reflected at the same point on the rear surface of the drop. These two rays, being deviated more than $D(i_c)$, will appear inside *the primary bow (for the secondary rainbow they appear* outside *the bow). It is clear that the two rays, on traversing the drop, will have followed paths which are not quite equal in distance, and so they will arrive at the eye's retina with a certain phase-difference and interfere. If the difference in the lengths of the paths is an integral multiple of the wavelength of a given color, the rays will be reinforced; if it is an odd multiple of half a wavelength, the rays will extinguish each other. Several positions are expected where reinforcement takes place, and also other intermediate positions where the rays annihilate each other — the familiar phenomenon of* Newton rings, *namely, the formation of a whole series of bows.*

Thus, there are potentially infinitely many bands of each hue in the primary bow, *the bands becoming fainter and narrower as the radii diminish. The spacing of the bands depends on the variations in the length of the path, and these are determined by the size of the drops. Ordinarily there is considerable overlapping in the bands of various colors, especially when the drops are not of uniform size; and this accounts for the fact that most people see only a single primary rainbow.*

If the drops are unusually minute (as in a fine mist), the interference bands may become so intermingled that the result is a superposition of all colors, that is, a white bow.

For large drops of rain, only one brightly colored primary rainbow is usually seen; but Young found that supernumerary bows are clearly visible when the raindrops are uniformly sufficiently small, and noted that there was a regularity in their spacing which corresponds to that of Newton's rings. Young actually advanced this phenomenon as an argument supporting his doctrine of interference[144].

Young's theory, however, was unable to explain 18^{th} *century observations that the radius of the bow is not constant, but rather varies considerably; and the explanation had to wait for another* 35 *years. In the meantime, Young's work did not receive the recognition it deserved, partly due to the discovery of the phenomenon of polarization of light [**Malus** (1808); **Biot** (1812); **Brewster** (1815)].*

Indeed, the observation that light from the two rainbow arcs is almost entirely polarized in the planes which pass through the eye and the radii of the arcs, could not be fitted into Young's interference theory, since the latter was based (prior to 1816) on the wrong concept of light as a sound-like longitudinal motion. But in 1816 *both* **Fresnel** *and* **Young***, independently, finally saw that polarization made it necessary to abandon this preconception to conceive instead of light as a transverse vibration, in which the displacements take place at right angles to the direction of propagation. They assumed that light is a bundle of transverse waves in planes variously oriented, and that in reflection and refraction some of the planes of vibration are screened out to leave a beam which is wholly or partially polarized. This idea saved the wave theory from the incubus presented by the non-interference of polarized light, for transverse vibrations in different planes could scarcely be expected to affect each other.*

Since transverse vibrations were regarded as incompatible with the fluid state, Fresnel was forced to assume that the luminiferous ether behaves like an elastic solid; with an elasticity greater than that of steel. But scientists found it difficult to believe that the heavenly bodies are moving resistlessly through such a solid, and it was not until 1838 *that the Newtonian emission theory gave way to the wave theory of Young and Fresnel.*

Neither Young nor Fresnel gave the adequate mathematical exposition which was needed for the formation of the rainbow. The definitive explanation of the rainbow was to a large extent the work of three Cambridge men, not

[144] The interference theory of the rainbow made clear why the bow is brighter near the earth and why the supernumerary arcs seem to appear near the highest part of the bow; raindrops tend to increase in size as they fall, and the results of Young showed that where the drops are uniformly larger, there will the bow be brighter, but unaccompanied by supernumerary arcs.

one of whom was primarily a mathematician: ***George Biddell Airy*** *(1801–1892) was an astronomer,* ***William Hallowes Miller*** *(1801–1880) was a mineralogist, and* ***Richard Potter*** *(1799–1886) was a chemist and physicist with medical training. Their theory was further refined by other Cambridge scholars and professors, notably* ***Stokes****,* ***Larmor*** *and* ***Rayleigh****.*

The final major assault on the rainbow problem was started by ***Potter*** *(1835): He integrated all the former concepts of* ***Descartes*** *(limiting ray),* ***Huygens*** *(wave front),* ***Newton*** *(dispersion), and* ***Young*** *(interference) into a single mathematical theory. To this he added a central idea which escaped the notice of his predecessors — the caustic[145] wave front formed in the raindrop: following the refraction at the concave surface of the drop, the wave-front is no longer rectilinear, but curvilinear. In fact, some of the rays intersect others even before they strike the rear surface. Descartes had traced the path through the drop of one ray at a time, and so he failed to call attention to this intersection. These rays, after the first refraction, form a caustic by refraction.*

Potter then found that the orthogonal trajectory of the rays reflected from the rear concave surface of the drop is an s-shaped curve, with an equation approximately of the form $y = kx^3$. *Finally, the wave front which emerges from the drop after the second refraction consists of two convex portions, mutually tangent but with unequal radii of curvature, which form a cusp at a point slightly below the Cartesian limiting ray. The Cartesian rainbow band can therefore be thought of as a caustic, and the rays of each color have their own caustic, each one corresponding to a colored band in the rainbow as explained by Newton in different terms.*

Potter called attention to the fact that close to the caustic, the nearly parallel rays will exhibit the interference phenomenon of which Young had pointed out, creating the Newton-rings pattern. Consequently, the intensity of illumination does not fall of monotonically as one departs from the effective ray, as Descartes believed; the decline in intensity is oscillatory.

The precise analytical expression for the intensity of illumination at each and every point of the region brightened by the bow (as a function of the angular deviation of the ray from the least-deviated ray) was given by ***Airy***

[145] *Caustic surface* — the envelope of a family of reflected or refracted rays. An example of a *caustic curve*, which is a plane section of a caustic surface, can easily be seen by noting the bright arcs formed by *reflected* light rays on the bottom of a teacup. If the equations of the tangent lines (rays) forming the caustic are known, the equations of the orthogonal trajectories (wave-fronts) and envelopes (caustics) can be found by the methods of advanced calculus. Caustics can be formed either by reflection or by *refraction*, usually by *intersecting rays*.

(1836) in his diffraction theory of the rainbow. He found that the intensity of light is given by the square of an integral which since has come to be known as "Airy's rainbow integral". His calculations showed that the region of greatest brightness (as viewed by any particular observer) lie appreciably within the radius computed on the basis of geometrical theory.

Airy also showed that the radius of the primary bow (and not only the colors and spacing of the arcs) varies with the size of the drops. Moreover, whereas Descartes, Young and Potter maintained that there should be no light whatever returned to the eye at an angle greater than that of the least deviated ray, Airy's calculations show that this assumption is erroneous[146] *and that* diffraction *must be taken into account in any complete theory of the rainbow.*

Miller *(1841) extended Airy's analysis to include the secondary bow and performed experiments which verified the results of Airy.* ***Stokes*** *(1850) derived a more expeditious device for calculating the values of Airy's rainbow integrals, and calculated intensities of illumination sufficient to place the first fifty maxima.*

The theory of Potter and Airy ignored the finite size of the sun's disc; throughout the computations of the rainbow integral the light was assumed to come from a point source. ***Keiichi Aichi*** *(b. 1880, Japan) and* ***Aikischi Tanakadate*** *(b. 1856) extended the Airy theory for a circular source of light (1904). They had been struck by the fact that, according to the theory of Airy, one should anticipate numerous supernumerary arcs, whereas in nature the bow generally is accompanied by only a very limited number. They suspected, from some approximations, that this discrepancy might be accounted for by fact that the sun is not a point source of light. Their elaborate analysis showed that, for a* finite source, *the supernumerary arcs of the natural rainbow lose most of their color, especially for large drops. They demonstrated that the degree of luminous intensity depends on the breadth of the source, as well as on the size of the drops.*

Since 1945 *rainbows have been "used" for the first time as a means of calculating how large the drops of water in a cloud are. The results were used in aircraft icing investigations, where the free-water content and the size of drops becomes a matter of immediate concern; a camera "rainbow recorder" was used, both in natural clouds and in experimentally-controlled*

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[146] Diffraction occurs for negative values of the Airy parameter $z \approx 2\theta \left\{\frac{a^2}{h\lambda^2}\right\}^{1/3}$. Airy also showed that, as far as the rainbow is concerned, there is no need to integrate over the cusped wave-front that emerges from the drop, but over the *simpler s-shaped caustic-curve* that results from the reflection at the back of the drop.

fog chambers, to find the difference in viewing angle between the principal bow and the first supernumerary arc: the rainbow-calculated drop diameters differed by as little as 2 to 5 percent from those computed by other means. Airy's theory proved, however, inadequate in the range of drop sizes from 10 to 15 microns, and a new approximation for the equation of the generating caustic wave-front was derived.

The story of the rainbow had passed from Iris to Mathesis through a mythological *state, a* reflection *stage, a* refraction *state, a* geometrical *stage, a* dispersion *state, an* interference *stage, and a* diffraction *stage. But although much is known about the production of the rainbow, little has been learned about its perception; our knowledge of what goes on between the eye and the brain when one sees a rainbow is pretty much in a state of flux. Which part of what we see is due to physical factors, and which is due to purely entoptic reasons — is still unknown. As long as man by nature desires to know and yearn for beauty, just so long will Iris continue to inspire both exact science and romantic literature. For poets the rainbow had served as a ubiquitous source of inspiration, but mathematics has also given the bow a beauty which only the deeply initiated can fully appreciate.*

1836–1855 CE **Nicholas Joseph Callan** (1799–1864, Ireland). Priest, scientist and inventor. Created the first *induction-coil* (1836), which led to the modern transformer [ahead of **Ruhmkorff** (1851)].

Callan was influenced by the work of **William Sturgeon**, who invented (1825) the first electromagnet and by the discoveries of **Michael Faraday** (1831) and **Joseph Henry** (1832) concerning electromagnetic induction. Working from 1834 on, Callan employed a horseshoe-shaped iron-bar and wound it with thin insulated wire (primary coil) and then wound thick insulated wire over the winding of the thinner wire (secondary coil). He discovered that, when a DC current (sent by a battery) through the primary coil was interrupted, a high voltage was produced in the *open* secondary coil. In doing so he constructed what is today known as *autotransformer*. Callan's induction-coil also used a "breaker", consisting of a rocking wire that repeatedly dipped into a small cup of mercury. A clock mechanism was used to interrupt the current in the primary coil 20 times a second. It generated a 40-cm spark in the open secondary coil over an open-circuit voltage of some $600,000$ Volt.

Like Cavendish before him, Callan made an independent discovery of Ohm's law (Ohm, 1827). In applied science he discovered several types of

galvanic battery and influenced the study of high-voltage electricity. He also constructed one of the first DC electric motors.

In 1838, Callan stumbled on the principle of the self-excited dynamo; moving his electromagnet in the earth's magnetic field, he found he could produce electricity without a battery. In his words, he found that "by moving with the hand some of the electromagnets, sparks are obtained from the wires coiled around them, even when the engine is in no way connected to the voltaic battery". The effect was feeble so he never pursued it, and the discovery is generally credited to **Werner Siemens** (1866).

Callan was born at Darver, Ireland. After ordination as priest (1823) he went to Rome where he obtained a doctorate in divinity (1826) at the Sapienza University. While at Rome he became acquainted with the experiments of Galvani and Volta. In 1826 he was appointed to the chair of Natural Philosophy in Maynooth University (near Dublin) and remained in that post until his death.

Unfortunately, his name was forgotten and his inventions were attributed to other scientists: Maynooth was a theological university where science was marginal in the curricula. Callan's colleagues often told him that he was wasting his time. In such an atmosphere, Callan's pioneering work was soon forgotten after his death, and **Ruhmkorff** (who like all instrument makers, put his name on every instrument he made) got into the textbooks and thus received the pioneering credit for the induction-coil. It was never challenged until Callan's publications were rediscovered in 1936 and first put into physics textbooks in 1953.

1836–1858 CE **Robert Remak** (1815–1865, Germany). Neurologist and biologist. Made important discoveries in nerve and muscle diseases (1859). Developed new cell theory for animals, emphasizing protoplasm as cell substance and that cells are formed by division of existing cells. Showed (1845) that there are only three layers present in the early development of the embryo which he named: ectoderm, mesoderm and endoderm.[147]

Remak was born in Posen (Poznan), the oldest of the five children of Salomon Meyer Remak, who ran a tobacco shop and lottery office, and Friedrike

[147] In this he revised earlier theory of **Christian Pander** (1820) and **Karl von Baer** (1826) who first maintained that an embryo has heterogeneous structural layers, called *germ layers*, which always give rise to the same physiologically differentiated adult tissues. Remak emphasized that the formation of the germ layers occurs by *cell division*.

Caro[148]. The family were Orthodox Jews and in 1815 Poznan had returned from Polish to Prussian sovereignty by the Congress of Vienna. Remak received his earliest education at home and enrolled at the University of Berlin (1833) to study medicine. His pioneering studies on the nerve tissue (1836) gained him the M.D. (1838).

Although Remak wished to make a career in teaching, the way was barred to him, since in Prussia at that time Jews were not admitted to that profession. He therefore continued his laboratory research, and in 1839 discovered ganglion cells in the human heart. This finding seemed to him to explain the relatively autonomous action of the heartbeat. During 1843–1856, Remak applied many times for a teaching position, but in spite of his growing fame and the intervention of **Alexander von Humboldt** and other eminent friends on his behalf, his repeating requests were refused. Finally (1859) he was appointed assistant professor at the University of Berlin, but this belated and meager recognition had no effect upon his subsequent career.

His son **Ernst Julius** became a professor of medicine at the Berlin University (1902).

His grandson **Robert** became an important researcher in number theory. The name REMAK is an acronym for **Rabbi Moshe Kordovero**[149][1512–1570, Safed, Israel], and the Remak family probably stemmed from the same Spanish-Italian ancestry.

1837–1838 CE Based on the discoveries of **Oersted** (electromagnetism, 1820), **Sturgeon** (electromagnet, 1825) and **John Daniell** (steady current cell, 1836), three men developed successful *wire-telegraphy*: in England, working together, **William Fothergill Cooke** (1806–1889) and **Charles Wheatstone** (1802–1875), and in the U.S.A. the painter **Samuel Finley Breese Morse** (1791–1872). The *Morse code*, patented by Morse in 1840, uses patterns of dots and dashes to represent letters, numerals, punctuation and other signs.

1837–1844 CE **Samuel Finley Breese Morse** (1791–1872, USA). Portrait painter and inventor. Developed the first successful electric

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148 Kisch, B., "Forgotten Leaders in Modern Medicine", *Trans. Amer. Phil. Soc. 44*, 227–296, 1954; Pagel, J., *Allgemeine deutsche Bibliographie 28*, 191–192, Leipzig, 1889

149 Provided the first complete and systematic theory of the *Kabbalah*.

telegraph[150] in the United States and invented the *Morse Code*, still used occasionally to send telegrams.

Morse was born in Charlestown MA, the eldest child of the Reverand Jedidiah Morse and his wife, Elizabeth Ann Breese. He graduated from Yale College (1810), went to London (1811) and studied two years at the Royal Academy of Arts. He returned home in 1815 and within the next ten years became a well-known portrait painter. His interest in telegraphy began in 1832 and after working at it for five years he demonstrated his equipment in 1837. His symbolic alphabet, known as the Morse code was invented in 1840. A line was constructed between Baltimore and Washington and the first message, sent in May 24, 1844, was "What hath God wrought!". Morse and his telegraph were known within 12 years throughout North America and Europe. In 1861 the United States were linked by telegraph from coast to coast. Electromagnetic waves were not yet discovered at that time. Morse was not the first to invent the telegraph, but he is known as the "father" of the telegraph because he *created a new industry.*

1837–1853 CE **Heinrich Gustav Magnus** (1802–1870, Germany). Physicist and chemist. Investigated the motion of spinning spherically or cylindrically-shaped solids in a fluid (liquid or gas) and discovered an effect named after him ("*Magnus effect*")[151]. It is responsible for the "*curve*" of a served tennis ball or a driven golf ball, and affects the path of a spinning artillery shell. Analyzed (1837) gases in the blood and showed that a higher

[150] Independently, **William O'Shaughnessy** set up a 22 km demonstrator telegraph system in India, near Calcutta (1839) and later (1854) completed a 1300 km telegraph line in India, between Calcutta and Agra.

[151] In ideal-fluid aerodynamics (neglecting friction), the force exerted by the fluid on a finite rigid body moving with a constant velocity through it is zero, if the fluid closes behind the body (*d'Alembert's paradox*). The result implies that the so-called *drag-force* on the body due to fluid resistance is zero. [It also predicts a *zero lift force* for lifting bodies such as wings of an airplane!]
If however, a circulatory flow is superposed, such as occurs when the body is spinning, *Bernoulli's theorem* predicts the existence of a force that tends to divert the body from its straight trajectory. In the case of a cylinder (e.g. spinning artillery shell) with a clockwise rotation and moving to the right, the fluid velocity above the cylinder increases, whereas the velocity below it decreases. Consequently there is a low pressure ("*suction*") above it and a high pressure below it. The result is a lift on the cylinder [Rayleigh, 1876]. The lift on an airplane wing does not require a rotating body; the shape of the wing creates a velocity distribution with circulation, but no vortices. The lift is then caused by a Bernoullian pressure-gradient.

concentration of oxygen exists in the blood flowing in arteries than in that flowing in veins. This suggests that respiration takes place in the tissues[152].

Magnus was born in Berlin to parents of Jewish origin. He studied for a while under **Gay-Lussac** in Paris. In 1831 he returned to Berlin as a lecturer on technology and physics at the university, and in 1845 he became a full professor there.

1837–1859 CE **Gabriel Lamé** (1795–1870, France). Engineer and mathematician. Invented curvilinear coordinates. Made the following forecast of the scientific significance of coordinate systems:

"*Should anyone find it singular that we have been able to found a Course of Mathematics on the sole concept of a system of coordinates, he may be reminded that it is precisely these systems which characterize the phases and stages of science. Without the invention of rectangular coordinates, algebra might still be where Diophantos and his commentators left it, and we should lack both the infinitesimal calculus and analytic mechanics. Without the introduction of spherical coordinates, celestial mechanics would be absolutely impossible; and without elliptic coordinates, illustrious mathematicians would have been unable to solve several important problems of this theory... Subsequently the reign of general curvilinear coordinates supervened, and these alone are capable of attacking the new problems* [*of mathematical physics*] *in all their generality. Yes, this definitive epoch will arrive, but tardily*: *those who first recognized these new implements will have ceased to exist and will be completely forgotten — unless some archaeological mathematician revives their names. Well, what of it, provided science has advanced?*"

Lamé insistence on the importance of coordinates has been justified in modern physics. His early work (1839) on the conduction of heat in ellipsoids led him to discover the functions which bear his name.

Lamé's investigation in curvilinear coordinates led him into the field of number theory. In 1840 he was able to prove Fermat's Last Theorem for the case $n = 7$. In 1847 he developed a solution, in complex numbers, of the form $A^5 + B^5 + C^5 = 0$ and in 1851 a complete solution, in complex numbers, of the form $A^n + B^n + C^n = 0$.

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[152] This was later confirmed by the physiologist **Eduard Friedrich Wilhelm Pflüger** (1829–1910, Germany), who showed that the essential *chemical* changes of respiration occur in the tissues and cells rather than in the lungs. Finally **John Scott Haldane** (1860–1936, England) and **Joseph Bacroft** (1872–1947, England) elucidated the fine physical mechanism of respiration.

He also proved the following theorem: The number of divisions required to find the greatest common divisor of two numbers is never greater than 5 times the number of digits in the smaller of the numbers.

Lamé was born in Tours. He attended the École Polytechnique during 1813–1817. He then continued at the École des Mines from which he graduated in 1820. In the same year he accompanied **E. Clapeyron** to Russia. He was appointed director of the School of Highways and Transportation in St. Petersburg, where he taught the exact sciences. He was also busy planning roads, highways, and bridges that were built around that city.

In 1832 he returned to Paris and accepted the chair of physics at the École Polytechnique. In 1836 he was appointed chief engineer of mines. He also helped plan to build the first two railroads from Paris to Versailles and to St. Germain. In 1851 he became professor of physics and mathematics at the University of Paris.

It is difficult to characterize Lamé and his work. Gauss considered him the foremost French mathematician of his generation. French mathematicians, however, considered him too practical, while French scientists viewed him as too theoretical. Yet the work he began was generalized almost as soon as it appeared by such mathematicians as **Klein** and **Hermite**.

1837–1861 CE *Origins of the telephone*. In 1837, **C.G. Page** of Salem, Massachusetts, drew attention to the sound given of by an electromagnet at an instant when the electric current is closed or broken. He later discussed the musical note produced by rapidly revolving the armature of an electromagnet in front of the poles. In 1854, **Charles Bourseul** (Paris) recommended the use of a flexible plate which would vibrate in response to the varying pressure of the air, and thus open or close an electric circuit. A similar plate at the receiving station would be acted on electromagnetically, and thus produce as many pulsations as there are breaks in the current.

In 1861, **Johann Philipp Reis** (1834–1874, Germany) succeeded in transmitting speech and music electrically down a wire using a device he called 'das Telephon' in a lecture delivered before the physical society of Frankfurt. He described an apparatus in which he caused a membrane to open and close an electric circuit at each undulation, thus transmitting as many electric pulses through the circuit as there were periodic amplitude vibrations in the sound. These electric pulses were made to act on an electromagnet at the receiving station, which gave out a sound corresponding to the number of times it was magnetized or demagnetized per second. Reis could not, however, reproduce human speech with sufficient clarity. The suggestion of Bourseul and the experiments of Reis are founded on the idea that a succession of currents, corresponding in number to the successive undulations of the pressure on the

membrane of the transmitter, could reproduce at the receiving station sounds of the same character as those produced at the sending station. The professors to whom this invention was presented were not very impressed and his version of the "telephone" never received any financial support and no patent ensued.

Reis was born in the town of Gelnhausen, in Hesse-Cassel, Germany, where his father was a master baker and a petty farmer. Orphaned at an early age, he interrupted his high school education to become merchant but in 1855 he became a schoolteacher of mathematics and science.

1838 CE **Antoine-Augustin Cournot** (1801–1877, France). Economist and mathematician. Attempted to apply mathematics to solution of economic problems; pioneer in mathematical economics. In *Recherches sur les principles mathematiques de la theorie des richess* (1838) discussed supply and demand functions and introduced the concept of '*elasticity of demand*'. He also considered conditions for equilibrium with monopoly, duopoly and perfect competition. He considered the effect of taxes, treated as changes in production costs, and discussed problems of international trade. Conducted research in the theory of probability.

Cournot was professor at Lyons (1834), rector of academies at Grenoble (1835–1838) and Dijon (1848–1862).

1838 CE **Gerhardus Johannes Mülder** (1802–1880, Holland). Chemist. Coined the name *protein* from the Greek word for "first". He studied the chemical structure of the albuminous substances and concluded that they were built up of a basic building block to which various amounts of modifying structures were added. Mülder's speculation turned out to be not quite right, but the name remained.

1838–1839 CE **Matthias Jacob Schleiden** (1804–1881, Germany). Botanist. Recognized the importance of the cellular element of plants and stated that the *cell*[153] was the basic unit of life: an individual living and reproductive organ. The next year, the physiologist **Theodor Schwann** (1810–1882, Germany) advanced the same idea. Neither of them originated this concept. A number of other scientists had already come to believe that all organisms were made of cells. But from that time on, all biologists regarded the cell as the building block of life.

[153] In 1665, **Robert Hooke** (1635–1702) coined the word *cell* for the infrastructural unit of a piece of cork which he saw through his microscope.

The Cell (1831–1925)

The word cell *owes its existence to* ***Robert Hooke****, who first noticed the cellular structure of cork (1665). What Hooke really saw were dead cell walls in the bark of the cork oak. Other early microscopists soon observed cells in all kind of plants. Animals contained similar units, but these were harder to see because animal cells lack the thick walls that surround plant cells. Observers also reported the existence of many tiny* unicellular *organisms, each consisting of only one cell. Thus,* ***Antoine van Leeuwenhoek*** *observed (1674) bacteria 2 micron long, as well as blood cells and spermatozoa. More than 150 years later, in 1831, the appreciation of the cell as the basic unit of the organism was finally manifested through the works of* ***Robert Brown****, who coined the term 'cell nucleus'.* ***Schleiden*** *and* ***Schwann*** *soon followed (1838–1839) with the first theory on nucleus and cell formation. This theory states that:*

- *cells are the fundamental unit of life – the smallest entities that can be called "living"*
- *all organisms are made up of one or more cells.*

Robert Remak *(1845) was first to demonstrate that cells are formed by division of existing cells.* ***Rudolf Virchow*** *(1855) generalized and popularized Remak's discovery, using the aphorism:* Omnis cellula e cellula *- all cells from cells. He added a third statement:*

- *cells arise only by division of other cells. In other words, cells are the fundamental structural, functional and reproductive units of life.*

Why are there cells? The metabolism of a living organism requires a chemical environment different from any found in the nonliving world. A cell organizes an "environment in miniature" by maintaining strict control of the chemical composition within its boundaries. In the controlled environment of a cell, all the activities of life occur: acquiring energy; using this energy to maintain the chemical environment, to build organic molecules, to grow, and reproduce by division into two new cells.

Many organisms are unicellular, but most plants and fungi, and all animals, are multicellular*, composed of many cells. All cells must carry out certain basic activities; in addition, each cell of a multicellular organism makes a specialized contribution to the economy of the body as a whole. For example, a muscle cell in the heart is specialized to contract and help pump blood.*

Since it is deep within the body, it cannot capture its own food or obtain oxygen from the air, but must rely on other specialized cells, such as those of the digestive tract, lungs, and blood, to provide its food and oxygen.

Thus there is division of labor among the cells of a multicellular organism. Unicellular organisms may also be highly specialized for their own ways of life, with features much more complex than those in the cells of most plants and animals. A specialized cell is usually distinguished by the exaggeration or modification of one or more features common to most types of cells, rather than by possession of structures or chemicals that other calls lack. We can think of specialized cells as variations on the basic theme of cell structure and function.

An idealized "basic cell" has three main parts:

1. *The* plasma membrane, *covering the outside of the cell. (in plants, this lies just inside the nonliving cell wall.)*
2. *The* cytoplasm *(cyto = cell), containing water, various salts, and organic molecules. The cytoplasm also contains a variety of larger structures, collectively called* organelles, *which are the working parts of the cell. Many of these "little organs" are surrounded by membranes very similar to the plasma membrane. The cytoplasm fluid is crisscrossed by a barely visible network of* microtubules *involved in transporting molecules to and from the membrane.*
3. *The* cell nucleus *(in bacteria, the* nuclear area*), containing the cell's genetic material.*

The Plasma Membrane

Molecules and ions are in constant, random motion. Left to themselves, these substances would diffuse down their concentration gradients and eventually become uniformly distributed. However, the chemical composition and the physical environment is usually not completely appropriate for the biochemical reactions of life. Cells require higher concentrations of some substances and lower concentrations of others.

To remain alive, a cell must maintain chemical homeostasis *("same-standing"); that is, it must keep its internal chemical composition constant*

within the narrow limits suitable for life. However, a cell cannot create a suitable internal environment and then seal itself off from the world to avoid gaining or losing substances by diffusion. The biochemical reactions of metabolism require raw materials from outside the cell and generate waste products that must be expelled. Hence the cell must maintain homeostatis while continuously exchanging substances with its environment. Control of what substances enter and leave the cell is the task of the plasma membrane*, also called the* cell membrane *or* plasmalemma*.*

The idea of homeostatis is due to **Claude Bernard** *(1857) who coined the aphorism: "la fixité du milieu interieur c'est la condition de la vie libre" (free life depends on the constancy of the internal environment). Indeed, the plasma membrane plays an important role in homeostatis, the constancy of the internal environment at the cellular level.*

The plasma membrane is selectively permeable[154]*; that is, it permits some substances to pass more freely than others, and even prevents the passage of certain kinds of molecules, to which it is* impermeable*. Hence the concentration of ions, nutrients, and other substances inside the cell differ from those in the cell's environment.*

For biological systems to operate, some parts of organisms must be separated from other parts. On a cellular level, the outside of the cell has to be separated from the inside of the cell. "Greasy" lipid membranes *serve as the barrier. In addition to isolating the contents of the cell, membranes allow the selective transport of ions and organic molecules into the cell.*

All biological membranes consist of lipids[155]*and* proteins*. The actual kinds*

[154] Discovered by **Charles E. Overton** (ca 1895) and established further by the experiments of **I. Langmuir** (1905).

[155] *Lipids* are a class of biochemical compounds composed mainly of fats and oils. They are soluble in fat solvents such as benzene, ether, and chloroform, but insoluble in water. Lipids do not contain a common chemical unit, and their composition as well as their structure varies widely. Most lipids, however, are ethers of fatty acids. Lipids are essential constituents of almost all living cells. The more complex lipids are concentrated in brain and nerve tissues.
Lipids may be classified into the following categories:

I. *Simple lipids* that contain C, H, O, N, and P

(a) Phospholipids (phosphatides)

(1) Lecithins
(2) Cephalins
(3) Sphingosides

and proportions of molecules present depend upon the membrane, its location and function.

The plasma membrane is very thin, only about 7.5 *to* 10 *nanometers (nm) thick* (1 nm $= 10^{-9}$ m $= 10^{-6}$ mm)*. Therefore, it cannot be seen with a regular microscope. However, its existence was deduced from the behavior of cells long before the much more powerful electron microscope showed that it really is there. Electron micrographs show the membrane as a continuous double line surrounding the cell.*

The classic studies of membrane structure used the red cells of mammals (warm-blooded animals with fur or hair, including humans). These cells were chosen because of their simple structure. Mammalian red blood cells lack most of the internal components of other cells; they consist of a single, sac-like plasma membrane containing little other than the red, oxygen-carrying hemoglobin.

An experiment done in 1925, by **E. Gorter** *and* **F. Grendel***, established that plasma membranes contain a double layer of lipid molecules. These researchers broke open blood cells and separated the membranes from the*

(b) Glycolipids

II. *Derived lipids*

(a) Sex hormones

(b) Fat-soluble vitamins A, D, E, and K

Fats are esters of three molecules of fatty acid and one molecule of glycerol. Fats are called *triglycerides* because they are triesters of glycerol and three fatty acids. They are the most abundant as well as the most important class of lipids in the average diet (they comprise 10 percent of the body weight of mammals). Triglycerides are used to store energy, and they provide protection against heat loss and mechanical shock.

Many internal organs, such as the kidneys, are enveloped in a thick layer of fat to protect them from the effects of violent shock.

Among lipids found in biological membranes is *cholesterol* (a derived lipid) and *phospholipids.* The latter are complex esters composed of an alcohol, fatty acids, phosphoric acid, and a nitrogenous base. They are present in every tissue of the body, but especially in the nervous system.

In the energy economy of the cell, glucose reserves are like ready cash, whereas lipid reserves are like a fat savings account. The stored energy of lipids resides in the fatty acid chains of triacylglycerols. When there are excess calories, fatty acids are *synthesized* and stored in fat cells. When energy demands are great, fatty acids are *catabolized* to liberate energy.

hemoglobin. They dissolved the membrane lipids in acetone (the main component of nail polish remover). When the acetone-lipid preparation was mixed with water, the lipids rose to the top and spread over the water surface in a monolayer *only one lipid molecule thick. This lipid monolayer occupied two times the surface area calculated for the plasma membranes of the original red blood cells. Hence the researchers concluded that the plasma membrane's lipid must exist in a double layer, or* bilayer. *We now know that all biological membranes have double layers of lipid molecules. In fact, lipids found in biological membranes may form bilayers spontaneously even when they are removed from the cell.*

Membranes have two main functions: they form a physical barrier around a cell or organelle, and they control the passage of substances into or out of the enclosed area. Some aspects of the membrane permeability have been understood for a long time, but not until the 1970s did it become clear that substances cross biological membranes in only three distinct ways: by dissolving through the lipid layers, by being moved through the lipid layers by way of the membrane proteins, or by moving within a sac formed from part of the membrane. We shall consider each of these in turn.

Some molecules cross biological membranes by virtue of their interactions with the lipid molecules in the membrane. In essence, these substances dissolve in the lipid on one side of the membrane and emerge at the opposite face. These substances move by diffusion, each entering or leaving the cell according to its own concentration gradient. To study these interactions without interference from membrane proteins, researchers work with artificial lipid bilayers. The lipids in these bilayers behave essentially like lipids in intact biological membranes. A lipid bilayer's hydrophobic interior makes it relatively impermeable to ions and to many polar molecules. As a result, the plasma membrane prevents most of the water-soluble cell contents from escaping. However, small uncharged molecules can slip between the hydrophilic heads of the membrane phospholipids and will diffuse across the bilayer. The rate at which such a substance can diffuse through the lipid bilayer depends on its solubility in lipids and its molecular size.

Hence small, nonpolar molecules such as oxygen (O_2)*, nitrogen* (N_2)*, benzene, ethylene, and ether cross membranes rapidly. Uncharged polar molecules also cross the lipid bilayer rapidly if they are small enough. For example, ethanol and urea cross rapidly; glycerol, which is also uncharged but larger, crosses much more slowly, and glucose, twice the size of glycerol, can hardly cross an artificial lipid bilayer at all.*

Because water is relatively insoluble in lipids, it is somewhat surprising that water molecules cross lipid bilayers quite rapidly. This is partly because of the water molecule's small size, but it may also be that the molecules' unique

bipolar structure somehow permits it to pass the membrane's hydrophilic outer layers especially easily.

Artificial lipid bilayers are about 10^9 *(one billion) times more permeable to water molecules than to charged ions, even such small ones as sodium* (Na^+) *and potassium* (K^+)*. The inability of ions to penetrate the plasma membrane is partly due to their electric charge. In addition, ions in solution are* hydrated*, that is, surrounded by a layer of water molecules, which in effect makes them much larger.*

Many (particularly polar) molecules move rapidly across biological membranes even though they cross artificial lipid bilayers very slowly. Examples include various small ions, glucose, and amino acids. These substances are transported by membrane transport proteins *or* carriers*. Each transport protein is specific in that it transport only one or a few chemically similar substances.*

Some transport proteins merely move one type of solute across the membrane. In others, transfer of one solute depends upon the simultaneous transfer of a second kind of solute in the same or the opposite direction. Some proteins move their solutes in only one direction, while others work in both directions. Here we consider some of the more important types of protein transport systems.

In passive transport*, the membrane protein provides a means for an ion or molecule to cross the membrane, moving down its electrochemical gradient. It is well-known that molecules move according to their concentration gradients, going from areas where their concentration is greater to areas where it is less. However, a second factor also influences the movement of ions: the electrical environment.*

In most plasma membranes, transport proteins move ions in such a way that electrical charge is unequally distributed on the two sides of the membrane.

Therefore we say that an electrical potential difference*, or* membrane potential*, exists across the membrane. For instance, if the interior of a particular cell has an electrical potential of* -50 *millivolts (mV) compared with the exterior, its membrane has a membrane potential of* -50 *mV. So the membrane potential has this effect: positively charged ions tend to enter the cell readily, attracted by the excess negative charges there, but negatively charged ions tend to remain outside the cell, attracted by the external positive charge and repelled by the negatively charged interior.*

Now we can see that the diffusion of a given ion across a membrane depends on two factors: (1) the ion's own concentration gradient and (2) the overall electrical potential gradient across the membrane, which is the gradient of the

amalgamated concentrations of all electrically charged species present. Both these gradients together constitute the electrochemical gradient *for the ion, and this determines how rapidly the ion diffuses across the membrane.*

Perhaps the simplest case of passive transport occurs where membrane proteins form channels *through the lipid membrane. These channels contain an aqueous solution and permit small solutes to cross the bilayer by simple diffusion down their electrochemical gradients, avoiding the membrane's hydrophobic interior.*

While some of the channels are open all the time, others, called gated channels*, behave as if they have gates that open and close. Some gated channels open when a specific substance binds to a receptor on the plasma membrane. Others open in response to changes in the membrane potential. Still others open when the concentration of a particular ion inside the cell increases. Many "gates" close again automatically even if the stimulus that caused them to open is still present. Gated channels permit the membrane's permeability to change from time to time. This feature is vital, among other things, to the working of nerves and muscles.*

In facilitated diffusion*, a carrier protein combines with a specific solute and moves it from one side of the membrane to the other, down its electrochemical gradient. This in effect increases the membrane's permeability to the substance and so allows the substance to cross membranes faster than it otherwise would.*

An example is the system that facilities the diffusion of glucose into the cells of some tissues of vertebrates (animals with backbones). In the liver, the lens of the eye, and red blood cells, facilitated diffusion moves glucose across the plasma membrane in both directions by means of a carrier molecule. The carrier molecule is more likely to encounter and pick up a glucose molecule on that side of the membrane where glucose is more plentiful. When a cell is breaking down glucose quickly during respiration, the glucose concentration inside the cell falls; glucose is then more plentiful outside the cell, and it is moved into the cell more rapidly than it is moved out.

Facilitated diffusion is just as important in increasing the rate at which glucose leaves a cell. Cells in the liver, for instance, not only remove glucose from the bloodstream when the blood glucose level is high but also replenish the blood glucose when its level drops.

In active transport*, substances are moved either with or against their electrochemical gradients; this process requires the expenditure of energy. The source of energy for active transport may be the high-energy ATP molecule or the electrochemical gradient of an ion across a membrane. Common sources of such electrochemical energy are steep differences in sodium ion (Na^+) or*

hydrogen ion (H^+) concentration on the two sides of a membrane. Since there is a strong tendency for ions to move down a steep gradient, such a gradient represents a source of energy.

Among the many active systems are those that take up amino acids, peptides and potassium in the bacterium Escherichia coli. These substances move into the bacterium only in the presence of the appropriate carriers and of a source of energy.

Another example is a calcium pump, found in many cells, which pumps Ca^{2+} *out of cells and so keeps their internal* Ca^{2+} *concentration much lower than external levels. A spectacular pump found in the stomach wall is responsible for secreting "stomach acid": using the energy of ATP, it exports* H^+ *against a* pH *gradient of about a million to one! But one of the most widespread and best examples is the sodium-potassium pump.*

In order for a cell to maintain homeostatis, it must have strict control over its chemical content, which includes not only the absolute amounts of solutes but also their concentration. Thus the content of the solvent, water, in a cell must also be precisely regulated. Vital as water is to living cells, cells have known carriers or other direct means of transporting water in or out. Water seems to travel through the plasma membrane quite freely – faster, in fact, than any other substance.

Osmosis, the process by which water moves through a selectively permeable membrane, is a special case of diffusion: it involves the diffusion of a solvent, such as water, rather than the diffusion of substances dissolved in the solvent. In osmosis in living cells, water moves across a membrane from a weak, or dilute, solution into a strong, or concentrated, solution (this process is spontaneous since it increases the overall solute entropy).

A simple way to demonstrate osmosis is to separate distilled (pure) water from an aqueous solution by a membrane that is permeable to water but not to the solute. As time passes, the volume of the solution increases and that of the distilled water decreases. Therefore water must be moving by osmosis from the water, across the membrane, and into the solution.

What is the molecular mechanism for this entropy-increasing process? Water molecules can cross the membrane in either direction. However, the water molecules in the higher-concentration solution bump into the solutes and also experience forces attracting them to solute particles; this retards the movement of the water molecules in the solution, and so water moves into that side of the membrane solution faster than it moves out.

In a laboratory osmotic system, the net movement of water into the solution increases the height of the solution in the tube, and the weight of the column of solution exerts hydrostatic pressure. As water enters the solution,

the hydrostatic pressure builds up until it is pushing water molecules out as fast as they enter. The solution will remain at this level.

The extent of movement of water across a membrane can be predicted by knowing the osmotic potentials of the two solutions separated by the membrane. The osmotic potential *of a solution is its tendency to gain water when separated from pure water by an ideal selectively permeable membrane. A stricter definition of osmotic potential is that it is the negative of the* osmotic pressure, *which is the minimum pressure that must be applied to a solution to prevent it from gaining water when it is separated from pure water by an ideal selectively permeable membrane.*

The osmotic potential is expressed in negative terms: the more concentrated the solution, the lower (more negative) *its osmotic potential, and the greater its tendency to gain water from a solution with a higher* (less negative) *osmotic potential. The osmotic potential is the driving force of osmosis, since water tends to move in a downhill direction in terms of free energy, that is, in the direction of the lower osmotic potential,*

The osmotic potential in a system depends on the concentration of particles in the solution and on the attraction of water molecules to the particles, which slows the movement of the water molecules. There may be only one type of molecule dissolved in a solution, or there may be many, as in a living cell. Each molecule of a strong-electrolyte ionic substance dissociates into more than one particle in aqueous solution; $NaCl$ *dissociates into two particles,* $MgCl_2$ *into three, and so forth. The more particles there are in a solution, the lower the osmotic potential. If the solute particles are able to pass through the membrane, then the osmotic potential of the solution will gradually change as solute particles enter or leave it.*

We can now see how a cell can control its water content, and hence its volume. The cell can create a difference in osmotic potential across its membrane by the active transport of solutes, especially by means of the sodium-potassium pump. Water will then move by osmosis toward the side of the membrane where the osmotic potential is lower.

Cells behave as an osmotic system. A living cell has a selectively permeable plasma membrane, which encloses the cell's internal solution of various particles dissolved in water. To remain alive, the cell must be covered by at least a thin layer of water, which also has solutes dissolved in it. If this extracellular (extra = outside) *solution is in osmotic balance with the intracellular* (intra = within) *solution, no net exchange of water occurs between them, and the cell is said to be living in an* isotonic *solution.*

If the external solution is made more concentrated, so that the cell loses water to its environment, such an extracellular solution is said to be hypertonic

to the cell contents. And if the cell is placed in a solution dilute enough for the cell to gain water from outside, this environment is said to be hypotonic *to the intracellular solution.*

Some animal cells in dilute (hypotonic) solutions may take in so much water that their internal pressure ruptures the plasma membrane, allowing the cell contents to escape. This process is known as lysis *(bursting) of a cell. In the same situation, the rigid wall of a plant cell produces a* wall pressure *which opposes the outward pressure of swelling and makes most plant cells more resistant to swelling in a hypotonic solution.*

Many animals live in freshwater environment which are hypotonic to their cell cytoplasm. Why don't these animals take up so much water by osmosis that they swell up and burst? Most of a freshwater animal's body surface is covered by a layer of rather impermeable material, which retards water uptake. Such layers include the mucus of fish and worms or the wax-impregnated chitinous armor of aquatic insects and spiders. In addition, the excretory structures of such organisms have well-developed active transport mechanisms, which allow them to rid their bodies of water while conserving precious salts.

Freshwater protozoa (unicellular organisms that lack cell walls, such as Amoeba *and* Paramecium*) gain a great deal of water by osmosis. These organisms void excess water by way of specialized structures called* contractile vacuoles*, which accumulate water and then contract, squeezing the water back into the environment. Like all other freshwater organisms, protozoans face a scarcity of available salts. So, before the contractile vacuole expels its contents, salts are removed by active transport. Every 4 to 8 minutes the contractile vacuoles of a paramecium eject a volume of water equivalent to the volume of the entire cell!*

The cells of a multicellular organism communicate with their neighbors by the exchange of substances in the cytoplasm. Such transfers can be accomplished most effectively by direct cytoplasm-to-cytoplasm connections. In many cases we find plasma membranes arranged to permit such connections.

In plants, the cytoplasm of neighboring cells is often connected by strands of cytoplasm called plasmodesmata *(singular:* plasmodesma*). These cytoplasmic bridges pass through interstices in the cell walls between the two cells, and both the cytoplasm and the plasma membranes of these cells are essentially continuous with each other.*

Gap junctions *occur between many kinds of animal cells and are thought to permit passage of ions and small molecules from cell to cell. This cell-to-cell connection can be shown by placing microelectrodes inside two adjacent cells that are linked by gap junctions. The electrical resistance between the electrodes is very low, indicating that electrically charged substances can move*

unimpeded between the cells. In contrast, if the electrodes are placed so that one is inside and the other outside a cell, the electrical resistance measured is high. This shows that the flow of ions through the membrane is highly controlled.

The main electrically charged particle in biological systems are proteins and nucleic acids, which are too big to leave the cell (even through gap junctions), and small ions such as Na^+, K^+, *and* Cl^-. *The low electrical resistance between adjacent cells suggests that ions can move from one cell to another, probably through the intercellular channels of the gap junction. It is sometimes possible to confirm such a finding by tracing the movement of fluorescent or radioactive substances from one cell to another.*

In some tissues, the function of electrical coupling between cells is clear. For instance, coupling helps to coordinate the contractions of adjacent heart muscle cells. However, the function of this coupling is not yet understood in other cases, such as the many gap junctions in early animal embryos.

Electron micrographs of gap junctions show an array of proteins channels linking the cytoplasm of two neighboring cells. Each channel consists of two short, pipe-like sections, one in each plasma membrane, lined up so that they meet in the intercellular space to form a continuous passageway. One end of each "pipe" juts its counterpart on the opposite membrane. The walls of each "pipe" consists of six rather cylindrical membrane proteins arranged in a circle surrounding a channel about 1.5 *to* 2.0 *mm in diameter.*

The complex process by which a cell nucleus gave rise to two daughter nuclei was worked out during the 1870's and the 1880's, largely by German investigators who gave the first clear description of the appearance of chromosomes *in* mitosis. *These phenomena gained importance (1890) when* ***August Weismann*** *took up the theories of* hereditary *particles developed by* ***Hugo de Vries*** *and others, combined them with studies of cytology and embriogenesis, and developed the idea that nuclei, and particularly chromosomes, contained determinants which directed the development of the cells and hence controlled the characteristics of the whole animal or plant. Hence the nucleus, previously seen as a part of the active protoplasm of the cell, came to be viewed as a store of information.*

Summary

Cells must maintain the internal concentrations of all substances at appropriate levels. At the same time, cells must maintain a lively commerce

with their environments, taking in new raw materials for their metabolism and expelling waste products.

The plasma membrane regulates what enters or leaves the cell. It is permeable to many types of small molecules and ions, yet sufficiently impermeable to prevent the loss of such materials as nucleic acids, proteins, and polysaccharides.

A biological membrane consists of a fluid lipid bilayer with various proteins floating in it, some mobile in the bilayer and some anchored to stable cellular structures. Oligosaccharides are attached to some protein and lipid molecules, forming glucoproteins and glycolipids.

This basic structure has two properties crucial to membrane function. First, lipid bilayers spontaneously form closed compartments, thereby keeping the solutions inside and outside the membrane separate. Second, membranes are asymmetrical, with different lipid and protein components in each of the two layers, and with molecules oriented so that they consistently face one membrane surface or the other. For example, active transport molecules are oriented so that they move substances in only one direction.

A membrane's lipid bilayer is freely permeable to water. It also admits small, lipid-soluble molecules, which diffuse through the lipid layers according to their concentration gradients.

Most ions and polar molecules can cross the membrane only with the aid of protein transport molecules. Each protein is specific for a particular solute or a few closely related solutes. Channel proteins form aqueous channels through the membrane; some are gated so that they open in response to specific polar molecules and ions down their electrochemical gradients. Other proteins mediate active transport, which can move a solute either with or against its electrochemical gradient (used e.g. to transport ions such as Na^+ *or* H^+*). The sodium-potassium pump, powered by ATP, pumps* Na^+ *out of a cell and* K^+ *in. This pump is largely responsible for the membrane potential of a cell, and the electrochemical gradient of sodium that it creates also provides energy for the active transport of solutes such as glucose.*

When the cell acquires macromolecules or larger particles, the membrane surrounds them and pinches off to become a vesicle or vacuole inside the cell, by the process of endocytosis. Substances can be discharged from many cells by the opposite process of exocytosis.

Cells gain or lose water by osmosis. The membrane does not control the movement of water molecules directly; rather, it performs active transport of solutes and so creates an osmotic potential difference that will induce osmosis. The cell wall of a plant cell exerts a pressure that limits the cell's water

content. Many protozoans void excess water taken in by osmosis by means of a contractile vacuole.

The plasma membrane may be expanded to provide additional surface area for exchange of substances with the environment. The plasma membranes of adjacent animal cells may interact. Tight junctions seal membranes together and prevent seepage of substances between cells. Intermediary junctions and desmosomes provide mechanical strength by attaching the membranes of adjacent cells. Gap junctions act as "pipes" through animal cell membranes, providing for direct transfer of ions from cell to cell. In plants, direct transfer between cytoplasm of adjacent cells occurs by way of plasmodesmata.

1838–1840 CE **Ferdinand Minding** (1806–1885, Germany). Mathematician. A forerunner of hyperbolic non-Euclidean geometry. Contributed to the differential geometry of surfaces of constant curvature and was first to introduce the concept of the pseudosphere[156] and show that the hyperbolic

[156] *Pseudosphere*: A surface of revolution formed from the plane pursuit curve called *tractrix*. This curve was first studied by **Newton** (1676) and later by **Huygens** (1693), **Leibniz** (1693), **Johann Bernoulli** (1728), and **Liouville** (1850). Also called *Tractory* and *Equitangential curve*. It is the path of a particle pulled by an inextensible string whose other end moves along a line (x-axis, say). The simplest example is that of a toy-boat pulled by a boy with a string: he starts walking in a direction perpendicular to the string, always keeping the string taut. The tractrix is the boats' path in the water (ignoring the boy's height). Since the thread is stretched in the direction *tangent* to the curve, we can write $\frac{dy}{dx} = -\frac{y}{\sqrt{p^2-y^2}}$ where p is the length of the string. Integration yields the equation

$$x = p\,\mathrm{ch}^{-1}\left(\frac{p}{y}\right) - \sqrt{p^2-y^2}\,,$$

or in parametric form:

$$y = p\sin\phi, \qquad x = p\left[\log\tan\frac{\phi}{2} - \cos\phi\right], \qquad -\frac{\pi}{2} < \phi < \frac{\pi}{2},$$

where ϕ is the angle between the string and the x-axis (the asymptote).

Consider the surface of revolution formed by rotating the curve about the entire

formulas for triangles hold on the pseudosphere (1840). Minding was a student of **Gauss** and later became a professor at the University of Dorpat (now Tartu, Estonia).

Hyperbolic non-Euclidean geometry (nEg) can briefly be characterized by the fact that there are (at least) two straight lines passing through a point and parallel to the given line, and the sum of the angles of a triangle is less than π. In *Elliptic* nEg there does not exist a straight line parallel to a given one, and the sum of the angles of a triangle is more than π. The intrinsic geometry of surfaces with positive constant Gaussian curvature is, at least locally, identical with elliptic nEg.

Minding established the theorem (1839) that all surfaces with the same constant curvature are isometric; in particular, every surface of constant negative Gaussian curvature $K = -\frac{1}{\lambda^2}$ can be isometrically mapped onto a pseudosphere of pseudoradius λ.

Minding started from the standard relation for a spherical triangle of sides $\{a, b, c\}$ and angle A opposite the side a:

$$\cos\frac{a}{R} = \cos\frac{b}{R}\cos\frac{c}{R} + \sin\frac{b}{R}\sin\frac{c}{R}\cos A,$$

where R is the sphere's radius and $\frac{a}{R}$ is the angle subtended at the sphere center by side a, in radians, and similarly for $\frac{b}{R}$, $\frac{c}{R}$. In the plane limit $R \to \infty$ it reduces to the Euclidean law of cosines $a^2 = b^2 + c^2 - 2bc\cos A$. On a surface with constant curvature K, the above relation is generalized into

$$\cos(a\sqrt{K}\,) = \cos(b\sqrt{K}\,)\cos(c\sqrt{K}\,) + \sin(b\sqrt{K}\,)\sin(c\sqrt{K}\,)\cos A.$$

Applying this to surfaces with *negative* curvature $K = -1$, one finds $\operatorname{ch} a = \operatorname{ch} b \operatorname{ch} c - \operatorname{sh} b \operatorname{sh} c \cos A$, which holds for a geodetic triangle on a *pseudosphere.* This very equation was obtained earlier (1837) by **Lobachevsky** who did not realize that it holds on a pseudosphere. It is one of the great examples of noncommunication in mathematical history. Neither Minding nor Lobachevsky seem to have read the other's paper. *Nobody* seem to have read them both and realize that Lobachevsky's "imaginary geometry"

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asymptote. It has an area of $4\pi p^2$ and a volume of $\frac{2}{3}\pi p^3$ (Huygens showed in 1693 that they are finite). The *mean curvature* of the said surface (arithmetic mean of maximum and minimum curvatures at a point) is a *negative* constant $\left(-\frac{1}{p}\right)$. It is for this reason that the surface is called the *pseudosphere* with *pseudoradius p*. Because of this property the *pseudosphere* serves as a model for non-Euclidean hyperbolic geometry, just as a *sphere* does for non-Euclidean elliptic geometry.

was nothing more than the very real geometry of a particular surface. Hence, the importance of this result for hyperbolic geometry was totally missed. Perhaps it was clear that the pseudosphere cannot serve as a "plane", because it is infinite in only one direction. It was not until 1868, when **Beltrami** extended the pseudosphere to a true *hyperbolic plane*[157] — a surface *locally* like the pseudosphere but infinite in all directions — that hyperbolic geometry was finally placed on a firm foundation.

1838–1852 CE **Ferdinand Gottfried Max Eisenstein** (1823–1852, Germany). Mathematician. One of the most gifted mathematicians in Germany, of the two generations after Gauss.

Contributed to the fields of elliptic functions, algebra and number theory. In the latter he created the theory of cubic forms. His work led to several theorems for quadratic and biquadratic residues, cyclotomy and quadratic partition of prime numbers[158] and reciprocity laws. He was Gauss' favorite student and Gauss wrote of him to von Humboldt: "*Eisenstein belongs to those talents who are born but once in a hundred years*".

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[157] *Hyperbolic plane*: consider a circle of radius 2 in the x–y plane. Define a metric tensor w.r.t. a polar coordinate system (r, θ) at the origin:

$$g_{11} = \left(1 - \frac{r^2}{4}\right)^{-2}, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = r^2\left(1 - \frac{r^2}{4}\right)^{-2}.$$

The *Gaussian curvature*

$$K = -\frac{1}{\sqrt{g_{11}g_{22}}}\left[\frac{\partial}{\partial r}\left(\frac{1}{\sqrt{g_{11}}}\frac{\partial\sqrt{g_{22}}}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\sqrt{g_{22}}}\frac{\partial\sqrt{g_{11}}}{\partial\theta}\right)\right]$$

is found to be equal to -1, while the *geodesics* consist of Euclidean straight lines through the origin and circles whose center lies *outside* the boundary $r = 2$ and intersect $r = 2$ orthogonally. Thus, through a point not on a given geodesic circle, there exist infinitely many "parallel lines" (circles) to the given one.

[158] Prime numbers can be defined in fields other than integers. In the complex field $\mathbb{C}$ we have the *Gauss integers* $n + im$ (n, m integers) and the *Gauss primes* [primes of the form $4k-1$ over the integers are still primes in $\mathbb{C}$, but 2 and primes of the form $4k+1$ can be factored in $\mathbb{C}$, e.g. $2 = (1+i)(1-i)$, $5 = (2+i)(2-i)$, $13 = (2+3i)(3-3i)$, $19 = (4+i)(4-i)$, etc.].
Eisenstein defined the *Eisenstein integers* $n + \omega m$, where ω is the complex cube root of unity $[\omega = \frac{1}{2}(1 - \sqrt{-3})$; $1 + \omega + \omega^2 = 0]$. The prime 2 and primes of the form $6k-1$ are also *Eisenstein primes*, but 3 and primes of the form $6k+1$ can be factored, e.g. $3 = (1-\omega)(1-\omega^2)$, $7 = (2-\omega)(2-\omega^2)$, $13 = (3-\omega)(3-\omega^2)$, $9 = (3-2\omega)(3-2\omega^2)$, etc.

Eisenstein was born in Berlin to Jewish parents. He started his university studies while still in high school and on his second year of study he received his Ph.D. degree, honoris causa, from the University of Breslau.

Eisenstein suffered all his life from bad health. After leaving school he traveled with his parents to England where they were looking for a better life. While in Ireland Eisenstein met **Hamilton**, who gave him a copy of a paper that he had written on Abel's work on the impossibility of solving quintic equations. This stimulated Einstein to do research in mathematics and on his return to Germany he enrolled at the University of Berlin. In 1844 Eisenstein went to Göttingen for a short time and met **Gauss**.

Throughout this year he published no fewer than 25 papers in German mathematical journals. In 1852 he became a professor of mathematics at the University of Berlin and was elected to the Prussian Academy of Science. He was an influential teacher of **Riemann** in Berlin.

But his health worsened after 1847. Soon he spent most of his time in bed. Eisenstein spent a year in Sicily in an attempt to improve his health but after his return to Germany he died of pulmonary tuberculosis at the age of 29. The collected works of Eisenstein were published by Gauss already during his lifetime (1847) [his *autobiography* was published in Volume **40** of *Zeitschr. f. Math. u. Physik*, pp. 143–200, 1895].

1839, Aug 19 *Official date for the start of photography*

The Academy of Science and the Academy of Fine Arts met jointly in Paris to debate one last time the process invented by **Daguerre** (1839) and (after the French state purchased the inventor's patent at the price of a life annuity) made the world a gift of that process.

Evolution of early photography (1826–1851)

The discovery of photography during the first third of the 19th century was the response of the technological era to the new tastes of a middle class of considerable economic means. Indeed, manual graphic techniques such as drawing, etching and lithography must have seem outmoded to people living in an age when steam machine energy was being pressed into the service of

capitalist production, when mechanical looms were gradually replacing labor force, and when railroads were on the verge of bringing distant regions within the reach of one another, thus endowing humanity with new mobility.

Above all, such manual and therefore largely subjective forms of representation hardly corresponded with the objective vision of the world and the environment to which the rational positivism of the times aspired.

Strictly speaking, the basic principles of photography had been known for some time. They only had to be combined and the final missing link supplied. The working of the camera obscura, for example, had been known since ancient times and had also served as an aid to artists interested in achieving more true-to-life perspective in their work since the Renaissance. Similarly, the sensitivity to light of silver salts was a known fact by, at the latest, the time of ***Johann Heinrich Schulz*** *(1727). What remained lacking up until the end of the 18th century was a socially rooted interest in obtaining pictures by mechanical means and a way of making permanent the camera obscura "sun pictures".*

Various separate developments took place during the first half of the 19th century: in 1826, the Frenchman ***Nicéphore Niepce*** *had succeeded in taking a photograph of the view from his workroom window by using bitumen coatings on a copper plate; since 1834, England's* ***Henry Fox Talbot*** *had been successfully experimenting with sensitized paper negatives; and by 1837,* ***Louis Jacques Mondé Daguerre*** *had succeeded in producing the type of photograph which was named after him: the daguerreotype.*

The international reaction to the publication of the technique by the French Academies (1839) was as immediate as it was tremendous – a proof that Western industrial society had been waiting for just such a technical means of producing pictures. People were enchanted with the way the process brought out every detail of a picture equally, and with the method's speed. Although lack of color was disappointing, what fascinated and finally convinced most people was the mechanical, almost automatic nature of the process.

The potential range of applications of the new medium was soon recognized: architectural and landscape photography, art reproductions, portraiture, and a tool of astronomy and photometry.

The widely traveled ***Alexander von Humboldt*** *reflected on the usefulness of a camera on journeys*[159].

[159] The long exposure times required in 1839 left the achievements of portraits in the realm of the utopia. For example, after a first visit to Daguerre (Feb 1839), Humboldt mentions exposure times taking 10 minutes! Even in 1842, after

From a technical point of view, it was the invention of the negative by ***Talbot*** *which laid the cornerstone for the popularization of photography. Both in theory and in practice, negatives affording countless positives was a winning formula. At the start, however, Talbot's brown-spotted "catlotypes" could hardly measure up to the finely lined, steel-gray and sharply defined daguerreotypes of the day. Even stiffer competition was afforded by the colored motifs in the stereoscopic daguerreotypes which so enthused the public and even now remain astonishing. But steady improvement in the technique from approximately 1840 onwards soon made its basic superiority apparent.*

One of the first commercial photographers was ***Antoine-Jean-Francois Claudet*** *(1797–1867, France and England), who was a French glass merchant, living in High Holborn, England. He learned the details of the daguerreotype process from its inventor, and bought from him a license to operate in England. In 1841 he set up a studio on the roof of the Adelaide Gallery (near the Nuffield Centre), behind St. Martins in the Fields Church, London, and later on in two other sites in London.*

In 1847 glass was used for the first time as an emulsion support, while albumen paper, long appreciated for its sheen, was put on the market in 1850. Finally, the arrival of the colodion wet-plate process (1851) at last made available photographic material combing a high level of light-sensitivity with sharp definition. By the middle of the 19th century, the main steps had been taken w.r.t. both the practical aspects of the medium and the over-widening scope of its subject matter.

In addition to the various chemical-technical improvements, there were also new discoveries and refinements of optical features and the cameras themselves.

1839 CE **Charles Goodyear** (1800–1860, U.S.A.). A New-England hardware merchant; discovered a method of vulcanizing rubber.

The rubber in use at that time became hard in the winter and sticky in the summer. Without having any idea of what he was doing (he was no chemist), Goodyear began a series of experiments to try to improve the properties of rubber. He worked with rubber-sulfur mixtures, because he had heard of other similar experiments. When he accidentally spilled one of his

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the introduction of improved lenses and more sensitive photographic plates, exposure times were rarely less than 30 seconds.

concoctions on a hot stove, he found that he had created a superior type of rubber whose properties did not alter with heat or cold. Thus was the process of *vulcanization* born.

Unfortunately, Goodyear himself reaped no benefit from it. Before his discovery, his life had been marked by debtor's prison and bankruptcy, after it by patient litigation and the borrowing of huge sums to promote his invention. He died not quite 60 years old and hundreds of thousands of dollars in debt.

1839 CE **James Maccullagh** (1809–1847, Ireland). Irish mathematician and physicist. Produced an elastic model of the ether as a solid, the potential energy of which depends only on the rotation of a volume element, thus explaining the single wave velocity $c = (\mu/\rho)^{1/2}$. In terms of the later Maxwell theory, his displacement vector $\boldsymbol{u}$ corresponds to the magnetic field vector, and $\mu\,\mathrm{curl}\,\boldsymbol{u}$ to the electric field vector [μ = rigidity, ρ = density]. He also derived the gravitational potential of a finite body at an external point in terms of its principal moments of inertia.

Maccullagh was born near Strabane, Ireland. After a brilliant career at Trinity College, Dublin, he held the chair of mathematics during 1832–1843 and in 1843 was transferred to the chair of natural philosophy. Overwork induced mental disease, and he died at his own hands in 1847.

The Principle of Conservation of Energy

The failure to comprehend the distinction between velocity and acceleration retarded the study of dynamics for centuries. The study of heat was retarded by a somewhat similar confusion between temperature and heat, and by the further misapprehension that heat was a substance. Discovery of the true relationships involved some of the most illustrious names in theoretical and experimental physics.

The principle of conservation of kinetic and potential energy for rigid bodies was intuitively recognized already by ***Galileo Galilei*** *(1564–1642), following his experiments with bodies in free fall. The generalization of this principle to the entire field of mechanics is due to* ***C. Huygens*** *(1629–1695),* ***G.W. Von Leibniz*** *(1646–1716),* ***J. Bernoulli*** *(1667–1748) and* ***L. Euler*** *(1707–1783).*

*The concept of work originated with **J.V. Poncelet** (1788–1867). The word energy occurs already in **Aristotle**'s writings. It was introduced into the language of science by **T. Young** and **William Rankine** (1853). **Robert Mayer** (1840), **James Prescott Joule** (1843), **William Robert Grove** (1846), **H.L.F. Helmholtz** (1847) and **Lord Kelvin** (1852) reformulated the energy principle to include thermal, electrical and chemical phenomena and found the proper numerical conversion factors.*

1839–1842 CE *The Chinese Opium War* with Great Britain. The Chinese government had long been alarmed by the flourishing trade in opium and had vainly tried to stop it. In 1839 it moved to confiscate and destroy the vast quantities of the drug stored in Canton. This provided Britain the rationale for taking over China, something that it had long desired. A punitive force, assisted by the fleet of the East India Company, invaded China. The Chinese were no match for the experienced Westerners. Britain lost 500 troops; the Chinese lost 30,000. After three years of intermittent fighting, the Chinese were forced to agree to Britain's terms as laid down in the Treaty of Nanking (1842).

Opium had been used in China for medical purposes for centuries, but there was only a small amount of addiction among the people. China had little use for Western goods and ideas; its society was stable, and the vast country supplied all its own needs.

Europe, however, was extremely interested in the goods of China. Its *tea*, *silks*, *spices*, and *porcelain* commanded high prices on a market fascinated by the Orient.

The conquest of Bengal (1773) by Britain and the development of the World's finest merchant fleet allowed England's East India Company to obtain a foothold in the China trade and, by 1800, to monopolize it; after delicate negotiations with the Chinese government, a small island off the shores of Canton was established as a basis for trade. But the Chinese insisted that all goods be paid for in *silver*. By 1810–1815, China had acquired a good share of the silver of Europe which, because of scarcity, was rising rapidly in price, thus reducing the profits of British merchants.

Moreover, the Napoleonic wars have dwindled the gold reserves of Britain to a degree that it could not pay for the imported tea from China. Obviously, there had to be something that the Chinese wanted (except silver) and, with

unerring intelligence, the British decided to sell Indian-made *opium* to the Chinese.

This required two preliminary steps: first, making the Chinese masses addicted to the drug, and second, securing a safe ocean route for trafficking the drug from India to China. These two objects were simultaneously achieved: on the one hand a British military expedition, occupied *Java* and controlled the *Straits of Malacca* (1811). In 1819 **Thomas Stamford Raffles** (1781–1826) founded *Singapore*.

Simultaneously opium was given away, and as addiction spread, prices rose accordingly. Because of its absolute control over India, the East India Company subverted the agriculture of Benares, Baher, and other areas of India to the growing of the poppy and the production of opium. Poppy cultivation was compulsory, and since the production of food crops was limited, the people of these India provinces were reduced near to starvation.

As addiction in China rose to astronomical proportions, silver began to move out of the country back into Europe where its price fell and the profits of the East Indian Company soared. Since silver was the currency of China, taxes went unpaid and internal business was disrupted. Based on Confucian and Tao ideals, Chinese society was grounded in the philosophy of self-discipline and a hierarchical arrangement of duties to family and emperor. The addict to his habit sacrificed family, duty, and self-discipline, and the fabric of Chinese society and government began to collapse. Nevertheless, the British government refused to order the East India Company to stop the opium trade; tax revenues from India, tea duties, and opium sales were simply too profitable.

1839–1846 CE **Christian Friedrich Schönbein** (1799–1868, Germany). Chemist. Discovered *ozone* (1839) and *guncotton* (1846), a powerful explosive, which he prepared and applied as a propellant in fire-arms.

Schönbein was born at Metzingen, Swabia. After studying at Tübingen and Erlangen, he taught chemistry and physics at Germany and England, but most of his life he spent at Basel, where he was appointed full professor (1835–1868). He was a prolific writer and carried on a large correspondence with other men of science such as **Berzelius**, **Faraday**, **Liebig** and **Wöhler**.

1839–1846 CE **William Robert Grove** (1811–1896, England). Jurist, physicist and electrochemist-inventor. "*Father of the Fuel Cell*".

A *fuel cell* is a device that produces electricity by combining hydrogen and oxygen — the reverse process of *electrolysis*. In his classic "*On the Correlation of Physical Forces*" (1846), he enunciated the *principle of conservation of energy* a year before the German physicist **Hermann von Helmholtz**.

Grove invented two cells of special significance. His first cell consisted of zinc, in dilute sulfuric acid and platinum in concentrated nitric acid, separated by a porous pot (*Grove Cell*).

Grove's nitric acid cell was the favorite battery of the early American telegraph (1840–1860), because it offered strong current output. This cell had nearly double the voltage of the first Daniell cell. As telegraph traffic increased, it was found that the Grove cell discharged poisonous nitric dioxide gas. Large telegraph offices were filled with gas from rows of hissing Grove batteries. As telegraphs became more complex, the need for constant voltage became critical and the Grove device was necessarily limited (as the cell discharged, nitric acid was depleted and voltage was reduced). By the time of the American War, Grove's battery was I replaced by Daniell battery.

His second cell, a "gas voltaic battery" was the forerunner of modern *fuel cells*. He produced the first fuel cell in 1839 basing his experiment on the fact that sending an electric current through water splits the water into its component parts of hydrogen and oxygen. So, Grove tried reversing the reaction — combining hydrogen and oxygen to produce electricity and water. This is the basis of a simple fuel cell. The term "fuel cell" was coined later in 1889 by **Ludwig Mond** and **Charles Langer**, who attempted to build the first practical device using air and industrial coal gas.

Grove was born at Swansea, South Wales. He was educated by private tutors and then at Brasenose College, Oxford, and also studied law at Lincoln's Inn and was called to the bar in 1835. His scientific career flourished while he was a professor of physics at the London Institution (1839–1864). At that period he also invented the earliest form of a filament lamp intended for use in mines.

He was elected FRS in 1840 and was one of the leaders of the reform movement. His law career was resumed in 1879, when he became a Judge at the Court of Common Pleas. He moved to the High Court of Justice in 1880 and became a Privy Councilor in 1887. Grove was knighted in 1872.

In 1846 he published his book on *The Correlation of Physical Forces*, the leading ideas of which he had already put forward in his lectures: its fundamental conception was that each of the manifested energy-forms of nature — light, heat, electricity, etc — is definitely and equivalently convertible into each other, and that where experiment does not give full equivalent, it is because the initial energy has been dissipated, not lost, by conversion into heat.

1839–1876 CE **John William Draper** (1811–1882, U.S.A.). Pioneer scientific photographer, photochemist, historian and author. His contributions:

- Made portrait photography possible by his improvements (1839) on Daguerre's process. Made first telescopic daguerreotype of the moon (1840) and the sun's diffraction spectrum (1844). Took photographs of the solar spectrum (1876) and anticipated development of *spectrum analysis.*
- Showed that plants grown in solution of sodium bicarbonate can liberate oxygen in light (1844).
- Investigated the dependence of the color of a heated substance upon its temperature.

Draper was born at St. Helen near Liverpool, England. He studied at the University of London. Went to the U.S.A. (1831) and continued his studies at the medical school of the University of Pennsylvania (1835–1836). Professor of chemistry at the University of the City of New York (1838–1882).

His son **Henry Draper** (1837–1882, U.S.A.), astronomer, was professor at the University of the City of New York (1860–1882). Built and mounted a 28-inch reflector (1869) with which he did pioneering work in stellar spectroscopy: obtained the first photograph of the stellar spectrum of *Vega* (1872), and the *Orion Nebula* (1880).

1840 CE **Germain Henri Hess** (1802–1850, Switzerland and Russia). Chemist, Formulated *Hess's Law*[160], which states that the net heat evolved or absorbed in any chemical reaction depends only upon the initial and final stages, It was a forerunner of the more complete law of the conservation of energy.

Hess was born in Geneva, Switzerland and became a professor of chemistry at St.Petersburg, Russia (1830–1850).

160 *Hess's Law*: If a reaction is carried out in stages, the algebraic sum of the amounts of heat evolved in separate stages (heat absorbed being reckoned negative) is equal to the total evolution of heat when the reaction occurs directly.

e.g.: the *heat of combustion* of carbon to carbon dioxide: $C+O_2 = CO_2 + 94$ kilocalories; the *heat of combustion* of carbon monoxide to dioxide: $CO+\frac{1}{2}O_2 = CO_2 + 67.8$ kilocalories. By subtracting the second of these equations from the first, we find the *heat of formation* of carbon monoxide: $C+\frac{1}{2}O_2 = CO+26.2$ kilocalories.

Table 4.2: Mail services and Newspapers throughout history

900 BCE	China has an organized postal service for government use
500 BCE	Persia has a form of pony express
59 BCE	Julius Caesar ordered posting of *Acta Diurna*
100 CE	Roman couriers carry government mail across the empire
1200 CE	European monasteries communicate by letter system
1300 CE	Incas and Aztecs employ courier runners to carry messages over their kingdoms roads at top speed of ca 400 km/day
1305 CE	The Taxis family began a private postal service in Europe
1450 CE	First newsletters began circulating in Europe
1533 CE	A postmaster in England
1609 CE	First regularly published *newspaper* appeared in Germany
1627 CE	France introduced registered mail
1650 CE	A daily newspaper in Leipzig, Germany
1840 CE	First postage stamps sold in Britain

1840 CE **Joseph Max Petzval** (1807–1891, Hungary). Mathematician and optician. Contributed to the design of precision optical systems through his work on lenses and aberrations. These had great impact in the design of modern cameras; Petzval produced an achromatic portrait lens that was vastly superior to the simple meniscus lens then in use. *Petzval curvature*, *Petzval surface*, *Petzval theorem. Petzval condition* and *Petzval lens* are named after him.[161]

[161] To dig deeper, see:

- Born, M. and E. Wolf, *Principles of Optics*, Macmillan: New York, 1964, 808 pp.

The discovery of photography (1839) by Daguerre was chiefly responsible for early attempts to extend the Gaussian theory. Practical optics, which until then was mainly concerned with the constructions of *telescope objectives*, was confronted with the task of producing objectives with *large apertures and large fields*. Petzval attacked with considerable success the related problem of supplementing the Gaussian formulae by terms involving *higher powers of the angular inclination of rays to the axis*. Unfortunately, Petzval's extensive manuscript on the subject was destroyed by thieves; what is known about his work comes mainly from semi-popular reports.

He was professor at the University of Vienna and worked for much of his life on the *Laplace transform*, being influenced by the work of **Liouville**. He pioneered the application of the Laplace transform to the solution of linear differential equations although he did not use contour integration to invert the transform.

1840 CE **Friedrich Gustav Jacob Henle** (1809–1885, Germany). Anatomist and pathologist. First to argue that infectious diseases are transmitted by living organisms which can reproduce. In his work *Pathologische Untersuchungen* (Pathological investigations) he presented an early version of germ theory of disease in which parasite living matter can be transmitted through contact or through the atmosphere. His contention was proved by his pupil Robert Koch 40 years later.

Henle was born at Fürth, a grandson of a rabbi, and was baptized at the age of 11. He took his doctors degree in medicine at Bonn (1832) and latter became a professor at Heidelberg (1844).

1840 CE Polio first identified or described with accuracy.

1840–1862 CE *Cholera* spread worldwide; fatalities were in the millions.

1840–1865 CE **Karl Friedrich Schimper** (1803–1867, Germany). Naturalist and poet. Proposed many of his scientific ideas in poetic form, including botany, geology, and the formation of the Alps during the Ice Age. A pioneer of modern plant morphology. Originated the modern concepts of the Ice Age and the climatic cycles. Formulated the theory of phyllotaxis.

Schimper was born in Mannheim. He studied theology at Heidelberg (1822) and medicine at Munich (1829) but failed to secure an academic post nor any other regular appointment. He never married despite two engagements and eventually moved to Schwetzingen where he died of dropsy.

His cousin, **Wilhelm Philipp Schimper** (1808–1890) was a botanist. Studied at the University of Strasbourg, where he became Director of the local Natural History Museum (1835). Identified the *Paleocena Period* in earth's

history (1874). His son **Andreas Franz Wilhelm Schimper** (1856–1901) was also a botanist and a professor at Basel (1898–1901). Proved (1880) that starch is a source of stored energy for plants and a product of photosynthesis.

1840–1870 CE **Joseph Whitworth** (1803–1887, England). Mechanical engineer and inventor; a leader in tool design and manufacture. Invented measuring machines and found a method of milling and testing plane surfaces. Introduced a *system of standard measures*, gauges, and screw threads.

Whitworth was born in Stockport, England, and died at Monte Carlo.

1840–1895 CE Leading Western poets and novelists in the Age of Naturalism and Realism:

• Charles Dickens	1812–1870
• Ivan Turgenev	1818–1883
• Walt Whitman	1819–1892
• Charles Baudelaire	1821–1867
• Gustav Flaubert	1821–1880
• Feodor Dostoevsky	1821–1881
• Charles de Coster	1827–1879
• Henrik Ibsen	1828–1906
• Lev N. Tolstoi	1828–1910
• Bjornstjerne Bjornson	1832–1910
• Mark Twain	1835–1910
• Emil Zola	1840–1902
• Edmondo de Amicis	1846–1908
• Henryk Sienkiewicz	1846–1916
• Jens Peter Jacobsen	1847–1885
• August Strindberg	1849–1912
• Guy de Maupassant	1850–1893
• Robert Louis Stevenson	1850–1894
• Anton Chekhov	1860–1904

1841–1847 CE **Edward Forbes** (1815–1854, England). Naturalist and oceanographer. One of the first men to take a scientific interest in the ocean *depths*. As a naturalist on board the surveying ship *H.M.S. Beacon*, he did some dredging in the Aegean Sea, studying the distribution of flora and fauna and their relation to depth, temperature and other factors. His pioneering work led the way to the *Challenger* expedition.

Forbes was born at Douglas, in the Isle of Man. In 1854 he became professor of natural history in the University of Edinburgh, but died soon afterwards.

1841–1852 CE **James Prescott Joule** (1818–1889, England). Physicist. With **J.R. Mayer** and **H.L.F. Helmholtz** established the First Law of Thermodynamics and the mechanical equivalent of heat.

Joule began his work with the discovery of the rate of heat production by an electric current in a conductor and showed it to be proportional to the square of the current strength and the wire resistance (*Joule's Law*, 1841). In 1843, Joule read before the British Association at Cork his paper, entitled: "*The Caloric Effects of Magneto-Electricity and the Mechanical Value of Heat*".

In 1847, he generalized his former results and asserted equivalence and convertibility of heat, mechanical, electrical and chemical forms of energy, rendering some numerical conversion factors. In 1852 he established, with **W. Thomson**, the Joule-Thomson effect.

Joule was born at Salford, near Manchester. Although he received some instruction from **John Dalton** in chemistry, most of his scientific knowledge was self-taught.

1841–1852 CE **David Gruby** (1811–1898, France). Distinguished physician and pioneer in the fields of modern microbiology, veterinary protozoology, and parasitology. Created the field of pathological mycology of humans and pets. First to show that fungi can cause diseases in humans. His decisive and pioneering contribution to the development of microbiology and parasitology has been underestimated in the history of medicine and biology.

Gruby was born in a small village, near Novi-Sad, Hungary, the son of a poor Jewish farmer. He left his home at an early age and went to Budapest and from there to Vienna, where he studied medicine. In Paris (1835) he distinguished himself as a lecturer in the Museum of Natural History. He ceased his research activity in 1852 and dedicated all his time to his medical practice. He was the personal physician of Heinrich Heine and and Alexandre Dumas.

1841–1852 CE **Alexander Bain**[162] (1811–1877, Scotland). A mechanical genius of the first order who came before his time. Clockmaker, inventor, telegraphy pioneer and the 'father of the *facsimile' (fax)*, which can be said to be a primitive forerunner of television. Proposed facsimile

[162] Not to be confused with his namesake and country sake **Alexander Bain** (1818–1908), philosopher and psychologist

telegraph[163] transmission system (1843). Made the first *electric clock* (1841). Invented the first *chemical telegraph* (1846). Details of his inventions are:

- *Electrical clock*: Electromagnetic pendulum is kept going by an electric current instead of springs and weights. He improved on this idea in following patents, and also proposed to derive the motive electricity from an 'earth battery', by burying plates of zinc and copper in the ground.

- *Facsimile*: His method for sending a facsimile image cleverly explored the transmission of electrical signals over telegraph wires. The telegraph was a relatively new device in Bain's day but was rapidly gaining on popularity. Both amateur and professional inventors were trying their utmost to find new ways to use it.

  Bain's operated as follows: the sender write a message on a sheet of conducting tin in non-conducting ink. The sheet was then fixed to a curved metal plate and scanned by a needle controlled by a swinging pendulum. This 'scanner' read the text line by line, point by point at a rate of three lines per millimeter. It emitted an electrical signal, which registered at one strength as it passes through the images' black points (ink) and at another as it passed through the images' white points (absence of ink, i.e. metal). The two distinct signals traveled over the telegraph wire to the receiver where a *synchronized pendulum* controlled a stylus that marked out with Prussian blue ink on a paper soaked in potassium ferrocyanide – leaving behind images of black and white dots that had defined the original text.

  To ensure that both needles scanned at exactly the same rate (so that they would begin and end the scan lines at the same point) two extremely accurate clocks should be used. This, however, could not have been achieved in Bain's time. Other improved on Bain's invention in the years to come.

- *Chemical telegraph*: Bain recognized that the Morse telegraph was comparably slow in speed, owing to the mechanical inertia of the parts; and he saw that if signal currents were made to pass through a ribbon of paper soaked on a solution of iodide of potassium, a brown mark could be made at the point of contact due to liberation of iodine and consequently a very high speed could be obtained.

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163 Note that the first patent on a working fax machine had been filed and granted 33 years *before* Alexander Graham Bell patented his telephone, and even before Bell was even born!

> When the chemical telegraph was tried between Paris and Lille, the speed of signaling attained was 325 words/min as compared to 40 words/min on the Morse electro-magnetic instrument. Others later improved on the neglected method of Bain and reached a recording rate of 113 words/min.

Bain was born of humble parents in the little town of Thurso at the extreme north of Scotland. Learning the art of clock-making, he went to Edinburgh, and subsequently removed to London. By 1870, his royalties from patents for electric telegraph and clocks were exhausted, and he sank into poverty. Moved by this unhappy circumstances, **Lord Kelvin** and **William Siemens** obtained for him from the Prime Minister W. Gladstone (1873) a pension of 80 pound a year; but the beneficiary lived in such obscurity that it was a considerable time before his lodging could be discovered. The Royal Society had previously made him a gift of 150 pounds.

In his later years his health failed and he was removed to the Home for Incurables at Broomhill, Kirkintilloch, where he died. He was a widower, and had two children, but they were said to be abroad at the time, the son in America and a daughter on the continent.

Several of Bain's earlier patents were taken out in two names; owing to his poverty he was compelled to take a partner to share the patent fees. Considering the early date of his achievement, and his lack of education or pecuniary resource, we cannot but wonder at the strength, fecundity, perseverance and prescience of his creative faculty. Beyond a few facts in a little pamphlet (published by himself) there is little to be gathered about his life; a veil of silence had fallen alike upon his triumphs, his errors and his miseries.

1841–1873 CE **David Livingstone** (1813–1873, Scotland and Africa). Missionary, physician, geographer and explorer in Africa. No single African explorer has ever done so much for African geography as Livingstone during his thirty years' work: his travels covered one-third of the continent, extending from the Cape to the equator, and from the Atlantic to the Indian Ocean. He did his journeying leisurely, carefully observing, mapping and recording all that was worthy of note, with rare geographical instinct and with the eye of a trained scientific observer, studying the ways of the people, eating their food, living in their huts, and sympathizing with their joys and sorrows.

In all the countries through which he traveled his memory was cherished by the native tribes as a superior being. Indeed, in the annals of exploration of Africa, he stands preeminent above all. His example and death raised in Europe a powerful feeling against the slaver trade that through him it received its death-blow. The motto of his life was advice he gave some school children in Scotland: "Fear God, and work hard".

Livingstone was born at the village of Blantyre Works, in Lanarkshire, Scotland. Received a degree in medicine from Glasgow University (1840) and then became connected with the London Missionary Society. He went to South Africa (1841) to begin his missionary work. His aims were to convert African natives to Christianity, to put a stop to the slave trade, and to explore the mysterious African continent. He arrived at *Lake Ngami* (1849), *Zambezi River* (1851), and sighted *Victoria Falls* (1855). He explored *Lake Nyasa* region, the *Shire River* (1858), *Lake Shirwa* (1858), reached the southern end of *Lake Tanganyika* (1867) and *Lake Bangweulu* (1868).

Concern over his safety led to the expedition of **Henry Morton Stanley** and the two met near Lake Tanganyika (1871). Livingstone refused to return to the coast with Stanley (1872) and continued his travels for another year. Weakened by illness, he arrived at *Lake Bangweulu* where he died on April 30, 1873. He was later buried at Westminster Abbey in London.

1842 CE **Johann Christian Doppler** (1803–1853, Austria). Physicist. Discovered the law that determines the change of frequency of a moving source of mechanical radiation (*Doppler Effect*)[164]. Since its inception, this law became a major tool in determining translational and angular velocities in all branches of physics and astronomy where the moving bodies might be electrons, satellites, stars or galaxies, in the framework of classical physics, quantum physics and general relativity.

Doppler was born in Salzburg. He was educated at the Polytechnisches Institut in Vienna and became professor of experimental physics at the University of Vienna in 1850.

1842 CE **Julius Robert Mayer** (1814–1878, Germany). Physicist and physician. First to recognize that the law of conservation of energy goes beyond the framework of classical mechanics, without giving this idea a precise mathematical formulation. It is remarkable that in spite of inaccurate reasoning and data of limited quality, he was able to obtain a correct numerical result for the mechanical equivalent of heat. Thus, on account of his boldness, insight and intuition, it can be claimed that he was the father of the First Law of Thermodynamics[165].

Mayer, the son of the owner of an apothecary shop, was born at Heilbronn. He studied medicine at Tübingen, Münich and Paris, and after a journey to

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[164] Later found to apply to electromagnetic radiation as well, with the appropriate relativistic correction.

[165] On this point there was no agreement between **Sommerfeld** on the one hand and **Lord Kelvin**, **Maxwell** and **G.G. Stokes** on the other. The British physicists claimed that distinction for their countryman **J.P. Joule**.

Java in 1840 as surgeon of a Dutch vessel, in the East Indies, obtained a medical post in his native town. It was here, by a curious route, that he was led to the idea of the conservation of energy. That route involved medicine, not physics.

Letting out blood was a common medical cure of the time, and while letting the blood of sailors arriving at Java, Mayer noted that their blood was unusually red. He reasoned that the heat of the tropics reduced the metabolic rate needed to keep the body warm and therefore reduced the amount of oxygen that needed to be extracted from the blood. Accordingly, the sailors had a surplus of oxygen in their blood, causing its extra redness. This hypothesis, and its apparent validation, were taken by Mayer to support the link between heat and chemical energy, the energy released by the combustion of oxygen.

After deciding that there must be a balance between the input of chemical energy and the output of heat in the body, Mayer made a conceptual leap. Friction in the body, from muscular exertion, also produced heat, and the energy associated with this heat also had to be strictly accounted for by the intake of food and its content of chemical energy.

Mayer, being a physician and not a physicist, was at first not familiar with the principles of mechanics, and his first paper on energy had errors. It was rejected. Although disappointed, Mayer immediately took up the study of physics and mathematics, learned about kinetic energy, and submitted a new paper a year later. In 1842 he published a little paper "*Bemerkungen über die Kräfte der unbelebten Natur*" in which he expounded his ideas concerning conversion and conservation of energy. This paper did not receive much attention, but within the next few years other physicists, mainly **J.P. Joule** (1843–1849) and **H.L.F. Helmholtz** (1847) put the First Law on a much firmer foundation.

Despondent over his lack of recognition, Mayer attempted suicide in 1850. He suffered episodes of insanity in the early 1850s and was confined in asylums on several occasions.

After 1860, Mayer was finally given the recognition he deserved. Many of his articles were translated into English, and such well-known scientists as **Rudolph Clausius** in Germany and **John Tyndall** in England began to champion Mayer as the founder of the law of the conservation of energy.

From his marriage Mayer had 7 children, 5 of whom died in childbirth. He died of tuberculosis in 1878.

1842 CE **Joseph Alphonse Adhémar** (1797–1862, France). Mathematician. First to propose an astronomical theory of the ice ages based on the precession of the equinoxes.

Adhémar theorized that glacial climates occur whenever a hemisphere enters winter while at its farthest distance from the sun. Thus, every 11,000 years[166] (every half cycle) an ice age would occur, alternately in one hemisphere and then in the other; a series of abnormally cold winters would allow snow and ice to build up and would pitch the globe into an ice age.

In 1852, **Alexander von Humboldt** (1769–1859, Germany) pointed out that Adhémar's basic idea was incorrect: the average temperature of either hemisphere is controlled not by the number of *hours* of daylight and darkness, but the total number of calories of solar energy received each year. And, as **d'Alembert**'s calculations had demonstrated many years before, any decrease in solar heating that occurs during one season (earth farther from the sun), is exactly balanced by an increase during the opposite season, when the earth is closer to the sun. Therefore, the total amount of heat received by one hemisphere during the year is always the same as that received by the other.

Although Adhémar's theory was proved wrong, it was nevertheless an important step toward understanding the ice age mystery. The idea that *astronomical* phenomena such as the precession of the equinoxes might have a significant effect on the earth's climate was not forgotten, and would set the stage for further discoveries.

1842 CE **Samuel Earnshaw** (1805–1888, England). Mathematician. Showed that a set of physical point-objects (charges, masses, magnetic poles), governed by the classical long-range inverse-square law (electrostatic, magnetostatic, gravitational), cannot be maintained in a stationary stable equilibrium configuration. This is known as 'Earnshaw's Theorem'.

Earnshaw was born in Sheffield and graduated Senior Wrangler and Smith's Prizeman (1831) in Cambridge University. He worked there as tripos coach (1831–1847) and was also appointed to the parish church at St. Michael, Cambridge (1846).

He published several articles and books on classical physics and mathematics, but is best known for his article: "On the Nature of the Molecular Forces which Regulate the Constitution of the Luminiferous Ether" (Trans. Camb. Phil. Soc. 7, 97–112, 1842). Earnshaw's Theorem has consequences for levitation by means of electromagnetic fields, as it shows the impossibility of stable levitating permanent magnets without active control. Note that the

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[166] Previously (1754) it was shown by **d'Alembert** that the simultaneous precession of the equinoxes (due to the combined pull of the sun and the moon), and the precession of the earth's perihelion (due to the perturbation of the planets) cause the earth to undergo a *general* precession of the equinoxes with a period of 22,000 years.

case of a point charge in arbitrary static electric field is a simple consequence of Gauss' law: div$\boldsymbol{E} = -\Delta U = 0$ at field points, where $\boldsymbol{E}(\boldsymbol{r})$ is the electric field and $U(\boldsymbol{r})$ is the potential and curl $\boldsymbol{E} = 0$ in free space. Thus, at any equilibrium point, there must be some direction along which the equilibrium is unstable.

The theorem also means that even dynamic system of charges are unstable in the long term due to EM radiation. This lead, for some time, to the puzzling question of how matter is held together electromagnetically. The answer came via the quantum-mechanical structure of the atom. It was then discovered that the Pauli exclusion principle and the uncertainty principle are responsible for holding bulk matter in a rigid form.

1842–1843 CE (**Augusta**) **Ada Byron**, Countess of Lovelace; 1815–1852, England. An amateur mathematician[167] who wrote the first computer program for Babbage's *analytical machine*. It was a set of instructions for the machine to compute the Bernoulli numbers (the analytical engine never reached the stage of allowing the program to be run).

Ada was the only child of Lord and Lady Byron. She had considerable mathematical talent (in this she took after her mother, who was described by the poet as the 'Princess of Parallelograms'), and frequently visited Babbage while he was working on his engine. In 1840, Babbage gave a series of lectures in Turin. Among his audience was **L.F. Menabrea**[168], a young engineer officer on the staff of the Military Academy in that city. Menabrea wrote an account of Babbage's ideas and published it in a Geneva Journal in 1842. The paper was translated into English by Lady Lovelace, who added extensive notes of her own, and was published in Taylor's *Scientific Memoirs* in 1843. She had a remarkable grasp of Babbage's ideas and her lucid notes make fascinating reading even today.

Her notes are mainly concerned with the formulation of a schedule of instructions (the program) which will enable the machine to carry out a desired calculation automatically. Lady Lovelace went into the subject in considerable detail, and illustrated her points by describing several programs for performing

[167] Ada lost much of her fortune by using her computations to predict horse races.

[168] **Luigi Federico Menabrea** (1809–1896, Italy). Became professor of mechanics at the military academy and at the University of Turin (1842). Embarked in a political career which led him to become Italian Premier and Foreign Minister (1867). During this period of politics he still continued his scientific work, giving the first precise formulation of methods of structural analysis based on the principle of virtual work. Published, jointly with **J.L.F. Bertrand**, the first correct proof of the principle of least work (1870). This was later called the *Castigliano principle*.

advanced mathematical calculations, some of them of considerable sophistication. The two basic features of her programs are: the use of repetitive cycles, whereby the same set of instructions is executed over and over again, and the use of a jump (branching) instruction to enable the calculation to take one or the other of two alternative paths.

In one of her more elaborate programming examples she introduced a number in a certain register for the specific purpose of counting the number of repetitions of a group of instructions. This number is arranged to change sign when the desired number of cycles has been executed, and a jump instruction is inserted to cause the machine to move out of the loop at this point to the next part of the calculation.

Since the punched cards in the **Jacquard** loom (1805) pass through the mechanism in a fixed order which cannot be varied once the loom is set up, Lady Lovelace suggested the provision of an additional *hardware* facility to enable the *backing* of the cards of the analytical engine: the drum over which the train of cards passes must be able to *rotate in the reverse direction* in an amount determined by the program. She wrote:

"*The object of this extension is to secure the possibility of bringing any particular card or set of cards into use any number of times successively in the solution of one problem. The power of repeating the cards reduces to an immense extent the number of cards required. It is obvious that the mechanical improvement is especially applicable wherever* cycles occur in the mathematical operation, *and that, in preparing data for calculations by the Engine, it is desirable to arrange the order and combination of the processes with a view to obtain them as much as possible symmetrically and in cycles, in order that the mechanical advantages of the* backing *system may be applied to the utmost*".

Already in 1842, this remarkable pioneer of modern programming had a full grasp of the 'soul of the computer', as she put it in her unusual clarity: "*The Analytical Engine has no pretensions whatever to originate anything. It can do whatever we* know how to order it *to perform*".

The computer language "Ada" was named by the designers after her.

Ada was the first wife of Baron King, who in 1838 was made earl of Lovelace. They had two sons and a daughter.

1842–1845 CE Physicians in the United States first used *anesthesia* to ease pains during treatment of patients. **Crawford Williamson Long** (1815–1878), surgeon, was first to use *ether* vapor (1842), in Jefferson Ga, to knock a patience unconscious during operation. In 1845, he used ether for the first time in delivering a child; **William Thomas Green Morton** (1819–1868), dentist, administered ether (1846) during a surgical operation at Mass. General

Hospital (Boston). He did this at the suggestion of **Charles Thomas Jackson**[169] (1805–1880), physician, chemist and geologist at the Harvard Medical School. **James Young Simpson** (1811–1870, Scotland), physician, was first to use *chloroform* (1847) to reduce pain at childbirth. This was quicker and more effective than ether; Queen Victoria was one of the first women to be anesthetized during childbirth.

1842–1855 CE **Ludwig Otto Hesse** (1811–1874, Germany). Mathematician. Introduced the '*Hessian normal form*'[170] and also the '*Hessian function*' and the '*Hessian matrix*'[171] that appear in extremalization of real-valued functions and the theory of algebraic invariants.

Hesse was born in Königsberg. His studies at the University of Königsberg (1832–1840) were interrupted for an educational journey throughout Germany and Italy and his subsequent high-school teaching career. Hesse studied

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169 Practiced medicine in Boston (1832–1836) but abandoned medicine to work in chemistry and mineralogy (1836). Claimed to have pointed out to S.F.B. Morse the basic principles of the electric telegraph; also claimed the priority in discovery of guncotton (explosive). In 1852, the French Academy awarded a prize of 5000 francs jointly to Jackson and Morton. Both men claimed sole credit for the discovery, and Morton refused to share the prize with Jackson. A bitter quarrel and lawsuits followed, and Morton was ruined financially. Long did not get any credit since he published the account of his early discovery only in 1849.

170 The *vector form* of a plane in space is $\boldsymbol{x} \cdot \boldsymbol{x}_1 + d = 0$, where $\boldsymbol{x} = (x, y, z)$ and $\boldsymbol{x}_1 = (a, b, c)$, i.e. $ax + by + cz + d = 0$. If this equation is divided by the normalization factor $\pm(a^2 + b^2 + c^2)^{1/2}$, one arrives the *Hessian normal form*

$$x \cos\alpha + y \cos\beta + z \cos\gamma - p = 0,$$

where (α, β, γ) are the direction cosines of the normal $\boldsymbol{x}_1$, and p is the distance of the plane from the origin.

171 Let P be some particular point chosen as the origin of a coordinate system with coordinates $\boldsymbol{x}$. Then, any function f can be approximated by its Taylor series

$$f(\boldsymbol{x}) = f(P) + \boldsymbol{x} \cdot \operatorname{grad} f|_P + \frac{1}{2}\boldsymbol{x}\boldsymbol{x} \colon \operatorname{grad}\operatorname{grad} f|_P + \cdots$$

The matrix

$$(\operatorname{grad}\operatorname{grad} f)_{ij} \equiv \frac{\partial^2 f}{\partial x_i \partial x_j}|_P,$$

whose components are the second partial derivative matrices of the function, is called the *Hessian matrix* of the function at P. The determinant functional $H[f] = \|\nabla\nabla f\|$, known as the *Hessian determinant*, has found many applications in algebraic geometry.

For example, if in the bilinear form

mainly under **C.G.J. Jacobi**, who greatly stimulated his mathematical investigations.

He then taught at Königsberg (1845–1855), Heidelberg (1856–1868) and Münich (1868–1874). He became an ordinary professor in 1855 and among his students were **G. Kirchhoff**, **S.H. Aronhold**, **C. Neumann**, **A. Clebsch** and **R.O.S. Lipschitz**.

1842–1866 CE **Siemens**: Name of German brothers: inventors, electrical engineers and industrialists.

- **Ernst Werner von Siemens** (1816–1892); invented the *electroplating process* (1842); invented the *dial telegraph* (1846); laid an underground electric telegraph for the army (1847). Founded the Siemens firm for manufacture of telegraphic equipment and electrical apparatus (1847). Laid cables across the Mediterranean and from Europe to India; invented the self-excited generator (1866).
- **Karl Wilhelm (Charles William) Siemens** (1823–1893). Naturalized British citizen (1859). Invented the regenerative steam engine

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$$f(x_1, x_2) = a_0x_1^2 + 2a_1x_1x_2 + a_2x_1^2$$

we effect the linear transformation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix},$$

we obtain the new bilinear form

$$F(X_1, X_2) = A_0X_1^2 + 2A_1X_1X_2 + A_2X_2^2$$

where

$$A_0A_2 - A_1^2 = (a_0a_2 - a_1^2)D^2,$$

and

$$D = \det \begin{bmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{bmatrix} = \xi_1\eta_2 - \xi_2\eta_1.$$

It is also true that $H[F] = D^2H[f]$, where under an orthogonal transformation $D^2 = 1$; the Hessian functional is

$$H[f] = \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2 = \text{Hessian of } f = \det \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}.$$

(1847) and Siemens process for making steel (1861). Laid the first cable between Britain and the USA (1875).

- **Friedrich Siemens** (1826–1904). Invented (1856) a regenerative smelting oven widely used in the glass and steel making industries.
- **Karl von Siemens** (1829–1906). Organized and directed the Russian branch of the firm.

1843 CE **Pierre Alphonse Laurent** (1813–1854, France). Engineer and mathematician. Discovered the relationship between power series and analytic functions in a domain bounded by two concentric circles[172].

Laurent was born in Paris. After studying for two years at the École Polytechnique, he joined the army engineering corps and took part in the expeditions to Algeria. He then returned to France and spent about 6 years

[172] *Laurent series*: Let $f(z)$ be a function analytic in the ring-shaped region between two concentric circles C and C', of radii $R' < R$, and center a, and on the circles themselves. Then $f(z)$ can be expanded in a series of positive and negative powers of $z-a$, convergent at all points of the ring-shaped region:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n;$$

$$a_n = \frac{1}{2\pi i}\int \frac{f(\omega)}{(\omega-a)^{n+1}}d\omega,$$

for all values of n.
The integral is taken round any simple closed contour within the region. In the particular case where $f(z)$ is analytic inside C', all the $n<0$ coefficients are zero (by Cauchy's theorem), and the series reduces to a Taylor series.
In the neighborhood of a pole of the n^{th} order, the Laurent series are truncated at the negative power with exponent n. In a similar way, a function defined by a Laurent series has a pole of the n^{th} order at infinity when the terms with positive powers ends at the term with power n.
If $f(z)$ is regular in an arbitrary narrow annulus

$$1-\varepsilon < |z| < 1+\varepsilon \qquad (0<\varepsilon<1),$$

it can be represented there by the series $\sum_{n=-\infty}^{\infty} c_n z^n$, where

$$c_n = \frac{1}{2\pi i}\oint_{|z|=1}\frac{f(\zeta)d\zeta}{\zeta^{n+1}} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta \qquad (n=0,\pm1,\pm2,\ldots).$$

In particular, at the points $z=e^{it}$ on the *unit circle*, $f(z)$ is represented by

in the port of Le Havre, directing hydraulic construction projects. In the midst of these technical operations he submitted two memoirs to the Academy of Sciences, one of which dealt with the 'Laurent series expansion'. It was communicated by **Cauchy**, but due the negligence of the latter was never published by the Academy. It did not appear until 1863, when it was published in the Journal de l'École Polytechnique.

1843 CE **Heinrich Samuel Schwabe** (1789–1875, Germany). Apothecary and amateur astronomer. Discovered the 11-year cycle of solar activity, the sunspot cycle; also made (1831) the first known detailed drawing of the Great Red Spot on Jupiter.

In 1825, Schwabe began to study the sun and its sunspots. He spent 17 years looking at it (with the proper precautions to avoid blindness) and discovered that the number of spots rose and fell in what seemed a cycle of 10 years (more like 11, according to continuing studies by others). This contributed to the beginning of the science of *astrophysics*, the study of physical phenomena in stars and other objects in the universe.

1843 CE **John Stuart Mill** (1806–1873, England). Philosopher and logician. Delineated the foundations of *inductive logic* and the scientific method in his book *A System of Logic* (1843). He applied principles of Empiricism to the scientific method, interpreted as a system of inductive logic. Rounding out and perfecting Francis Bacon's inductive technique, he advocated induction as a new approach of problem-solving that would supersede the Aristotelian method of deductive logic.

Mill was born in London and educated completely by his father. He began to study Greek at the age of 3, and at 14, had mastered Latin, classical literature, logic, history, political economy and mathematics. He entered the East India Company at the age of 17. Like his father, he became director of the company. He retired after 33 years of service and was elected member of Parliament (1865).

$$F(t) = f(e^{it}) = \sum_{n=-\infty}^{\infty} c_n e^{int},$$

where

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} F(\theta)e^{-in\theta}d\theta \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

This is none other then the complex form of the Fourier series of $F(t)$. Thus, on the unit circle $z = e^{it}$, the Laurent series, considered as a function of the real variable t, is the *Fourier series* of the function $F(t) = F(e^{it})$.

The main task of the *System of Logic*[173] is the analysis of inductive proof. His canons of inductive methods for comprehending the causal relations between phenomena are valid under the assumption of the validity of the *law of causality*, which cannot be accepted except on the basis of induction — making the whole argument circular.

1843–1849 CE **Søren Aabye Kierkegaard** (1813–1855, Denmark). Philosopher. Attacked social and religious complacency. His assault on institutional Christianity and on traditional Western philosophy generated a crisis that produced a radically new way of philosophizing and made him the founder of a school that would later be called *Existentialism* – centered on the obsession with the particularity of human existence. To Kierkegaard, reality was personal, subjective – it began and ended with the individual. To him, **Hegel**'s system[174] was an immense fraud which, by its verbose techniques of reconcilia-

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[173] Mill presented five rules (canons) of inductive reasoning:

(1) *Rule of Agreement*: If two or more instances of the phenomenon under investigation have only one circumstance in common, the circumstance in which alone all the instances agree is the cause (or effect) of the given phenomenon.

(2) *Rule of Difference*: If an instance in which the phenomenon under investigation occurs, and an instance in which it does not occur, have every circumstance save one in common, that one occurring only in the former, the circumstance in which alone the two instances differ is the effect, or cause or a necessary part of the cause, of the phenomenon.

(3) *Joint Rule of Agreement and Difference*: If two or more instances in which the phenomenon occurs have only one circumstance in common, while two or more instances in which it does not occur have nothing in common save the absence of that circumstance, the circumstance in which alone the two sets of instances differ, is the effect, or cause, or a necessary part of the cause, of the phenomenon.

(4) *Rule of Residues*: Subduct (subtract) from any phenomenon such part as is known by previous inductions to be the effect of certain antecedents, and the residue of the phenomenon is the effect of the remaining antecedents.

(5) *Rule of Concomitant Variations*: Whatever phenomenon varies in any manner whenever another phenomenon varies in some particular, is either a cause or an effect of that phenomenon, or is connected with it through some fact of causation.

His rule (1) is almost identical to the hermeneutic Talmudic rule *Binyan-Av* of **Hillel** (ca 32 BCE). It was used by the latter as a tool of logical deduction from the juxtaposition of two legal sections.

[174] **Hegel** had treated the human individual as an insubstantial being of secondary importance to social institutions and the state. He saw God as a poetic myth anticipating in a primitive way the higher philosophical truth embodied in his own doctrine of the Absolute.

tion and rationalization, refused to confront the actual circumstances of man in the world, and in particular the fact of deaths and the remoteness and inscrutability of God.

Kierkegaard argued that actual existence, as we experience it in life, cannot be rationalized in Hegel's way: men and the world they inhibit cannot be tidily explained. Belief in God is not the solution to a theoretical problem but a free act of faith. In this framework he saw the Protestant Church as a means of perverting its original messages, and the very symbol of self-satisfied bourgeois snugness that stands between the individual and the truth. *Knowledge*, as Kierkegaard construes it, is always abstract but *existence* cannot be thought, because it is always concrete. Existence must, at its very core, be experienced as anguish or dread of the possibility of death at any moment.

Kierkegaard took the position that religion was a personal experience. He divided experience into three categories: *aesthetic*, *ethical*, and *religious*. The child is an example of an individual who lives almost exclusively at the aesthetic level: all choices are made in terms of pleasure and pain, and experience is ephemeral, having no continuity, no meaning, but being merely a connection of isolated, non-related moments. The *ethical* level of experiences involves choices, whenever conscious choice is made, one lives at the ethical level. At the *religious* level, one experiences a commitment to oneself, and an awareness of one's uniqueness and singleness. To live in the religious level means to make any sacrifice, any antisocial gesture that is required by being true to oneself.

These levels are not mutually exclusive but may coexist. He concluded that only when man experiences the suffering of firm commitment to the religious level of experience can he be considered truly religious. If religion then is a purely personal matter, truth is clearly subjective, quite separate from the "truth" of religious doctrine. Objective truth, such as that of geometry, is acquired by the *intellect*; subjective truth must be experienced by the total individual. One may *have* objective truth, but one must *be* religious truth.

His main works: *Fear and Trembling* (1843); *Either/Or* (1843); *The Concept of Dread* (1844);*The Sickness unto Death* (1849).

Kierkegaard lived most of his life in Copenhagen. In 1840 he became engaged to a 17-year old girl, but he broke off the engagement after about a year. Their affair continued to haunt him throughout his life. For two years he traveled in Germany (1840–1842) where he studied Hegelian philosophy at the Berlin university under Schelling with his classmates **Friedrich Engels**, **Ludwig Feuerbach** and **Michael Bakunin**. Kierkegaard died in the middle his violent battle against the Lutheran Church establishment in Denmark; he died a lonely man with hardly a follower.

There were riots at his funeral, caused by theology student's outrage at the way the Church tried to take over in death the man who had opposed it so bitterly with his last breath. He had wanted to have written on his tombstone simply "THE INDIVIDUAL".

In his short life, Kierkegaard wrote more than twenty-five books. After his death, his works slipped into obscurity.

He has been 'discovered' only in the 20th century, and has influenced both modern Protestant theology and the Existentialist philosophy (e.g. of **Heidegger**). **Jean-Paul Sartre** said of Kierkegaard: "I want to catch hold of him, and it is myself I catch".

Worldview XX: Kierkegaard

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"To seek objectivity is to be in error."[175]

* * *

"Existing is a form of doing, not a form of thinking."

* * *

"People demand freedom of speech to make up for the freedom of thought which they avoid."

* * *

"Wherever there is a crowd there is untruth."

* * *

"Life can only be understood backwards; but it must be lived forwards."

* * *

"The tyrant dies and his rule is over; the martyr dies and his rule begins."

* * *

"It is not true that the scientist goes after truth. It goes after him."

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[175] Compare with the cynical wit of **Oscar Wilde** (1856–1900): "It is only about things that do not interest one that one can give a really unbiased opinion, which is no doubt the reason why an unbiased opinion is always absolutely valueless."

Kierkegaard probably got this skeptical view from reading **Hume** (1711–1776) who held that no *moral* claim can ever be grounded in objective fact.

1843–1857 CE **Ernst Eduard Kummer** (1810–1893, Germany). Mathematician. Showed that the Fermat conjecture was true for all prime powers smaller than 100 except 37, 59 and 67. [First three '*irregular primes*'.] In his efforts to prove the conjecture, Kummer extended (1846) the notion of 'prime' numbers in the integers to general algebraic domains, preserving the existence of a unique factorization into prime factors. The prime factors in the more general domains are called 'ideal numbers'. In 1849 he extended the theory of Gaussian complex numbers.

1843–1857 CE **John James Waterstone** (1811–1883, Scotland). Physicist and engineer. First formulated the essential features of the kinetic theory of gases. Submitted to the Royal Society (1845) a speculative memoir on gases linking heat with molecular motion. In it he included a calculation of the ratio of specific heats at constant temperature and constant volume. The memoir was dismissed by the referees as '*nothing but nonsense*'. In 1892 it was reproduced in complete form by **Rayleigh** (1842–1919). However, many of Waterstone's key ideas had by then been published by **Clausius** and **Maxwell**. Waterstone's misfortune[176] resulted from the fact that the idea of *energy conservation* became accepted only in 1858. Thus the rejection of his work delayed progress by about 15 years.

Waterstone was born and educated in Edinburgh. He moved to London (1833) to do surveying for the railways, then took a job in the Hydrographers' Department of the Admiralty. Went to India (1839) as teacher of the East India Company's cadets in Bombay. He returned to Edinburgh (1857) to devote all his efforts to research. His work, however, was repeatedly rejected or ignored, causing him to withdraw from the scientific community.

Waterstone wrote other papers on gravitation, sound, capillarity, physiology, latent heat and various aspects of astronomy. He also estimated the temperature of the sun (1857).

1843–1858 CE **Haim Zelig Slonimsky** (1810–1904, Poland). Mathematician, astronomer and inventor. Made important contribution to the study of the Hebrew calendar (1852). Invented a novel calculating machine (1843) and developed a method of delivering simultaneously four messages via a telegraph wire. Befriended the German astronomers and mathematicians **Bessel**, **Crelle**, **Encke** and **Jacobi** and especially **Alexander von Humboldt** who remained his lifelong friend.

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[176] He did not even merit a mention in the *Britannica's* 11th edition (1910)!

Slonimsky was trained as a rabbinical Talmudic scholar up to his 18th year. After his marriage (1828), he self-educated for six years in the home of his father-in-law, publishing a number of books on various mathematical and astronomical subjects, and also mathematical papers in the Crelle Journal. His *Yesodei ha-Ibur* (Foundations of the Calendar, 1844) is still the seminal work in this field.

Slonimsky was born in Bialystok and later lived in Warsaw. The Russian government appointed him supervisor of the rabbinical academies (1862). In the same year he began publishing the Hebrew scientific weekly, *Hazefira*, which turned (1886) into daily newspaper. Slonimsky's grandchildren became prominent figures in the Polish and Russian literature.

1843–1865 CE **Claude Bernard** (1813–1878, France). Physiologist. Investigated chemical phenomena of *digestion*, discovering role of pancreas in digestion of fat and the glycogenic function of the liver; discovered regulation of blood supply by vasomotor nerves.

In an attempt to prove that animals could *synthesize* food materials in their bodies instead of having to obtain all nutrients from plant life, Bernard discovered that the liver could serve as a source of blood sugar (1843); in 1857 he indeed announced the isolation of *glycogen* from the liver[177]. Thus, the foundations for an understanding of *carbohydrate metabolism* had been laid, though the real comprehension of the reaction involved had to wait until the structure of the sugars had been worked out.

Bernard was a professor at the Sorbonne (1854), College de France (1855–1868), and the Musée d'Histoire Naturelle (1868–1878).

1843–1873 CE **Charles Hermite** (1822–1901, France). One of the eminent French mathematicians of the 19^{th} century. A professor at the Sorbonne (1869) and the teacher of **Picard** (1856–1911), **Borel** (1871–1956) and **Poincaré** (1854–1912).

Abel had proven in 1824 that the quintic equation cannot be solved by functions involving only rational operations and root extractions. One of Hermite's most surprising achievements (1858) was to show that this equation can be solved by *elliptic functions*. In 1873 he proved that e is transcendental[178].

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177 *Glycogen* was independently discovered (1857) by **Viktor Hensen** (1835–1924, Germany), a medical student who worked under **Rudolf Virchow** (1821–1902, Germany).

178 In a sense this is paradigmatic of all the discoveries of Hermite. By a slight adaptation of Hermite's proof, **Felix Lindemann** (1882) obtained the much more exciting transcendence of π. Thus, Lindemann, a mediocre mathemati-

Hermite was born in Dieuze, Lorraine, the sixth of seven children. His father, Ferdinand, a man of strong artistic inclination who had studied engineering, entrusted his draper's trade to his wife, Madeleine, in order to give full rein to his artistic bent. Around 1829 Charles' parents transferred their business to Nancy. They were not much interested in the education of their children, but Charles continued his studies in Paris; his mathematics professor was the same Richard who 15 years earlier had taught Évariste Galois. So, instead of seriously preparing for his examination Hermite, at the age of 17, read Euler, Gauss and Lagrange. He was thus admitted to the École Polytechnique in 1842 with the poor rank of 68. After a year's study, he was refused further study, because of a congenital defect of his right foot, which obliged him to use a cane.

At this time, Hermite resembled a Galois resurrected: Owing to the intervention of influential people the decision was reversed, but under conditions to which Hermite was reluctant to submit and he declined the paramount honor of graduating from the École Polytechnique, contenting himself with the career of a high school teacher. In 1847 he became acquainted with Jacobi's work on elliptic and hyperelliptic functions, and already in 1843, at the age of 20, he was able to generalize some of Abel's results, thus placing himself in the ranks of the first analysts. He communicated his discovery to **Jacobi**, who did not conceal his delight.

The association of Hermite with the École Polytechnique was resumed in 1848 and through the influence of **Pasteur** (1822–1895), a special position was created there for him. In 1869 he took over **J.M.C. Duhamel**'s chair as professor of analysis both at the École Polytechnique and the Sorbonne positions which he kept until 1876 and 1897 respectively. He was an honorary member of a great many academies and learned societies, and was awarded many decorations. His 70^{th} birthday gave scientific Europe the opportunity to pay homage in a way accorded very few mathematicians.

Hermite married the sister of **Joseph Bertrand** (1822–1900); one of his daughters married **Émile Picard** (1856–1941). Hermite was seriously ill with smallpox in 1856, and under **Cauchy**'s (1789–1857) influence became a devout Catholic. He studied Sanskrit and ancient Persian.

Throughout his life Hermite exerted a great scientific influence by his correspondence with other prominent mathematicians. If Hermite's work were

cian, became even more famous than Hermite — for a discovery for which Hermite had laid all the groundwork and that he had come within a gnat's eye of making. [The irrationality of π and e had previously been demonstrated by **Lambert** (1776).] Hermite also produced an 'artificial' new transcendental number $\sum_{n=0}^{\infty} 2^{-n!}$.

scrutinized more closely, one might find more instances of Hermitian preludes to important discoveries by others, since it was his habit to disseminate his knowledge lavishly in correspondence, in his courses, and in short notes. His correspondence with T.J. **Stieltjes**, for instance, consisted of at least 432 letters written by both between 1882 and 1894. Hermite's most important results have been so solidly incorporated into more general structures that they are rarely attributed to him.

Several of his purely mathematical discoveries had unexpected applications many years later in mathematical physics: *Hermitian forms* and *matrices* which he invented in connection with certain problems of number theory turned out to be crucial for **Heisenberg**'s 1925 formulation of quantum mechanics, and *Hermite polynomials* and *functions* appear in the solution of **Schrödinger**'s wave equation for a harmonic oscillator, as well as in solutions of the *classical* wave equation representing narrow beams.

1843–1876 CE **George Gabriel Stokes** (1819–1903, Ireland and England). A British mathematician and physicist with an extraordinary combination of mathematical prowess and experimental skill. His contributions range from optics, acoustics, and hydrodynamics to viscous fluid problems (a unit of viscosity is named for him).

Stokes was born in Skreen, Ireland. He entered Bristol College at 16 and matriculated at Pembroke College, Cambridge, in 1837. He became a Fellow of Pembroke College in 1841 and in 1849 received the Lucasian Professorship of mathematics at Cambridge, held by **Airy** from 1826. Baron since 1889, member of parliament (1887–1892) and president of the Royal Society (1885–1890).

In 1843 he gave a new deduction of the general equation of viscous flow (discovered by **Navier** in 1823; *Navier-Stokes equation*[179]). Anticipated the instability of laminar flow patterns. In 1847 he created the concept of *uniform convergence* of series. In 1849 he conceived the first mathematical model of a point source in an elastic solid ('luminiferous ether'), treating light as a transverse wave in the elastic ether.

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[179] The *Navier-Stokes equation* governs the motion of *Newtonian fluids* (viscous fluids for which the shearing stress in linearly related to its rate of deformation).

The laws of conservation of mass, linear momentum, and angular momentum lead directly to the two basic field equations:

Stokes' theorem was discovered by William Thomson (Lord Kelvin) and communicated to his friend Stokes in a postscript to a letter of July 2, 1850. Stokes replied that the result was very elegant and new to him and that he had constructed his own proof. He never claimed it as his own or published a proof, but he did include a question on it in the Smith's Prize Examination for 1854 [a competitive examination given to the best mathematics students at Cambridge University]. One of the students who took the 1854 examination, and who tied for first place on it, was **James Clerk Maxwell** (1831–1879). Stokes' theorem is of critical importance in electromagnetic theory and in the formulation of Maxwell's equations.

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{V}) = 0$$

(*equation of continuity*: $\rho(\boldsymbol{r}, t) =$ density, $\boldsymbol{V}(\boldsymbol{r}, t) =$ particle velocity),

$$\operatorname{div} \mathfrak{T} + \rho \boldsymbol{F} = \rho \frac{D\boldsymbol{V}}{Dt}$$

(*Euler's equation of motion* relative to an inertial frame,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{V} \cdot \boldsymbol{\nabla};$$

$\boldsymbol{F} =$ force per unit mass, $\mathfrak{T} =$ stress tensor).
The adequate stress tensor for isotropic linear viscous fluid (*Navier-Poisson law*) is derived on the basis of experimental evidence:

$$\mathfrak{T} = -p\mathfrak{I} + \left(\overline{\lambda} - \frac{2}{3}\eta\right) \mathfrak{I} \operatorname{div} \boldsymbol{V} + \eta(\nabla \boldsymbol{V} + \boldsymbol{V}\nabla),$$

where I is the unit dyadic, $\overline{\lambda}$ is the *bulk viscosity* and η the *shear viscosity*. Assuming uniform $\overline{\lambda}$, η, a substitution of the explicit form of $\mathfrak{T}$ in Euler's equation yields the *Navier-Stokes equation* (non-linear in $\boldsymbol{V}$):

$$\rho \frac{D\boldsymbol{V}}{Dt} = \rho \boldsymbol{F} - \operatorname{grad} p + \eta \nabla^2 \boldsymbol{V} + \left(\overline{\lambda} + \frac{1}{3}\eta\right) \operatorname{grad} \operatorname{div} \boldsymbol{V}.$$

We thus have 4 scalar equations in the 5 unknown functions $(\boldsymbol{V}, \rho, p)$. The *equation of state* $p = p(\rho)$ supplies the missing relation.

Introduced the *Stokes parameters* (1852), useful in the experimental determination of the state of polarization of a light beam. Derived the expression for the *drag force* on a sphere moving slowly (small 'Reynolds numbers') in a viscous fluid [$F = 6\pi a\eta u$; $a =$ sphere's radius[180], u its velocity, $\eta =$ dynamic viscosity. It was used by **Millikan** in his famous experiment to determine the charge of the electron (1910) and by **Einstein** in analyzing Brownian motion in external fields].

Stokes named and explained the phenomenon of *fluorescence* (1852).

Discovered the *Stokes phenomenon* (1857), namely — the discontinuity of the constants in the asymptotic expansion of integral functions. Stokes illustrated the change with the aid of Bessel functions whose orders are 0 and $\frac{1}{3}$, the latter being those associated with the **Airy** integral. On this discovery, **George Neville Watson** (1886–1965) remarked that "*the discovery was apparently one of those which are made at three o'clock in the morning*".

Stokes was first to derive an analytical expression for *group velocity*[181] (1876).

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180 This law is still valid in the form $F = A\eta u$ for non-solid and non-spherical objects, where the parameter A depends on the shape of the body, and upon its physical state. For example, $A = 4\pi a$ (air bubble), $16a$ (disk, moving face-on), $\frac{32}{3}a$ (disk, moving edge-on), $12a$ [disk, moving at random. The 'addition law' is

$$\left(\frac{1}{A}\right)_{\text{random}} = \frac{1}{3}\left(\frac{1}{A_x} + \frac{1}{A_y} + \frac{1}{A_z}\right)].$$

Thus we have:

$$A = \frac{4\pi a}{\log_e \frac{2a}{b} - \frac{1}{2}}$$

(Ellipsoid, $a \gg b$, moving lengthwise), and

$$A = \frac{8\pi a}{\log_e \frac{2a}{b} + \frac{1}{2}}$$

(Ellipsoid, $a \gg b$, moving sidewise).

181 Subsequently developed by **Rayleigh** (1877). It appears however that as early as 1839 **Hamilton** had made investigations into the velocity of advance of a finite train of waves in a dispersive medium, but his researches were only published in short abstracts and have been entirely overlooked until recently. Also in 1839, **George Green** derived the formula for the *phase velocity* of water waves in terms of wavelength ["*Note on the Motion of Waves in Canals*",

1843–1889 CE **Joseph-Louis-Francois Bertrand** (1822–1900, France). Mathematician. Known for his contributions to differential geometry, number theory and probability theory.

Conjectured (1845) that there is at least one prime number between n and $2n-2$ for $n > 3$. This was proved by **Chebyshev** (1850). His book *Calcul des probabilités* (1889) contains what Poincaré later called, *Bertrand's paradox*[182] and *Bertrand's coin problem*[183].

1844 CE **Johann Martin Zacharias Dase** (1824–1861, Germany). A calculating prodigy who calculated π correctly to 200 decimal places in less than two months; using the formula

$$\frac{\pi}{4} = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right),$$

with a series expansion for each arctangent [the '*Gregory-Leibniz*' formula is not suitable for practical calculation of π, since one would need 100,000 terms to calculate π to 5 decimal places].

Dase gave exhibitions of his extraordinary calculating prowess in Germany, Austria and England. During an exhibition in Vienna in 1840 he made acquaintance with **Schultz von Strassnitzky** (1803–1852, Austria), who urged him to make use of his powers for the calculation of mathematical tables.

Dase calculated the natural logarithms of the first 1,005,000 numbers, each to 7 decimal places, in his spare time in 1844–1847, when employed by the Prussian Survey. On the recommendation of **Gauss**, the Hamburg Academy of Sciences agreed to assist him financially, for a preparation of table of factors

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Trans. Camb. Phil. Soc. **7**, 87–95]. Later extensions have originated in **Kelvin**'s *method of stationary-phase* (1887).

[182] *Bertrand's paradox*: A chord is chosen randomly in a circle of radius r. What is the probability that the length X of the chord will be less than the radius r? The answer depends on the method for randomly choosing points to determine the chord. In this manner one is able to obtain *distinct answers* for the probability $P(X < r)$. The importance of the paradox lies in that it serves as a warning to all persons who adopt practical policies on the basis of theoretical solutions, without first establishing that the *assumptions* underlying the solutions are in good accord with the experimentally observed facts.

[183] Of 3 identical boxes, one contains 2 gold coins, one contains a gold and a silver coin, and the third contains 2 silver coins; a box is selected at random and a coin taken from it. Given that the chosen coin is gold, what is the probability that the other coin in the selected box is also gold?

of all numbers from 7 to 10 million. He died in 1861, after he had finished about half of it.

Calculating Prodigies of the 18^{th} and 19^{th} centuries

Among the self-taught calculators who showed their power in youth[184] *were:* ***Jedediah Buxton*** *(1707–1772, England). With the exception of his power of dealing with large numbers, his mental faculties were of low order and he remained throughout his life a farm laborer. He could calculate 2^{140}. When asked later to square this number, he gave the answer $2\frac{1}{2}$ months later, and he said he had carried on the calculations at intervals during that period (he could not read or write). In 1754 he reached London and was examined by various members of the Royal Society. It was suggested that he counted by multiples of 60 and of 15 and thus reduced the multiplication to addition.*

Of billions, trillions, etc. he had never heard, and in order to represent the high numbers required in some of the questions posed to him, he invented a notation of his own, calling 10^{18} a tribe*, and 10^{36} a* cramp*. He could stop in the middle of a piece of mental calculation, take up other subjects, and after an interval of weeks, could resume the consideration of the problem.*

Zerah Colburn *(1804–1840, U.S.A.) showed extraordinary powers of mental calculation while still less than 6 years old, which were displayed in a tour of America and later in London (1812). He was born at Cabut, Vermont, the son of a small farmer. At the age of 8 he could calculate 8^{16} in a few seconds. He gave the answer to such questions so rapidly that the gentleman who was taking them down was obliged to ask him to repeat them more slowly. His power of factorizing numbers less than a million was exceptional. In 1814, his English and American friends raised money for his education. With education, his calculating powers fell off.*

George Parker Bidder *(1806–1878, England) had mental capabilities similar to those of Colburn. He could calculate the square root of 119,550,669,121 in 30 seconds. Bidder later graduated from the University*

[184] Excluding *educated* prodigies who channeled their energy into rational mathematics, such as **Wallis**, **Ampère**, **Gauss**, **Ramanujan** and others.

of Edinburgh as a civil engineer and rose to high distinction. He retained his power of rapid mental calculation to the end of his life. Other members of his family have also shown exceptional powers of a similar kind as well as an extraordinary memory.

Jacques Inaudi *(1867–1939, Italy) was employed in his early years as a shepherd and was ignorant of reading and writing even in his teens. He could find integral roots of equations and could represent numbers less than* 10^5 *as a sum of four squares in a minute or two. He could mentally reproduce the sound of the declamation of the numbers' digits in his own voice, and was confused, rather than helped, if the numbers were shown him in writing. A number of 24 digits, having been read to him in 59 seconds, was memorized by its sound. His memory was excellent for numbers, but normal or subnormal for other things.*

Most of these calculating prodigies found it difficult or impossible to explain their methods. There are a few analyses by competent observers of the processes used, notably of Bidder on his own work and that of ***Darboux*** *of Inaudi.*

[Bidder performed multiplication, say of 397×173, *by forming the product* $(100 + 73 + 3)$ *and* $(300 + 90 + 7)$ *and adding up all the partial products. This method he used even when multiplying a 9 digit number by another 9 digit number.]*

Dase visualized recorded numbers, working in much the same way as with pencil and paper, while Bidder made no use of symbols and recorded successive results verbally in a sort of cinematographic way.

In multiplication of a number of n *digits, the strain on the mind varied approximately as* n^x *(measuring it by the time taken in answering the question) where* $x \sim 5$ *for Bidder and* $x \sim 3$ *for Dase.*

1844 CE **Hermann Günther Grassmann** (1809–1877, Germany). Mathematician. The harbinger of modern abstract algebra, especially vector and polyadic algebra.

In his book '*Ausdehnungslehre*' (1844) he developed a mathematical system involving a theoretical algebraic structure (calculus of linear extensions) on which geometry of any number of dimensions in affine and metric spaces could be based. He used invariant symbolism in which we now recognize *vector* and *tensor* (dyadic) notation. His "gap" products correspond to Gibbs'

later 'indeterminate products'. Vector addition and subtraction, the two major kinds of vectorial products, vector differentiation and the elements of the linear vector function were all presented in forms either equivalent or nearly equivalent to their modern counterparts.

His '*Ausdehnungslehre*' includes the concept of hypercomplex numbers and their algebras, and Hamilton's algebra and matrix algebra are just special cases of his broader concepts — which embraces even the tensor algebra of general relativity.

Grassmann never attended a university mathematical lecture, and the great mathematicians of his day such as **Gauss**, **Kummer**, **Möbius**, **Hamilton** and others, failed to appreciate the greatness of his achievement. Thus, his ideas were overlooked in the main during his lifetime, and their importance was not recognized until the twentieth century. A later generation utilized parts of Grassmann's structure to build up vector and dyadic analysis for affine and metric spaces. All in all, the geometrical tradition of Hamilton and Grassmann led to the extremely useful vector algebras of classical mechanics and mathematical physics and eventually to tensor algebra and calculus.

Furthermore, Grassmann's non-commutative algebra was implemented in the matrix mechanics of quantum theory by **Werner Heisenberg** (1901–1976, Germany, 1925). It seems probable that Grassmann did not anticipate any such outcome for his extremely general 'geometric algebra'[185].

Grassmann was a high-school teacher in Stettin, Germany. His father, Justus Günther Grassmann once said: "*I would be happy if Hermann became a gardener or a craftsman*".

1844 CE **Gabriel Gustav Valentin** (1810–1883, Germany). Physiologist and physician. Discovered that pancreatic juice breaks down food in digestion. Contributed to the physiology of metabolism, the digestive tract and the nervous system.

Valentin was born to Jewish parents in Breslau. He became a professor of physiology in the University of Bern (1836).

1844–1859 CE **Carl Friedrich Wilhelm Ludwig** (1816–1895, Germany). Physiologist. One of the founders of physiochemical school of physiology. Helped create an autonomous discipline of physiology, with its research schools, professional societies and specialized journals.

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185 In 1845 the French engineer **Adhémar, Comte de Saint-Venant** (1797–1866) exposed mathematical ideas similar to those which are present in the Grassmannian system. Among other things he defined the dyadic product of two vectors.

Ludwig was born in Witzenhausen, Hesse and studied at Marburg (though temporarily compelled to leave the university as a result of his political activities[186]), Erlangen, and the surgical school in Bamberg. He wasprofessor at Marburg (1846–1849), Zürich (1849–1855), Vienna (1855–1865), Leipzig (1865–1895).

Ludwig showed (1844) that the epithelium of the kidney tubules serve as a passive filter in urine production. Demonstrated the influence of nerves on the distribution of blood and on the secretion of the glands. Developed (1846) the *kymograph* — first physical device for a continuous recording of *blood pressure*[187] and other physiological or muscular processes. Proved (1854) that blood circulation is purely mechanical, such that no mysterious *vital processes* outside ordinary physics need to be involved. First to keep animal organs alive in vitro outside the body, which he achieved by pumping blood through them (1859). Devised medical instruments, useful especially in diagnostic technology. Energetic and influential teacher. Sought explanation of living processes in the paradigms of physics and chemistry (reductionism).

1844–1871 CE **Pierre-Ossian Bonnet** (1819–1892, France). Mathematician. Contributed to the differential geometry of curves and surfaces.[188] The field was opened by **Euler**, **Monge** and **Gauss**[189], but at the time was lacking a systematic treatment. Between 1840 and 1950, this challenge was taken

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[186] He had a stormy student career: dueling left him with a heavily scarred lip.

[187] Before the late 19^{th} century, blood pressure studies required sticking a tube directly into the arteries.

[188] For further reading, see:

- Struik, D.J., *Lectures on Classical Differential Geometry*, Dover Publications: New York, 1988, 232 pp.
- Weatherburn, C.E., *Differential Geometry of Three Dimensions*, Cambridge University Press: Cambridge, 1939, 268 pp.
- Mishchenko, A. and A. Fomenko, *A Course of Differential Geometry and Topology*, Mir Publications: Moscow, 1988, 455 pp.
- Kreyszig, E., *Differential Geometry*, Dover Publications: New York, 1991, 352 pp.

[189] *Gauss-Bonnet theorem* (Bonnet, 1848; known earlier to Gauss): If the Gaussian curvature K of a surface is continuous in a simply connected region A, bounded by a closed curve C composed of k smooth arcs making at the vertices exterior angles $\theta_1, \theta_2, \dots, \theta_k$, then:

$$\int_C K_g ds + \iint_A K dA = 2\pi - \sum_{i=1}^{k} \theta_i,$$

up by Bonnet and a group of younger French mathematicians, among them **Serret**, **Frenet**, **Bertrand** and **Puiseux**[190]. Bonnet demonstrated the invariance of the geodetic curvature under bending of the surface and stressed the usefulness of special coordinate systems, such as isometric and tangential coordinates.

Bonnet was born at Montpelier. He studied at the École Polytechnique and became a teacher there in 1844. He succeeded the astronomer LeVerrier to a Sorbonne chair in 1878.

1844–1890 CE **John Fowler** (1817–1898, England). Civil engineer. Pioneer in the construction of railway systems (including bridges and deep tunneling 'tubes') in England, Italy, Egypt and Sudan.

Fowler was born at Wadsley Hall, near Sheffield and flourished in an era of railway construction initiated by the Stepensons. In 1890 he completed the *Forth bridge* with his partner Benjamin Baker.

1845–1867 CE **Robert William Thomson** (1822–1873, Scotland). Engineer and inventor. Invented the *vulcanized rubber pneumatic tire*[191]. He patented his invention in 1845, and it was successfully tested in London. However, it was abandoned because it was thought too expensive for common use. The tire was re-invented by John Dunlop in 1888.

Thomson also patented the *fountain pen* (1849) and a steam traction engine (1867). He was born in Stonehaven, Scotland.

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where K_g represents the geodetic curvature of the arcs.

This theorem is an application of Green's theorem, known from the theory of line integrals and surface integrals in the plane, to the integral of the geodetic curvature.

[190] **Victor Alexandre Puiseux** (1820–1893, France). Mathematician. Furthered Cauchy's work on functions of complex variable. First to distinguish *poles*, *essential singularities* and *branch points*.

[191] It consisted of inflexible casings around an *inner tube* and was designed for vehicles pulled by *animals*. They were ousted after a few years by *solid tires*. The **Michelin brothers** (France) were the first to fit *motor vehicles* with tires with *inner tubes* (1895).

The Real Number System

The middle of the 19^{th} *century saw the main thrust of the program for the arithmetization of analysis, which started with* ***d'Alembert*** *(1754),* ***Lagrange*** *(1797) and* ***Cauchy*** *(1821). In the first stage of this process, the foundations of the real number system were rigorized. This was done in several different ways.*

One of the methods starts with the positive integers as undefined concepts, states some axioms concerning them, and then uses them to build a larger system consisting of the positive rational *numbers (quotient of positive numbers). The positive rational numbers, in turn, are used as a basis for constructing the positive* irrational *numbers (such as* $\sqrt{3}$, π, *etc.). The final step is the introduction of the negative real numbers and zero. The most difficult part of the whole process is the transition from the rational numbers to the irrational numbers.*

Although the need for irrational numbers was apparent to the ancient Greeks from their study of geometry, satisfactory methods for constructing irrational numbers from rational numbers were not introduced until late in the 19^{th} *century.*

Three different theories were outlined by ***Karl Weierstrass*** *(1815–1897),* ***Georg Cantor*** *(1845–1918) and* ***Richard Dedekind*** *(1831–1916). In 1889, the Italian mathematician* ***Giuseppe Peano***[192] *(1858–1932) listed 5 axioms for the non-negative integers that could be used as the starting point of the whole construction:*

(1) *Zero is a number.*

(2) *If* a *is a number, the successor of* a *is a number.*

(3) *Zero is not a successor of a number.*

(4) *Two numbers of which the successors are equal are themselves equal.*

(5) *If a set* S *of numbers contains zero and also the successor of every number in* S*, then every number is in* S *(axiom of induction).*

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[192] For further reading, see:

- Kline, M., *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, 1990, 1211 pp.

Here, the postulational method attained a new height of precision, with no ambiguity of meaning and no concealed assumptions.

1845–1881 CE **Leopold Kronecker** (1823–1891, Germany). Distinguished mathematician and mathematical philosopher who planted the seed of *intuitionism*[193] in modern mathematics (although his views should not be confused with those of the present-day movement). In general Kronecker adhered to an arithmetical approach to algebra, via a postulational treatment of algebraic structures in terms of various number fields, and insisted that arithmetic and analysis be based on the whole numbers[194].

He categorically rejected the real number construction of his day on the ground that they cannot be achieved through finite processes only, and he

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[193] *Intuitionism* asserts that mathematics is built solely on *finite* constructive methods, employing a *finite* number of steps [e.g. a Galois field having a finite number of elements, as for example the field of integers modulo a prime number]. For the intuitionists, an entity whose existence is to be proved must be shown to be constructible in a finite number of steps.

Intuitionism stresses that mathematics has priority over logic; the objects of mathematics are constructed and operated upon in the mind by the mathematician, and it is impossible to define the properties of mathematical objects simply by establishing a number of axioms.

[194] *Kronecker's theorem in one dimension* (1884): If ν is irrational, α is arbitrary, and N and ϵ are positive, then there are integers n and p such that $n > N$ and $|n\nu - p - \alpha| < \epsilon$.

The theorem implies that there are integers n for which $n\nu$ is as near as we please to any number in $(0, 1)$. Alternatively, if ν is irrational, then the set of points $(n\nu)(\text{mod } 1)$ is *dense* in the interval $(0, 1)$. The theorem has a simple application to a plane geometrical-optics problem: a ray of light leaves a point inside a square, the sides of which are reflecting mirrors. What is the nature of the path? The equivalent geometrical theorem then states: Either the path is closed and periodic or it is dense in the square, passing arbitrarily near to every point in the square. A necessary and sufficient condition for *periodicity* is that the angle between a side of the square and the initial direction of the ray should have a rational tangent.

Kronecker himself proved his theorem for the more general case of a space of K dimensions. Later, **Harald Bohr** (1934) and **Georgii Fedoseevich Voronoi** (1868–1908) extended the theorem to spaces of infinite number of dimensions.

called for an arithmetical revolution that would ban the irrational numbers as nonexistent(!)[195]. In analysis, Kronecker openly criticized his contemporaries (especially **Weierstrass** and **Cantor**) in lectures and conversation. He believed that mathematics should deal only with finite numbers and a finite number of operations.

Kronecker made significant contributions to algebra: with **Kummer** and **Dedekind** he invented the modern theory of algebraic numbers. They did for higher arithmetic and the theory of algebraic equations what **Gauss**, **Lobachevsky** and **Riemann** did for geometry, in emancipating it from the narrow Euclidean dogma. Thus the creators of the theory of algebraic numbers have unified the separate theories of equations, algebraic curves and surfaces, and numbers into one firm supersystem based on a firm background of postulates.

In addition, Kronecker investigated the curvature of hypersurfaces in Euclidean space in n-dimensions (1869) and introduced (1881) his famous δ symbol. $[\delta_{ij} = 1 \text{ if } i = j;\ \delta_{ij} = 0 \text{ if } i \neq j.]$

He used the method of residues and the integral

$$\int \frac{e^{\frac{2\pi i}{m} z^2} dz}{1 - e^{2\pi i z}}$$

to render a simple proof of the *Gauss sum*

$$\sum_{s=0}^{m-1} e^{\frac{2\pi i}{m} s^2} = \frac{i + i^{1-m}}{i+1}\sqrt{m}.$$

Gauss himself devoted several painful years to determine the exact form of this sum. He later deduced from it the law of quadratic reciprocity for real primes.

Kronecker was born at Liegnitz, Prussia, of Jewish parents. At school he excelled in Greek, Latin, Hebrew, philosophy and mathematics. His mathematical talent appeared early under the expert guidance of Kummer. He acquired a broad liberal education in the Greek classics, painting and sculpture, and was an accomplished pianist and vocalist. Upon entering the University of Berlin in 1841 he came in contact with **Dirichlet**, **Jacobi**, **Weierstrass**, **Steiner** and **Eisenstein**. After taking his Ph.D. degree at the age of 22, he spent the years 1845–1853 managing a successful farming business.

[195] His motto: "*Die ganze Zahl schuf der liebe Gott, alles übrige ist Menschwerk*" (God made the integers, men made the rest.)

Until the last decade of his life, Kronecker was a free man with obligations to no employer. From 1861 to 1883 he conducted regular courses at the University of Berlin, principally on his personal researches. In 1883 Kummer retired, and Kronecker succeeded him as ordinary professor.

Kronecker was of very small stature and extremely self-conscious about his height. In fact he attacked rigorously anyone whose mathematics he disapproved.He believed that the mathematical analysis of Weierstrass, based on his conception of irrationals as defined by infinite sequences of rationals, is all wrong. His finitism obviously embarrassed Weierstrass, but it was **Cantor** whom he wounded most seriously. Not only did Kronecker stand in the way of a position for Cantor in Berlin, but he sought to undermine the branch of mathematics that Cantor was creating. In 1884 Cantor suffered the first of the nervous breakdowns that were to recur throughout the remaining 33 years of his life.

Kronecker died of bronchial illness in Berlin. On his death bed he converted to Christianity.

Analysts at his time regarded his views as excessively metaphysical. After a temporary decline, his views reappeared in a new form in the works of **Poincaré** (1902–1906) and **Brouwer** (1908). This school of intuitionism has gradually strengthened with the passage of time. It won over some eminent present-day mathematicians, and has exerted great influence on all thinking concerning the foundations of mathematics.

1846 CE, Sept 23 **Johann Gottfried Gale** (1812–1910, Germany), astronomer, discovered the planet *Neptune* using predictions of its position by **Urbain LeVerrier** (1811–1877, France) and **John Couch Adams** (1819–1892, England).

The discovery of Neptune was a dramatic and spectacular achievement of mathematical astronomy. The very existence of this new member of the solar system, and its exact location, were demonstrated with pencil and paper; there was left to observers only the routine task of pointing their telescopes at the spot the mathematicians had marked.

1846 CE **Hugo von Mohl** (1805–1872, Germany). Botanist. Pioneer in the field of plant cell structure and physiology. His meticulous observations were the first attempts at *cytochemistry*; he identified a substance he called *protoplasm*. Mohl was the first person (1846) to use the term protoplasm in cell biology. He was the first to clearly explain *osmosis*.

Mohl was born in Stuttgart and studied medicine at Tübingen. Professor of Physiology at Bern (1832–1835) and of Botany at Tübingen (1835–1872).

1846 CE, Aug 10 The *Smithsonian Institution* founded by act of Congress in Washington D.C., with a $100,000 bequest from English chemist and mineralogist **James Smithson** (1765–1829). It is a federal chartered nonprofit corporation of scientific, educational, and cultural interests, established for the "*increase and diffusion of knowledge among men*". The Smithsonian conducts scientific research and publishes the results of studies, explorations, and investigations. It preserves and displays items representing aeronautics and space exploration, science and technology and natural history.

James Smithson (known until 1801 as James Louis Macie) was born in Paris, the illegitimate son of Hugh Smithson Percy, 1^{st} Duke of Northumberland, and Elizabeth Macie. The mineral *smithsonite* (calamine) is named after him.

1846 CE **Ernest Heinrich Weber** (1795–1878, Germany). Anatomist and physiologist. Founded *experimental psychology*, studying the *response* of humans to *physical stimuli*. Professor at Leipzig (1818–1878). Established the empirical law (1846):

"Noticeable differences in sensation occur when the increase of stimulus is a constant percentage of the stimulus itself".

If s is the magnitude of a measurable stimulus and (Δs) the increase just required for discrimination, then the ratio $r = \frac{\Delta s}{s}$ is constant. This applies to sound, light and taste reception[196].

Weber's law is at best a good approximation to reality. It fails when s is either too small or too large.

1846–1885 CE **Louis Pasteur** (1822–1895, France). Distinguished chemist, microbiologist and humanist. Pioneered in the field of modern stereochemistry in proving the existence of *optical isomers* (1846) and explaining the phenomenon.

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[196] *Example*: Assume a person holds a weight of 20 grams in his hand and that he is tested for the ability to distinguish between this weight and a slightly higher weight. Experiments show that a person is not able to discriminate between 20.5 g and 20 g, but that he finds 21 g to be heavier than 20 g most of the time. Now, a person cannot reliably discriminate between 41 g and 40 g. The detectable increase is 2 g instead of 1 g. It is found that, in general, discrimination is possible if s is increased by 5 percent of the original value. The following list of r-values may illustrate the sensitivity of human senses:

visible brightness	1:50	(s =light intensity)
tone	1:10	(s =sound intensity)
smell for rubber	1:8	(s =number of molecules)
taste for saline solution	1:4	(s =concentration of solution)

Discovered that fermentation of wine and beer is caused by *microorganisms* (yeast), not by chemical means, as previously supposed and proved that these organisms do not arise by spontaneous generation (1856–1871). Determined that excess fermentation could be eliminated by boiling the liquid or filtering the microorganisms (1856). Discovered the bacilli causing two distinct diseases of *silkworm* and found a method of preventing spread of the disease (1868), thus saving the silk industry in France. Extended his theory of fermentation to the germ theory of disease (1862–1885) and developed effective inoculation against several specific diseases: *chicken cholera* (1880), *anthrax* (1882) and *rabies* (1885). Identified the bacteria streptococcus (1879). Invented the process of milk '*pasteurization*' (1885).

Pasteur was born at Dôle, Franche-Comté. In 1838 he was sent with a friend to Paris, to a preparatory school for the École Normale. But being a nervous and excitable boy, his health broke down, and he returned home, to Arbois. He then continued his education at the Royal College of Besancon. His admittance to the École Normale was hampered by a low grade in chemistry (1842). This only increased his incentive for a serious study of chemistry. After his brilliant solution of the isomeric problem (1846) which had baffled the greatest chemists and physicists of the time, he was immediately appointed professor of chemistry at the faculty of science at Strasbourg, where he soon married Mlle Laurent.

In 1854 he was appointed professor of chemistry and dean of the Faculty of sciences at Lille. In his inaugural address he used significant words, the truth of which was soon manifested in his case: "*In the field of observation, chance only favors those who are prepared*".

The diseases of beer and wine had from time immemorial baffled all attempts at cure. Pasteur one day visited a brewery containing both sound and unsound beer. He examined the yeast under the microscope, and at once saw that the globules from the sound beer were nearly spherical, while those from the sour beer were elongated; and this led him to a discovery the consequence of which have revolutionized chemical as well as biological science. It was the beginning of a series of experimental researches in which he proved conclusively that the notion of spontaneous generation was a chimera.

Up to this time the phenomenon of fermentation was considered strange and obscure. Explanations had indeed been put forward by men as eminent as **Berzelius** and **Liebig**, but they lacked experimental foundation. This was given in the most complete degree by Pasteur. For he proved that various changes occurring in the several processes of fermentation are invariably due to the presence and growth of minute organisms.

In a series of delicate and intricate experiments Pasteur was able to show that when the atmospheric germs are absolutely excluded, no changes take

place. The application of these facts to surgical operations has revolutionized surgical practice in Pasteur's own time.

Pasteur left Lille in 1857 to become the director of the École Normale in Paris (1857–1867). His discoveries on fermentation inaugurated a new era in the brewing and wine-making industries. Empiricism, hitherto the only guide, was replaced by exact scientific knowledge; the connection of each phenomenon with a controllable cause was established. Yet, in spite of rising fame and success, he still had to withstand grave opposition from powerful foes in the academy.

His powers of patient research and exact observation were about to be put to a severe test: An epidemic of a fatal character had ruined the French silk producers. Up to that time he had never seen a silkworm, and hesitated to attempt so difficult a task; but at the reiterated request of his friends he consented, and in June 1865 went to the south of France for the purpose of studying the disease on the spot. In September of the same year he was able to announce results which pointed to the means of securing immunity from the epidemic, thus bringing back prosperity to the silk trade of France.

In 1880 Pasteur attacked the problem of chicken cholera, an epidemic which destroyed 10 percent of the French fouls; after the application of inoculation the death-rate was reduced to below one percent.

Next came the successful attempt to deal with the fatal cattle scourge known as *anthrax*. Many million of sheep and oxen all over the world have been treated by Pasteur's method, and the rate of mortality reduced from 10 to less than one percent. It is estimated that the monetary value of these discoveries was sufficient to cover the whole cost of the war indemnity paid by France to Germany in 1870.

The most spectacular of Pasteur's anti-microbial wars was launched against the dread disease of *hydrophobia* in man and of *rabies* in animals. This was accomplished in spite of the fact that the virus causing the disease had not been identified. Here again, the method of inoculation proved to be successful. On the 14^{th} of November 1888, the 'Institut Pasteur' was founded. Thousands of people suffering from bites from rabid animals, from all lands, have been treated in this institute, and the death-rate from this disease has been reduced to less than one percent[197].

Pasteur brought to microbiology the spirit and logic of the exact methods of physics and chemistry. This enabled him to bring under the domain of scientific laws the phenomenon of disease. Rich in years and honors, but

[197] **Paul Muni** (1895–1967; born Muni Weisenfreund in Lemberg, Austria) played the character of Pasteur in the movie "*The Story of Louis Pasteur*" (1936).

simple and affectionate in his demeanor, this great benefactor of humanity passed quietly away near St. Cloud on the 28^{th} of September 1895.

In 1874 Pasteur said: "Life, as is known to us, is a direct result of the *asymmetry*[198] of the universe or of its indirect consequences. The universe is asymmetric."

Now, at that time, the only known asymmetry pertaining to this comment was that of optical isomers in the field of organic chemistry. From our present vantage point this reads as a prophetic statement because life, physics, matter and even the fabric of the vacuum which we inhabit, are known to stem from spontaneous breaking of a string of symmetries. e.g.: *time-reversal*, *electroweak gauge symmetry* and *chiral symmetry*.

In biology, the fundamental symmetry of the double helix molecule is a case in point.

Worldview XXI: Louis Pasteur

* * *

"Let me tell you the secret that has led me to my goal: my strength lies solely in my tenacity."

* * *

"Travailler, travailler toujours."

* * *

[198] *Symmetry* is the Greek word $\Sigma YM\text{-}METPIA$ = "the same measure".

"Blessed is he who carries within himself a god and an ideal and who obeys it — an ideal of art, of science, of gospel virtues. Therein lie the springs of great thoughts and great actions."

* * *

"In the field of observation, chance favors the prepared mind."

* * *

"Science owns no fatherland."

* * *

"Unfortunate are those scientists who have only clear thoughts in their heads!"

* * *

"There does not exist a category of science to which one can give the name applied science. There are science and the applications of science, bound together as the fruit of the tree which bears it."

* * *

At the inauguration of his institute (1888) he closed his oration with the following words:

"Two opposing laws seem to me now in contest. The one, a law of blood and death, opening out each day new modes of destruction, forces nations to be always ready for the battle. The other, a law of peace, work and health, whose only aim is to deliver man from the calamities that beset him. Which of these two laws will prevail, God only knows. But of this we may be sure, that science, in obeying the law of humanity, will always labor to enlarge the frontiers of life."

* * *

"Où en êtes-vous? Que faites-vous? Il faut travailler" (on his death-bed, to his devoted pupils, watching over him).

The Spontaneous Generation Controversy (340 BCE–1870 CE)

"Omne vivium ex Vivo."

(Latin proverb)

*Although the theory of spontaneous generation (abiogenesis) can be traced back at least to the Ionian school (600 B.C.), it was Aristotle (384–322 B.C.) who presented the most complete arguments for and the clearest statement of this theory. In his "On the Origin of Animals", **Aristotle** states not only that animals originate from other similar animals, but also that living things do arise and always have arisen from lifeless matter. Aristotle's theory of spontaneous generation was adopted by the Romans and Neo-Platonic philosophers and, through them, by the early fathers of the Christian Church. With only minor modifications, these philosophers' ideas on the origin of life, supported by the full force of Christian dogma, dominated the mind of mankind for more that 2000 years.*

According to this theory, a great variety of organisms could arise from lifeless matter. For example, worms, fireflies, and other insects arose from morning dew or from decaying slime and manure, and earthworms originated from soil, rainwater, and humus. Even higher forms of life could originate spontaneously according to Aristotle. Eels and other kinds of fish came from the wet ooze, sand, slime, and rotting seaweed; frogs and salamanders came from slime.

Rather than examining the claims of spontaneous generation more closely, Aristotle's followers concerned themselves with the production of even more remarkable recipes. Probably the most famous of these was ***van Helmont's*** *(1577–1644) recipe for mice. By placing a dirty shirt into a bin containing wheat germ and allowing it to stand 21 days, live mice could be obtained. Another example was the slightly more complicated but equally "foolproof" recipe for bees. By killing a young bullock with a knock on the head, burying him in a standing position with his horns sticking out of the ground, and finally sawing off his horns one month later, out will fly a swarm of bees.*

The more exact methods of observation that were developed during the seventeenth century soon led to a realization of the complex nature of the anatomy and life cycles of certain living organisms. Equipped with this better understanding of the complexity of living organisms, it became more difficult for some to accept the theory of spontaneous generation. This skepticism signaled the beginning of three centuries of heated controversy over a theory that had gone unchallenged for the previous 2000 years. What is more significant is that the controversy was to be resolved not by powerful arguments but by ingeniously designed, simple experiments.

The controversy went through four phases:

I. Redi (1688) Vs. Aristotelian school and Church dogma

Redi *was first to use carefully controlled experiments to test the theory of spontaneous generation. He put some meat in two jars. One he left open to air (the control); the other he covered securely with gauze. At that time it was well recognized that white worms would arise from decaying meat or fish. Sure enough, in a few weeks, the meat was infested with the white worms but only in the control jar which was not covered. This experiment was repeated several times, using either meat or fish, with the same result. On closer examination he noted that common houseflies went down into the meat in the open jar, later the white worms appeared, and then new flies. Redi reported that he had observed the flies deposit their eggs on the gauze; however, worms developed in the meat only when the eggs got to the meat. He therefore concluded from his observations that the white worms did not arise from the putrid meat. The worms developed from the eggs that the flies had deposited. The white worm then was the larva of the fly, and the meat served only as food for the developing insect.*

Redi's experiment provided the impetus for testing other well-established recipes. In all cases that were examined carefully, it was demonstrated that the living organism arose not by spontaneous generation, but from a parent. Thus it was shown that the theory of spontaneous generation was based on a combination of the weakness of the human eye and bits and snatches of information gathered by accidental observation. The early biologists had seen earthworms coming out of the soil and frogs emerging from the slime of pond water, but they had not been able to see the tiny eggs from which these organisms arose. Because their observations had not been systematic, they had not seen how the mice invaded the grain bin in search of food, so they thought that the grain produced the mice. Based on the more exact methods of observation, the evidence that supported the theory of spontaneous generation of animals and plants was largely demolished by the end of the seventeenth century.

II. Spallanzani Vs. Needham (1767–1768)

As soon as the discoveries of Leeuwenhoek[199] *became known, the proponents of spontaneous generation turned their attention to these microscopic organisms and suggested that surely they must have formed by spontaneous generation. Finally, experimental "proof" for this notion was published in 1748 by an Irish priest,* ***John Tuberville Needham*** *(1713–1781).*

Needham reported that he had taken mutton gravy fresh from the fire, transferred it to a flask, heated it to boiling, stoppered it tightly with a cork, and then set it aside. Despite boiling, the liquid became turbid in a few days. When examined under a microscope, it was teeming with microorganisms of all types. The experiments were repeated by and gained the support of the famous French naturalist, ***Georges Louis Le-clerc, Comte de Buffon*** *(1707–1788). Needham's demonstration of spontaneous generation was generally accepted as a great scientific achievement, and he was immediately*

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[199] The development of *microscopy* started with **Janssen** (1590) and continued with **Hooke** (1660), **Leeuwenhoek** (1676) and **Zeiss** (1883). Just as the theory of the abiogenesis of higher organisms was being refuted, the controversy was reopened, more heated than ever, because of the discovery of microorganisms by Antony van Leeuwenhoek. Leeuwenhoek patiently improve his microscopes and developed his techniques of observation for 20 years before he reported any of his results.

elected into the Royal Society of England and the Academy of Sciences of Paris.

Meanwhile in Italy, Lazzaro Spallanzani (1729–1799) performed a series of brilliantly designed experiments of his own that refuted Needham's conclusions. Spallanzani found that if he boiled the food for one hour and hermetically sealed the flasks (by fusing the glass so that no gas could enter or escape), no microorganisms would appear in the flasks. If, however, he boiled the food for only a few minutes, or if he closed the flask with a cork, he obtained the same results that Needham reported. Thus he wrote that Needham's conclusions were invalid because (1) he had not heated the gravy hot enough or long enough to kill the microorganisms, and (2) he had not closed the flask sufficiently to prevent other microbes from entering.

Count Buffon and Father Needham immediately responded that, of course, Spallanzani did not generate microorganisms in his flasks because his extreme heating procedures destroyed the vegetative force *in the food and the* elasticity *of the air. Regarding Spallanzani's experiments, Needham wrote, "from the way he has treated and tortured his vegetable infusions, it is obvious that he has not only much weakened, and maybe even destroyed, the vegetative force of the infused substances, but also that he has completely degraded ... the small amount of air which was left in his vials. It is not surprising, thus, that his infusions did not show any sign of life."*

Rather than engage in theoretical arguments over the possible existence of these mystical forces, Spallanzani returned to the laboratory and performed another set of ingenious experiments. This time he heated the sealed flasks to boiling not for one hour, but for three hours, and even longer. If Needham was right, this treatment should certainly have destroyed the vegetative force. As Spallanzani had previously observed, nothing grew in these heated, sealed flasks. However, when the seal was broken and replaced with a cork, the broth soon became turbid with microbes. Since even three hours of boiling did not destroy anything in the food necessary for the production of microbes, Needham could no longer argue that he had killed the vegetative force by the heat treatment.

Spallanzani continued to perform experiments that led him to the conclusion that properly heated and hermetically sealed flasks containing broth would remain permanently lifeless. He was, however, unable to answer adequately the criticism that by sealing the flasks he had excluded the "vital forces" in the air that Needham claimed were also necessary ingredients for spontaneous generation. With the discovery of oxygen gas in 1774 and the realization that this gas is essential for the growth of most organisms, the possibility that spontaneous generation could occur, but only in the presence of air (oxygen), gained additional support.

III-1. Schwann Vs. Berzelius, Liebig and Wholer (1836–1839) – the fermentation controversy

The art of brewing was developed by trial and error over a 6000-year period and practiced without any understanding of the underlying principles. From long experience, the brewer learned the conditions, not the reasons, for success. Only with the advent of experimental science in the eighteenth and nineteenth centuries did man attempt to explain the mysteries of fermentation. Let us, then, from our vantage point in time, trace the observations, experiments, and debates from which evolved our present understanding of fermentation and biological catalysis.

For centuries, fermentation had a significance that was almost equivalent to what we would now call a chemical reaction, an error that probably arose from the vigorous bubbling seen during the process. The conviction that fermentation was strictly a chemical event gained further support during the early part of the nineteenth century, when French chemists led by ***Lavoisier*** *and* ***Gay-Lussac*** *determined that the alcoholic fermentation process could be expressed chemically by the following equation:*

$$\underset{\text{glucose}}{C_6H_{12}O_6} \rightarrow \underset{\text{ethyl alchohol}}{2C_2H_5OH} + \underset{\text{carbon dioxide}}{2CO_2}$$

It was, of course, known that yeast must be added to the wort in order to ensure a reproducible and rapid fermentation. The function of the yeast, according to the chemists, was merely to catalyze the process. All chemists agreed that fermentation was in principle no different from other catalyzed chemical reactions.

Then in 1837, the French physicist ***Charles Cagniard-Latour*** *and the German physiologist* ***Theodor Schwann*** *independently published studies that indicated yeast was a living microorganism. Prior to their publications, yeast was considered a proteinaceous chemical substance. The reason the two workers came up with the same observations at approximately the same time is most likely due to the production of better microscopes.*

It should be mentioned that one of the reasons it was difficult to ascertain whether or not yeast is living was because, like most other fungi, yeast is not motile. The organized cellular nature of yeast was discovered only when improved microscopes became available. Schwann and Cagniard-Latour also observed that alcoholic fermentation always began with the first appearance of yeast, progressed only with its multiplication, and ceased as soon as its growth stopped. Both scientists concluded that alcohol is a by-product of the growth process of yeast.

The biological theory of fermentation advanced by Cagniard-Latour and Schwann was immediately attacked by the leading chemists of the time. The eminent Swedish physical chemist ***Jons Jakob Berzelius*** *reviewed the two papers in his Jahresbericht for 1839 and concluded that microscopic evidence was of no value in what was obviously a purely chemical problem. According to Berzelius, nothing was living in yeast.*

This opinion was supported by ***Justus von Liebig*** *and* ***Friedrich Wöhler****. Liebig argued that:*

1. *Certain types of fermentation, such as the lactic acid (souring of milk) and acetic acid (formation of vinegar) fermentations, can occur in the complete absence of yeast.*
2. *Even if yeast is living, it is not necessary to conclude that the alcoholic fermentation is a biological process. The yeast is a remarkably unstable substance which, as a consequence of its own death and decomposition, catalyzes the splitting of sugar. Thus, fermentation is essentially a chemical change catalyzed by breakdown products of the yeast.*

Liebig's views were widely accepted, partly because of his powerful influence in the scientific world and partly because of a desire to avoid seeing an important chemical change relegated to the domain of biology. And so the stage was set – biology against chemistry – for the entrance of Louis Pasteur.

III-2. Pasteur Vs. Liebig and Berzelius (1857–1860)

In 1851, ***Pasteur*** *published his first paper on the topic of fermentation. The publication dealt with lactic acid fermentation, not alcoholic fermentation. Utilizing the finest microscopes of the time, Pasteur discovered that souring of milk was correlated with the growth of a microorganism, but one considerably smaller than the beer yeast. During the next few years, Pasteur extended these studies to other fermentative processes, such as the formation of butyric acid as butter turns rancid. In each case he was able to demonstrate the involvement of a specific and characteristic microorganism; alcoholic fermentation was always accompanied by yeasts, lactic acid fermentation by nonmotile bacteria, and butyric acid fermentation by motile rod-shaped bacteria. Thus, Pasteur not only disposed of one of the opposition's strongest arguments, but also provided powerful circumstantial evidence for the biological theory of fermentation.*

Now Pasteur was ready to attack the crucial problem, alcoholic fermentation. Liebig had argued that this fermentation was the result of the decay of

yeast; the proteinaceous material that is released during this decomposition catalyzes the splitting of sugar. Pasteur countered this argument by developing a protein-free medium for the growth of yeast. He found that yeast could grow in a medium composed of glucose, ammonium salts, and some incinerated yeast. If this medium is kept sterile, neither growth nor fermentation takes place. As soon as the medium is inoculated with even a trace of yeast, growth commences and fermentation ensues. The quantity of alcohol produced parallels the multiplication of the yeast. In this protein-free medium, Pasteur was able to show that fermentation takes place without the decomposition of yeast. In fact, the yeast synthesizes protein at the expense of the sugar and ammonium salts. Thus Pasteur concluded in 1860:

"Fermentation is a biological process, and it is the subvisible organisms which cause the changes in the fermentation process. What's more, there are different kinds of microbes for each kind of fermentation. I am of the opinion that alcoholic fermentation never occurs without simultaneous organization, development and multiplication of cells, or continued life of the cells already formed. The results expressed in this memoir seem to me to be completely opposed to the opinion of Liebig and Berzelius."

Pasteur argued effectively, and more important, all the data were on his side. Thus the vitalistic theory of fermentation predominated until 1897, when an accidental discovery by ***Eduard Buchner*** *(1860–1917) demonstrated that the alcoholic fermentation of sugars is due to action of enzymes contained in the yeast.*

The controversy was thus finally resolved and the door was thrown open to modern biochemistry.

IV. Pasteur and Tyndall Vs. Pouchet (1859–1885)

The spontaneous generation controversy was brought to a crisis in 1859 when ***Felix Archimède Pouchet*** *(1800–1872), a distinguished scientist and director of the Museum of Natural History in Rouen, France, reported his experiments on spontaneous generation. Pouchet claimed to have accomplished spontaneous generation using hermetically sealed flasks and pure oxygen gas. These experiments, he argued, demonstrated that "animals and plants could be generated in a medium absolutely free from atmospheric air and in which therefore no germ of organic bodies could have been brought by air."*

The impact of Pouchet's experiments on his contemporaries was so great that the French Academy of Sciences offered the Alhumpert Prize in 1860 for

exact and convincing experiments that would end this controversy once and for all. Pasteur first set out to demonstrate that air could contain numerous microorganisms. From his microscopic observation, Pasteur concluded that there are large numbers of organized bodies suspended in the atmosphere. Furthermore, some of these organized bodies are indistinguishable by shape, size, and structure from microorganisms found in contaminated broths. Later he showed that these organized bodies that collected on the cotton fibers not only looked like microorganisms, but when placed in a sterile broth were capable of growth!

Pasteur's second series of experiments provided further circumstantial evidence that it was the microbes on floating dust particles and not the so-called vital forces that were responsible for sterilized broth's becoming contaminated. In these experiments, Pasteur carried sterile-sealed flasks to a wide variety of locations in France. At the various sites, he would break the seal, allowing air to enter the flask. The flask was immediately resealed and brought back to Paris for incubation. The conclusion from these numerous experiments was that where considerable dust existed, all the flasks would become turbid. For example, if he opened sterile flasks in the city, even for a brief period, they all became turbid, whereas in the mountainous regions, especially at high altitudes, a large proportion of the flasks remained sterile.

His third and most conclusive experiment utilized the now famous swan-neck flask. As a result of the experiments described, Pasteur hypothesized that the source of contamination was dust. If true, then it should be possible to keep a broth sterile even in the presence of air as long as the dust is kept out. In order to test this hypothesis. Pasteur constructed several bent-neck flasks. After placing broth into the flask, he boiled the liquid for a few minutes, driving the air from the orifice of the flask. As the flask cooled, fresh air entered the flask. Despite the fact that the broth was in contact with the gases of the air, the fluid in the swan-neck flask always remained sterile. Pasteur reasoned correctly that the dust particles that entered the flask were absorbed onto the walls of the neck and never penetrated into the liquid. As an experimental control, Pasteur demonstrated that nothing was wrong with the broth. If he broke the neck off the flask or tipped liquid into the neck (in both cases dust would enter the broth), the fluid soon became turbid with microorganisms.

With these simple, ingenious experiments, Pasteur not only overcame the criticism that air was necessary for spontaneous generation but he was also able to explain satisfactorily many of the sources (dust) of the contradictory findings of other investigators. Although Pasteur's conclusions gained wide support in both the scientific and the lay communities, they did not convince all the proponents of spontaneous generation.

Pouchet and his followers continued to publish reports of spontaneous generation. They claimed their techniques were as rigorous as those of Pasteur. Where Pasteur failed to obtain spontaneous generation they succeeded in every case. For example. they carefully opened 100 flasks at the edge of the Maladetta Glacier in the Pyrenees Mountains at an elevation of 10,850 feet. In this region which Pasteur had found to be almost dust free, all 100 of Pouchet's flasks became turbid after a brief exposure to the air. Even when Pouchet used swan-neck flasks, there was growth.

To Pasteur, this disagreement no longer revolved around the interpretation of experiments; rather, either Pouchet was lying or his techniques were faulty, Pasteur had complete faith in his own procedures and results and had no respect for those of his opponents. Thus he challenged Pouchet to a contest in which both of them would repeat their experiments in front of their esteemed colleagues of the Academy of Science. Pouchet accepted the challenge with the added statement, "If a single one of our flasks remains unaltered, we shall loyally acknowledge our defeat."

A date was set, and the place was to be the laboratory in the Museum of Natural History at the Jardin des Plantes, Paris[200]*. Pasteur arrived early with the necessary apparatus for demonstrating his techniques. Newspaper photographers and reporters were also on hand for this event of great public interest. However, Pouchet did not show up, and Pasteur won by default. It is difficult to ascertain whether Pouchet was intimidated by Pasteur's confidence or, as he later stated, he refused to take part in the "circus" atmosphere that Pasteur had created, and that their scientific findings should instead be reported in the reputable scientific journals. At any rate, in Pouchet's absence, Pasteur repeated his experiments in front of the referees with the same results he had previously obtained. As far as the scientific community was concerned, the matter was settled*[201]*. The law* Omne vivium ex vivo *(All life from life) also applied to microorganisms.*

In retrospect, however, the most ironic aspect of this extraordinary contest was not that Pouchet failed to show up, but rather that if he had appeared, he would have won! *Pouchet's experiments are reproducible. Pouchet performed his experiments in the following manner: He filled swan-neck flasks with a*

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[200] **Henri Milne-Edwards** (1800–1885), a French naturalist and zoologist (then a professor at the Museum and from 1864, its director) lent political and scientific support to Pasteur during the Pasteur-Pouchet debate. He wrote important works on crustaceans, mollusks, and corals and wrote a major opus on comparative anatomy and physiology.

[201] Yet, the Pasteur-Pouchet debate had a chilling effect on French evolutionary research for decades.

broth made from hay, boiled them for one hour, and then allowed the flasks to cool. He obtained growth in every flask. Pasteur's experiments differed in only two respects. Pasteur used a mixture of sugar and yeast extract for broth and boiled it for just a few minutes. Pasteur never obtained growth in his swan-neck flasks. The reason for their contradictory results was not understood until 1877, 17 years later.

*Mainly because of the careful work of the English physicist **Tyndall** (1820–1893), Pouchet's experiments could be explained without invoking spontaneous generation. Tyndall found that foods vary considerably in the length of boiling time required to sterilize them. For example, the yeast extract and sugar broth of Pasteur could be sterilized with just a few minutes of boiling, whereas the hay medium of Pouchet required heating for several hours to accomplish sterilization. Tyndall postulated that certain microorganisms can exist in heat-resistant forms, which are now referred to as spores. Furthermore, studies by Tyndall and the French bacteriologist **Ferdinand Cohen** revealed that hay media contain a large number of such spores. Thus the contradictory results of Pasteur and Pouchet were due to differences in the broths they used.*

Tyndall went on to demonstrate that nutrient medium containing spores can be sterilized easily by boiling for one-half hour on three successive days. This procedure of discontinuous heating, now called Tyndallization*, works as follows: The first heating kills all the cells that are not spores and induces the spores to germinate (in the process of germination, the spores lose their heat resistance as they begin to grow); on the second day, the spores have germinated and are thus susceptible to the heating. The third day heating "catches" any late germinating spores. Thus, with the publication of Tyndall's work, all the evidence that supported the theory of spontaneous generation was destroyed. Since that time, there has been no serious attempt to revive this theory.*

It should be pointed out, however, that by its very nature, the theory of spontaneous generation cannot be disproved*. One can always argue that the conditions necessary for spontaneous generation have not yet been discovered. Pasteur was well aware of the difficulty of a negative proof, and in his concluding remarks on the controversy, he merely showed that spontaneous generation had never been demonstrated.*

There is no known circumstance in which it can be affirmed that microscopic beings came into the world without germs, without parents similar to themselves. Those who affirm it have been duped by illusions, by ill-conducted experiments, and by errors that they either did not perceive, or did not know how to avoid.

1847 CE **Augustus De Morgan** (1806[202]–1871, England). Mathematician and logician, a contemporary of **Boole**. Laid the foundation of modern *symbolic logic* and developed new technology for logical expressions. Formulated *De Morgan's laws*. Introduced and vigorously defined the term *mathematical induction*. He endeavored to reconcile mathematics and logic, but compared with Boole, his impact on modern mathematics and its applications is small[203], and he is remembered mainly as a logical reformer. He is most noteworthy as the founder of the *logic of relations* and as a developer of the *algebra of logic* which reconstructed the logic of Aristotle upon a mathematical basis.

De Morgan was born in India, and taught at University College in London during 1836–1866. Although a convinced theist, he never joined a religious congregation. He renounced his professorship in 1866 when a colleague was denied a chair at University College because he was a unitarian.

The Basic Ideas of Topology

I. Polyhedra and surfaces[204]

A simple polyhedron is a body enclosed by faces, all of which are plane polygons (some examples of polyhedra are: pyramid, prism, frustum). It has

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[202] De Morgan was always interested in odd numerical facts; thus in 1849, he noticed that he had the distinction of being x years old in the year x^2 $(x = 43)$.

[203] Nevertheless, he shall be remembered in mathematics proper due to his discovery of the summation formula:

$$\sum_{n=1}^{N} \frac{x^{2^{n-1}}}{x^{2^n} - 1} = \frac{1}{x-1} - \frac{1}{x^{2^N} - 1} \qquad (x \neq 1).$$

[204] For further reading, see:

- Cundy, H.M., *Mathematical Models*, Oxford University Press, 1961, 286 pp.
- Coxeter, H.S.M., *Regular Polytopes*, Dover, 1973, 321 pp.
- Fauvel, T. et al (eds), *Möbius and his band*, Oxford University Press, 1993, 172 pp.

no holes, and can be continuously deformed into a sphere. A convex polyhedron[205] *is said to be regular if its faces are regular and congruent polygons (e.g. cube, tetrahedron). The study of polyhedra held a central place in Greek geometry, which already recognized most of their salient geometrical features. Greek geometers correctly concluded that the only polygons that can occur as faces of a regular polyhedron are the regular polygons having 3, 4 or 5 sides, bringing the total number of possible regular polyhedra to five.*

Now, all five of these possible forms actually exist. They were well known as early as ***Plato*** *(ca 390 BCE), and he gave them a very important place in his Theory of Ideas, which is why they are often known as the "Platonic Solids"*[206]*. The most important data on the regular polyhedra are given in Table 4.3 (L = length of edge, R = radius of circumsphere).*

While the sphere encloses the most volume of all shape having a given surface area, the tetrahedron, of all polyhedra, encloses the least volume with a given surface area [this ratio is equal to $(\frac{1}{12}a^3\sqrt{2})/a^2\sqrt{3} = \frac{a}{12}\sqrt{\frac{2}{3}}$, where a is the side length]. Table 4.3 suggests that for simple polyhedra $V - E + F = 2$, a fact first stated by ***Descartes*** *(1635), proved incompletely by* ***Euler*** *(1751)*

- Henle, M., *A Combinatorial Introduction to Topology*, Dover: New York, 1994, 310 pp.
- Flegg, H.G., *From Geometry to Topology*, Dover: New York, 2001, 186 pp.

[205] The designation '*convex*' applies to every polyhedron that is entirely on one side of each of its faces, so that it can be set on a flat table top with any face down. Although convexity is *not* a topological property it *implies* a topological property, since every convex polyhedron is necessarily simple.
There is a peculiar difference between the convex and the non-convex polyhedra: whereas every closed convex polyhedron is rigid, there are closed non-convex polyhedra whose faces can be moved relative to each other.

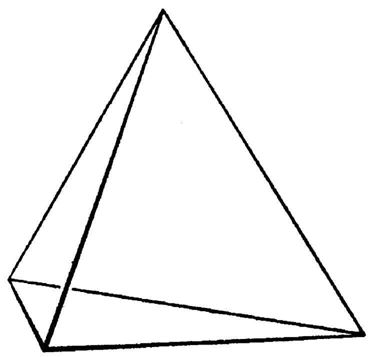
[206] It seems probable that **Pythagoras** (c. 540 BCE) brought the knowledge of the cube, tetrahedron and octahedron from Egypt, but the icosahedron and the dodecahedron have been developed in his own school. He seems to have known that all five polyhedra can be inscribed in a sphere. These solids played an important part in Pythagorean cosmology, symbolizing the five elements: *fire* (tetrahedron), *air* (octahedron), *water* (icosahedron), *earth* (cube), *universe or earth* (dodecahedron). The Pythagoreans passed on the study of these solids to the school of Plato. Euclid discusses them in the 13th book of his *Elements*, where he proves that no other regular bodies are possible, and shows how to inscribe them in a sphere. The latter problem received the attention of the Arabian astronomer Abu al-Wafa (10th century CE), who solved it with a single opening of the compass.

for convex polyhedra, and proved generally by **Cauchy** *(1811). It may have been known to* **Archimedes** *(ca 225 BCE), although the Greeks usually associated geometrical properties with measurements and not with mere counting.*

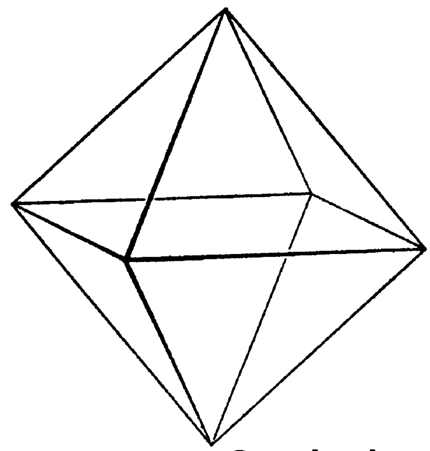
We have extant specimens of icosahedral dice that date from about the Ptolemaic period in Egypt. There are also a number of interesting ancient Celtic bronze models of the regular dodecahedron still extant in various museums. There was probably some mystic or religious significance attached to these forms. Since a stone dodecahedron found in northern Italy dates back to a prehistoric period, it is possible that the Celtic people received their idea from the region south of the Alps, and it is also possible that this form was already known in Italy when the Pythagoreans began to develop their teachings in Crotona.

Table 4.3: REGULAR POLYHEDRA

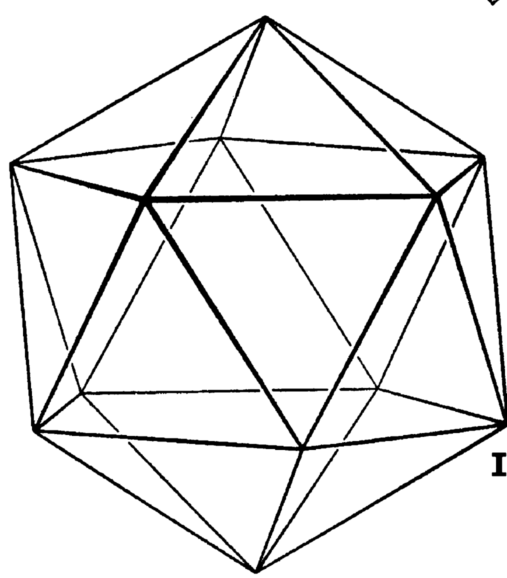
NAME OF POLYHEDRON	POLYGONS FORMING THE FACES	NUMBER OF				$\frac{R}{L}$
		VERTICES V	EDGES E	FACES F	FACES MEETING AT A VERTEX	
Tetrahedron	*Triangles*	*4*	*6*	*4*	*3*	$\frac{\sqrt{6}}{4}$
Octahedron	*Triangles*	*6*	*12*	*8*	*4*	$\frac{1}{\sqrt{2}}$
Icosahedron	*Triangles*	*12*	*30*	*20*	*5*	$\frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}}$
Cube (Hexahedron)	*Squares*	*8*	*12*	*6*	*3*	$\frac{\sqrt{3}}{2}$
Dodecahedron	*Pentagons*	*20*	*30*	*12*	*3*	$\frac{1}{2}\sqrt{\frac{5+3\sqrt{5}}{2}}$



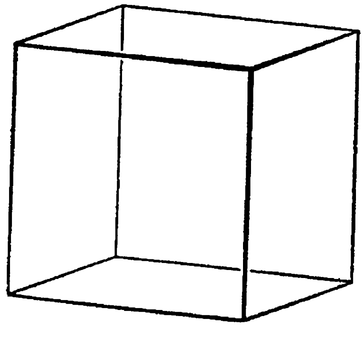
Tetrahedron



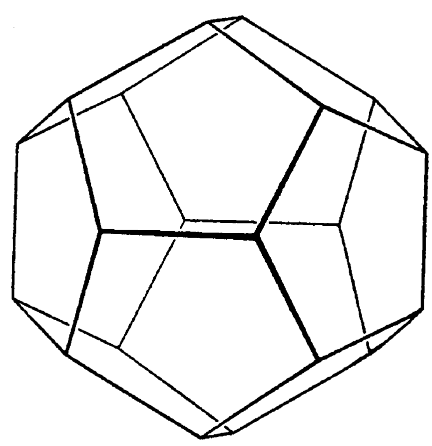
Octahedron



Icosahedron



Cube



Dodecahedron

The five regular polyhedrons attracted attention in the Middle Ages chiefly on the part of astrologers. At the close of this period, however, they were carefully studied by various mathematicians. Prominent among the latter was ***Pietro Franceschi****, whose work De Corporibus Regularibus (c. 1475) was the first to treat the subject with any degree of thoroughness. Following the custom of the time* ***Pacolli*** *(1509) made free use of the works of his contemporaries, and as part of his literary plunder he took considerable material from this work and embodied it in his De Divina Proportione.*

Albrecht Dürer*, the Nürnberg artist, showed how to construct the figures from a net in the way commonly set forth in modern works.*

Thus, Platonic and Archimedean polyhedra have sparked the imagination of creative individuals from Euclid to Kepler to Buckminster Fuller[207]*. These polyhedra are rich in connections to the worlds of art, architecture, chemistry, biology, and mathematics. In the realm of life, the Platonic Solids present themselves in the form of microscopic organisms known as radiolaria.*

Three other groups of polyhedra drew the attention of mathematicians throughout the ages:

- *Archimedean Solids: characterized by having all their angles equal and all their faces regular polygons, not necessarily of the same species. Archimedes' own account of them is lost. Thirteen such solids exist mathematically, some realized in crystalline forms: truncated tetrahedron (8 faces); cuboctahedron (14); truncated cube (14); truncated octahedron (14); rhombicuboctahedron (26); icosidodecahedron (32); truncated icosahedron ($V = 60$, $E = 90$, $F = 32$); snub cube (38); rhombicosidodecahedron (62); snub dodecahedron (92). Recently, the truncated icosahedron showed up in chemistry as the molecule C_{60}, known as a fullerene (after* ***Buckminster Fuller****).*

- *Kepler-Poinsot Polyhedra have as faces congruent regular polygons, and the angles at the vertices all equal, but their center is multiply enwrapped by the faces (convex polyhedra).*

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207 American engineer and inventor (1895–1983); among his numerous inventions is his *geodesic dome* structure (1947), based on 3-dimensional structural principles that were developed to achieve maximum span with a minimum material. His designs find parallels to such natural molecular geometries as the *tetrahedron* and the *truncated icosahedron* (C_{60}, named "Buckeyball" or "Fullerene" in his honor).
Fuller also built the geodetic dome at the American Pavilion in the 1970 World Fair in Montreal.

Four such solids exist: small stellated dodecahedron ($F = 12$, $V = 12$, $E = 30$); great dodecahedron ($F = 12$, $V = 20$, $E = 30$); great icosahedron ($F = 20$, $V = 12$, $E = 30$). They were described and studied by ***Kepler*** *(1619),* ***Poinsot*** *(1810),* ***Cauchy*** *(1813) and* ***Cayley*** *(1859).*

- *Semi-regular Polyhedra: solids which have all their angles, faces, and edges equal, the faces* not *being regular polygons. Two such solids exist: rhombic dodecahedron, a common crystal form; and semi-regular triacontahedron.*

On the basis of Euler's formula it is easy to show that there are no more than five regular polyhedra. For suppose that a regular polyhedron has F faces, each of which is an n-sided regular polygon, and that r edges meet at each vertex. Counting edges by faces, we see that

$$nF = 2E;$$

for each edge belongs to two faces, and hence is counted twice in the product nF; but counting edges by vertices,

$$rV = 2E,$$

since each edge has two vertices. Hence from $V - E + F = 2$ we obtain the equation

$$\frac{2E}{n} + \frac{2E}{r} - E = 2$$

or

$$\frac{1}{n} + \frac{1}{r} = \frac{1}{2} + \frac{1}{E}.$$

We know to begin with that $n \geq 3$ and $r \geq 3$, since a polygon must have at least three sides, and at least three sides must meet at each polyhedral angle. But n and r cannot both be greater *than three, for then the left hand side of the last equation could not exceed $\frac{1}{2}$, which is impossible for any positive value of E. Therefore, let us see what values r may have when $n = 3$, and what values n may have when $r = 3$. The totality of polyhedra given by these two cases yields the number of possible regular polyhedra.*

For $n = 3$ the last equation becomes

$$\frac{1}{r} - \frac{1}{6} = \frac{1}{E};$$

r can thus equal 3, 4, or 5. (6, or any greater number, is obviously excluded, since $1/E$ is always positive.) For these values of r we get $E = 6, 12$, or 30,

corresponding respectively to the tetrahedron, octahedron, and icosahedron. Likewise, for $r = 3$ we obtain the equation

$$\frac{1}{n} - \frac{1}{6} = \frac{1}{E},$$

from which it follows that $n = 3, 4$, or 5, and $E = 6, 12$, or 30, respectively. These values correspond respectively to the tetrahedron, cube, and dodecahedron.

While Euler's formula is valid for simply-connected polyhedra (regular and truncated polyhedra, pyramids, prisms, cuboids, frustums, crystal-lattice unit cells of various kinds) which are all topological spheres, *it fails for solids with holes in them and non-convex star-polyhedra. Thus, Kepler (1619) described the small and great* stellated dodecahedra *with $V = 12$, $F = 12$, $E = 30$, $\quad V - E + F = -6$ and* **Lhuilier** *(1813) noticed that Euler's formula was wrong for certain families of solid bodies. For a solid with g holes Lhuilier showed that $\quad V - E + F = 2 - 2g$.*

Consider for example a non-simply-connected polyhedron such as the prismatic block, consisting of a regular parallelepiped with a hole having the form of a smaller parallelepiped with its sides parallel to the outer faces of the block. Introducing just enough extra edges and faces to render all faces simply-connected polygon interiors (rectangles and trapezoids), this polygon is seen to have $V = 16$, $\quad E = 32$ and $\quad F = 16$ such that $\quad V - E + F = 0$. This corresponds to Lhuilier's formula with $g = 1$.

To understand the significance of the number g and its role in the topological classification of surfaces[208]*, we compare the surface of the sphere with that of a torus. Clearly, these two surfaces differ in a fundamental way: on the sphere, as in the plane, every simple closed curve separates the surface into two disconnected parts. But on the torus there exist closed curves that do not separate the surface into two parts — for example, the two* generator *circles on the torus surface. Furthermore, such a closed curve cannot be continuously shrunk to a point — whereas* any *closed curve on a* sphere *can be so shrunk. This difference between the sphere and the torus marks the two surfaces as belonging to two topologically distinct classes, because this shows that it is impossible to deform one into the other in a continuous way.*

Likewise, on a surface with two holes we can draw four *closed curves each of which does not separate the surface into disjoint components; these can be*

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[208] For the time being, we consider only *two-sided* and *closed* surfaces — i.e., we assume the surface has no boundary and that an ant, walking on one of its two sides, can never reach the opposite side without puncturing the surface. A 2-sided surface is also known as an *oriented* surface.

chosen to be the four generator curves (two per hole). Furthermore, one can draw two (non-intersecting) closed curves that, drawn simultaneously, still do not separate the two-hole surface. The torus is always separated into two parts by any two non-intersecting closed curves. On the other hand, three closed non-intersecting curves always separate the surface with two holes.

These facts suggest that we define the genus of a (closed and 2-sided) surface as the largest number of non-intersecting simple closed curves that can be simultaneously drawn on the surface without separating it. The genus of the sphere is 0, that of the torus is 1, while that of a 2-holed doughnut is 2. A similar surface with g holes has the genus g. The genus is a topological property of a surface and thus remains the same if the surface is deformed. Conversely, it may be shown that if two closed 2-sided (oriented) surfaces have the same genus, then one may be continuously deformed into the other, so that the genus $g = 0, 1, 2, \ldots$ of such a surface characterizes it completely from the topological point of view.

For example, the two-holed doughnut and the sphere with two "handles" are both closed surfaces of genus 2, and it is clear that either of these surfaces may be continuously deformed into the other. Since the doughnut with g holes, or its equivalent, the sphere with g handles, is of genus g, we may take either of these surfaces as the topological representative of all closed oriented surfaces of genus g.

Suppose that a surface S of genus g is divided into a number of regions (faces) by marking a number of vertices on S and joining them by curved arcs. As stated above, it has been shown that

$$V - E + F = 2 - 2g,$$

where V = number of vertices, E = number of arcs, and F = number of faces or regions[209]*. The topological invariant on the L.H.S. is usually denoted χ and is known as the Euler characteristic of the surface (this invariant admits a generalization to even-dimensional manifolds of dimension higher than two). We have already seen that for the sphere, $V - E + F = 2$, which agrees with the above equation, since $g = 0$ for the sphere.*

Another measure of non-simplicity which is used in the classification of surfaces will emerge from the following example. Consider two plane domains: the first of these, a, consists of all points interior to a circle, while the second, b, consists of all points contained between two concentric circles. Any closed

[209] An outline of the proof: S can be constructed from a particular partitioning of the sphere by *identifying* $2g$ distinct sphere faces pairwise. This reduces E and V by the same integer, and reduces F by $2g$, thus resulting in a reduction of $V - E + F$ by $2g$ from its sphere value (2), as claimed.

curve lying in the domain a can be continuously deformed or "shrunk" down to a single point within the domain. *A domain with this property is said to be* simply connected. *The domain b, however, is not simply connected. For example, a circle concentric with the two boundary circles and midway between them cannot be shrunk to a single point within the domain, since during this process the curve would necessarily sweep through the center of the circles, which is not a point of the domain. A domain which is not simply connected is said to be* multiply connected. *If the multiply connected domain b is cut along a radius, the resulting domain is simply connected.*

More generally, we can construct domains with two "holes". In order to convert this domain into a simply connected domain, two cuts are necessary. If $h-1$ non-intersecting cuts from boundary to boundary are needed to convert a given multiply connected planar domain D into a simply connected domain, the domain D is said to be h-tuply connected. The degree of connectivity of a domain in the plane is an important topological invariant of the domain. The number h is called the connectivity number *assigned to every surface. It extends also,* mutatis mutandis, *to 3-dimensional bodies.*

As an example, consider a closed, non-self-intersecting polygon (a chain*) consisting of edges of a polyhedron. If the* surface *of the polyhedron is divided into two separate parts by every such closed chain of edges, we assign the connectivity $h=1$ to the polyhedron. Clearly, all simple polyhedra have connectivity 1, since the surface of the sphere is divided into two parts by every closed curve lying on it. Conversely, it is readily seen that all polyhedra with connectivity 1 can be continuously deformed into a sphere. Hence the simple polyhedra are also called* simply connected.

A polyhedron is said to have connectivity h if $h-1$ is the greatest possible number of chains that, when simultaneously drawn, do not cut the surface in two. Since $h-1=2$ for the prismatic block, its connectivity is $h=3$.

We thus set $h=1$ for the sphere and $h=3$ for the torus. Surfaces of higher connectivity can be constructed by flattening a sphere made of a plastic material, cutting holes into it, and identifying *(sewing together) each pair of stacked hole-boundary closed curves.*

We shall call such surfaces pretzels. *It can be proved that a pretzel with g holes (i.e. a g-handle surface) must have connectivity $h=2g+1$.*

On a general surface, the curves can be chosen more freely than on a polyhedra, where we restricted the choice to chains of edges. Various other definitions can be given for the connectivity of surfaces – for example, the following:

On a closed surface of connectivity h, we can draw $h - 1$ closed *curves without cutting the surface in two, but every system of h closed curves cuts the surface into at least two separate parts. On a closed surface of connectivity $h = 2g + 1$ there is at least one set of g closed, mutually non-intersecting curves – and no set of more than g such curves – having the property that the curves in the set do not cut the surface in two when drawn simultaneously.*

All the polyhedra and closed surfaces we have considered thus far had odd *connectivity numbers h and* even *Euler characteristics ($\chi = 2 - 2g$), related by the formula $\chi = 3 - h$. If we extend both concepts to surfaces with boundaries (i.e. open) — with χ still defined as $V - E + F$ and h now defined as the maximal number of simultaneous cuts (along closed* or *boundary-to-boundary open curves) leaving the surface connected — the formula becomes*[210] *$\chi = 2 - h$. And for such surfaces, χ and h may be both even* or *both odd.*

The numbers χ, g and h are all topological invariants. *So is the orientability/non-orientability property, which we explain next.*

The question arises whether there are any closed *(boundary-less) surfaces at all with* even connectivities *or* odd χ values; *or whether there are boundary-less surfaces for which genus and connectivity are* not *related by $h = 2g + 1$. Indeed, such surfaces* do *exist and are called* one-sided *(or* non-orientable*) surfaces.*

Hitherto we have been dealing with "ordinary" surface, i.e. those having two sides. This restriction applied to closed surfaces like the sphere or the torus and to surfaces with boundary curves, such as the disc, a sphere with two holes (i.e. with two discs removed) – equivalent to a cylinder – or a torus from which a single disc has been removed.

The two sides of such a surface could be painted with different colors to distinguish them. If the surface is closed, the two colors never meet. If the surface has boundary curves, the two colors meet only along these curves. A bug crawling along such a surface and prevented from puncturing it or crossing boundary curves, if any exist, would always remain on the same side.

***Möbius** made the surprising discovery that there exist surfaces with only* one *side. The simplest such surface is the so-called Möbius strip (Figure 2), formed by taking a long rectangular strip of paper and pasting its two ends*

[210] Also, the formula $h = 2g + 1$ does *not* always apply for a non-closed surface. For instance, a cylinder with g handles – equivalent to a g-handle sphere with two discs cut out – has $h = 2g + 2$; for $g = 0$ (a simple cylinder) $h = 2$, since it can be cut once ($1 = h - 1$) while maintaining connectedness — e.g. from boundary to boundary along the cylinder axis.

together after giving one end a half-twist. A bug crawling along this surface, keeping always to the middle of the strip, will return to its original position upside down and on the opposite side of the surface! The surface is thus indeed one-sided when considered globally*; only local portions of it can be said to have two sides. The Möbius strip also has but one edge, for its boundary consists of a single closed curve. The ordinary two-sided surface formed by pasting together the two ends of a rectangle without twisting has two distinct, disconnected closed boundary curves; topologically it is a cylinder (or a sphere missing two discs).*

If this surface is cut along a plane separating the two closed boundary-curves, it falls apart into two *such disjoint cylinder surfaces, each with a new closed-curve component to its boundary. Like the cylinder, the Möbius strip has a continuous family of closed curves in its interior, each having the property of not being continuously deformable to a single point. And, as in the case of the cylinder, all such curves of unit* winding-number *(i.e. consisting of a single component if the surface is cut back into the original rectangle) can be deformed into each other, and are thus topologically equivalent.*

However, unlike the cylinder, if the Möbius strip is cut along one of its non-shrinkable closed curves, we find that it remains in one piece[211]*. It is rare for anyone not familiar with the Möbius strip to predict this behavior, so contrary to one's intuition of what "should" occur. If the surface that results from cutting the Möbius strip along the middle is again cut along its middle, two separate but intertwined strips are formed.*

The connectivity of the Möbius strip is $h = 2$*, just as the untwisted open cylinder. It also may be characterized by means of another important topological concept which can be formulated as follows: Imagine every point of a given surface (with the exception of boundary points, if any) to be enclosed in a small closed curve that lies entirely on the surface. We then try to fix a certain sense (handedness) on each of these closed curves in such a way that any two curves that are sufficiently close together have the same sense. If such a consistent determination of sense of traversal is possible in this way, we call it an* orientation *of the surface and call the surface* orientable.

While all two-sided surfaces are orientable, one-sided surfaces are not. Thus the classification of surfaces into two-sided and one-sided surfaces is identical to the classification into orientable and non-orientable surfaces.

[211] The cut strip is in fact equivalent to a rectangular strip subjected to *two* half-twists before identifying its two (short) opposite sides — both half-twists being in the same sense. This strip is topologically equivalent to a cylinder, yet cannot be deformed into it without self-intersection if embedded in 3-D space ($\mathbb{R}^3$).

It is easy to see that a surface is non-orientable if and only if there exists on the surface some closed curve such that a continuous family of small oriented circles whose center traverses the curve will arrive at its starting point with its orientation reversed.

The Möbius strip is an open *one-sided surface and does not intersect itself. But it can be proven that all one-sided* closed *surfaces embedded in* $\mathbb{R}^3$ *(Euclidean 3-dimensional space) have self-intersections. However, the presence of curves of self-intersection need not represent a topological property in the sense that in some cases it can be transformed away by deformation, or eliminated by defining the surface* intrinsically *(without embedding it in a 3-D* $\mathbb{R}^3$ *space), or else by embedding it in an* $\mathbb{R}^n$ *space with* $n > 3$*. If this is not the case we say that the surface has* singular points *which are a topological property.*

This raises the question of whether there can exist any one-sided closed surface (2-D intrinsic manifold) that has no singular points. Such a surface was first constructed mathematically by Felix Klein, as follows. Consider an open tube (cylinder). A torus[212] *is obtained from it by bending the tube until the ends meet and then gluing (identifying) the boundary circles together. But the ends can be welded in a different way:*

Taking a tube with one end a little thinner than the other, we bend the thin end over and push it through the wall of the tube, molding it into a position where the two circles at the ends of the tube have nearby and concentric positions. We now expand the smaller circle and contract the larger one a little until they meet, and then join them together (Fig. 7). This does not create any singular points and gives us Klein's surface, also known as the Klein bottle. It is clear that the surface is one-sided and, in any $\mathbb{R}^3$ *embedding, intersects itself along a closed curve where the narrow end was pushed through the wall of the tube.*

The connectivity number of the Klein bottle is 3, like that of a torus. It can be shown that any closed, one-sided surface of genus g *is topologically*

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212 *Torus*: a surface (intrinsic or embedded in $\mathbb{R}^3$). The *intrinsic* torus is a rectangle with opposite ends identified without twists (Fig. 9(f)). An $\mathbb{R}^3$-embedded torus is generated by revolving a circle about a line (in its plane) that does not intersect the circle. One of its parametric representations in Gaussian surface coordinates (u, v) is

$$\boldsymbol{r}(u,v) = [(a + b\cos v)\cos u; \quad (a + b\cos v)\sin u; \quad b\sin v],$$
$$a > b > 0; \qquad 0 \le u < 2\pi, \quad 0 \le v < 2\pi.$$

a and b are the two radii of the $\mathbb{R}^3$ torus, while the coordinates u, v are azimuths along two generating circles.

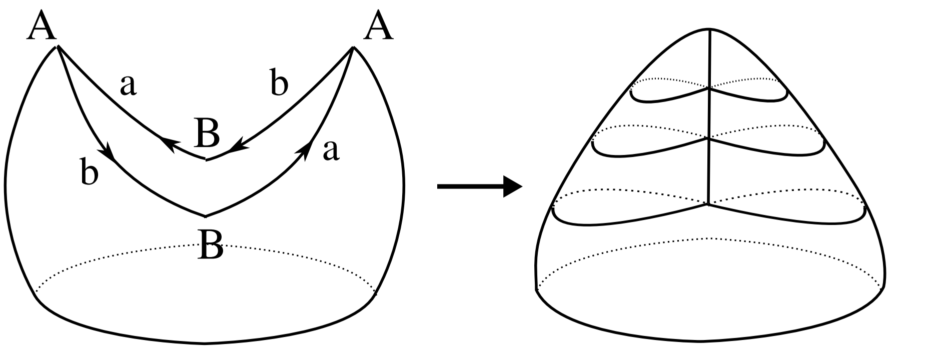


Fig. 1: Sewing up a cylinder to yield a representation of the Möbius strip as a topological sphere with cross-cap. The two copies of point A are identified, and similarly for B and the directed arcs a, b

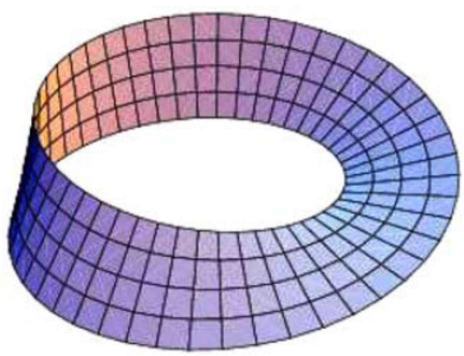


Fig. 2: An embedding of the Möbius strip in $\mathbb{R}^3$

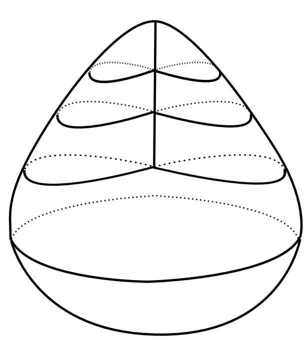


Fig. 3: The Real Projective Plane (sphere with one cross-cap)

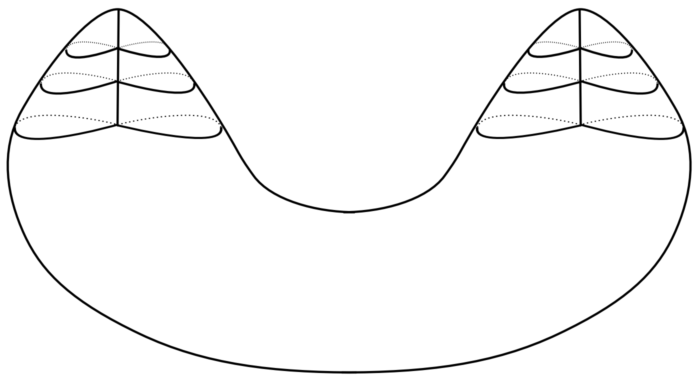


Fig. 4: Klein bottle represented as a topological sphere with two cross-caps

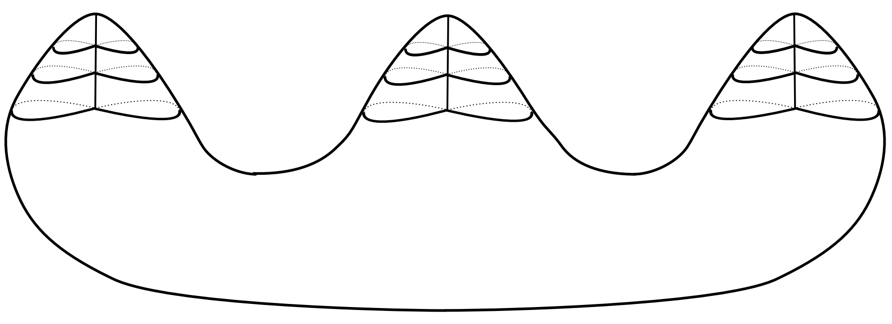


Fig. 5: The triple-crosscap surface (topological sphere with three cross-caps)

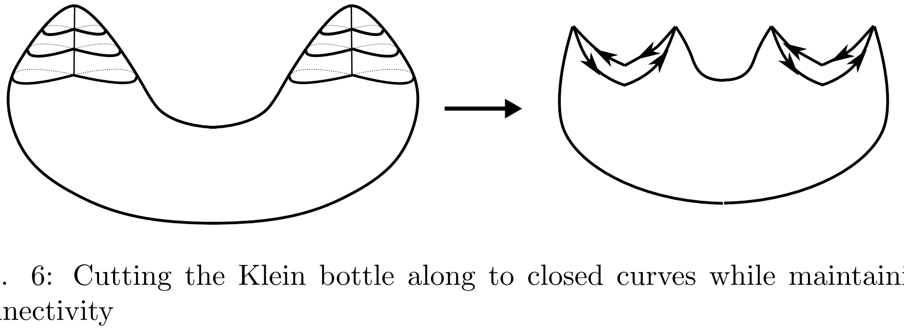


Fig. 6: Cutting the Klein bottle along to closed curves while maintaining connectivity

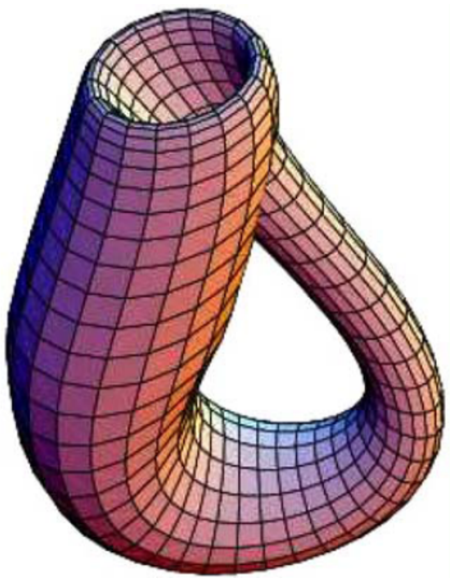
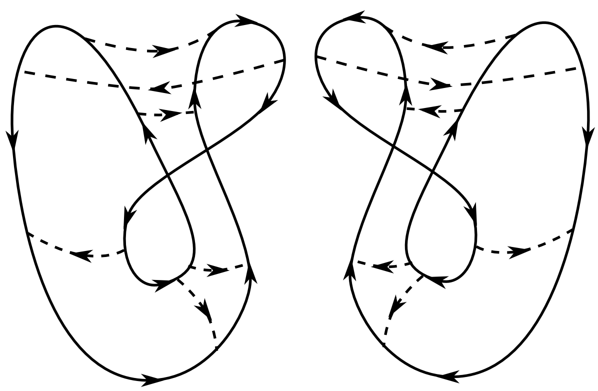
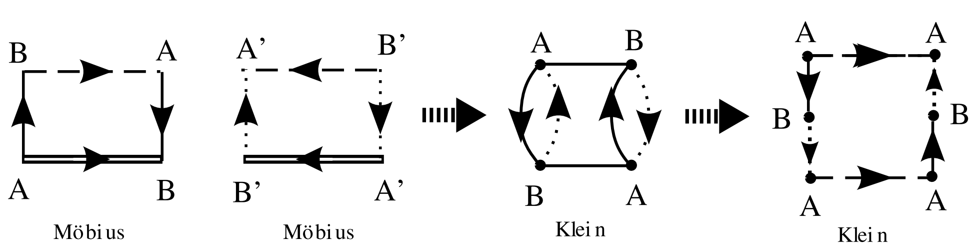


Fig. 7: A Klein bottle represented as a self-intersecting, nonsingular embedding (*immersion*) in $\mathbb{R}^3$



(a) Two Möbius strips with their boundaries sewn (identified) yield a Klein bottle.



(b)

Fig. 8

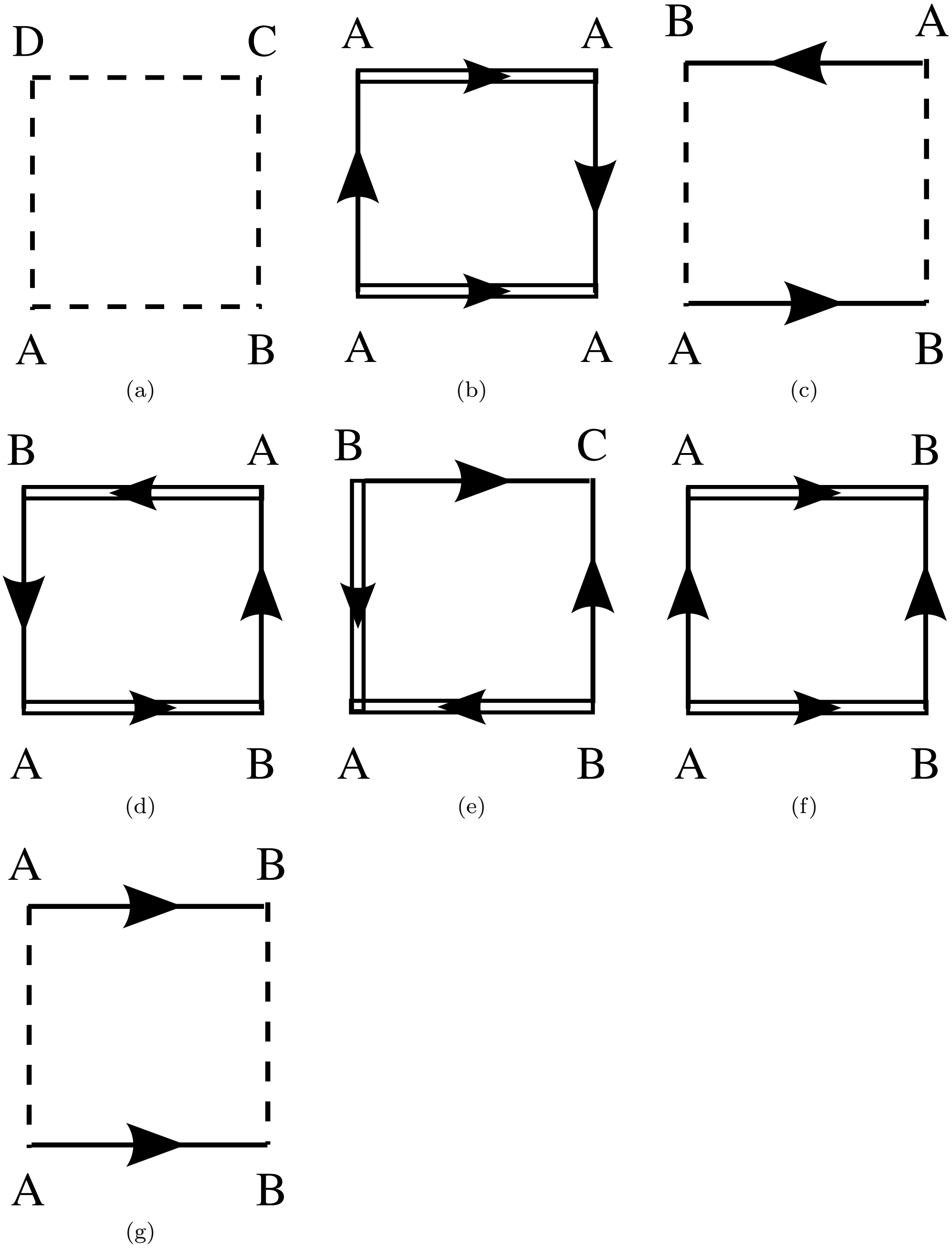


Fig. 9: Surfaces (open and closed, orientable and non-orientable) obtained from a rectangle by identifying (sewing) edges in various ways. Broken-line edges are not identified; arrows (single-line or double lines) are identified with same-type arrows (head with head and tail with tail). (a): topological disc; (b): Klein bottle; (c): Möbius strip; (d): real projective plane; (e): sphere; (f): torus; (g): cylinder (tube)

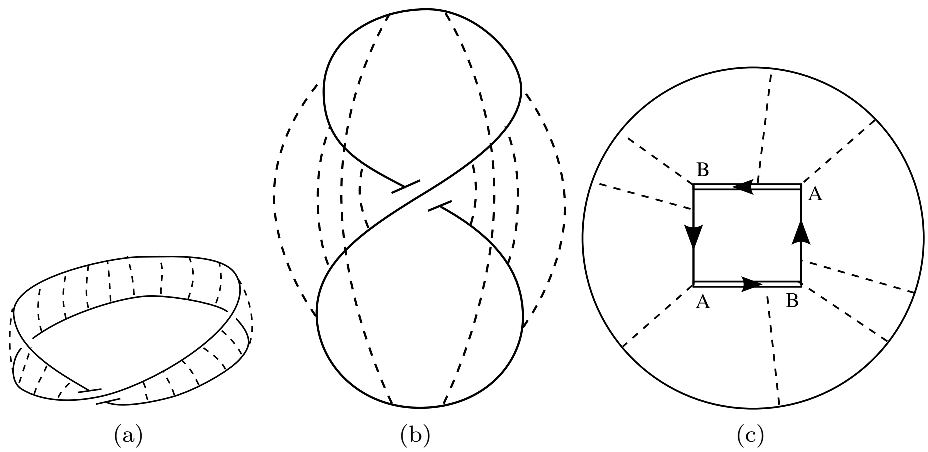


Fig. 10: Deforming the *standard* Möbius-strip representation into a *disk-with-crosscap*

equivalent to a sphere from which g discs have been removed and replaced by local topological constructs called cross-caps.[213]

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213 Cross-Caps: Modular building blocks for non-oriented surfaces.

The Möbius strip is the basic unit of non-oriented surfaces. It can be represented as described above: by half-twisting a rectangle and then identifying a pair of opposite sides (Figs. 2 and 9(c)). In the latter, the two corners labeled "A" are identified – "sewn" together – as are the two corners labeled "B"; and the two solid edges are identified in accordance with the arrow-indicated directions. But in this representation the Möbius strip has a complicated-looking boundary. Figures 1 and 10(a) to 10(c) show how to continuously deform this into the standard disc-with-crosscap representation. Unfolding the solid-line closed boundary of 10(a) yields 10(b) (the dotted lines indicate the Möbius-strip interior). Then, untwisting the boundary of 10(b) into a circle yields Fig. 10(c). In 10(c) it was necessary to cut the surface along some closed curve ABA to avoid intersections among the broken lines. This cut results in the rectangle depicted in 10(c), in which the two single-solid lines are identified along their arrows — as are the two double-solid lines. Re-sewing the cut ABA, as depicted in Fig. 1, results in a disc with one crosscap. This is a convenient representation of the Möbius strip, because its boundary is a simple curve (a circle if we wish), and also because any non-orientable surface (open or closed) is topologically equivalent to a sphere with some number of local discs removed, with some or all of these discs replaced by cross-caps. On the other hand — as clearly seen in Fig. 1 — the cross-cap representation of the Möbius strip makes it self-intersecting in a 3-D embedding (though it has no singular point).

From this it easily follows that the Euler characteristic $V - E + F$ of such a surface is related to g by the equation

$$V - E + F = 2 - g.$$

Thus for the Klein bottle, $g = 2$. For an orientable closed surface we have $\chi = 2 - 2g$, while for a non-orientable closed surface $\chi = 2 - g$.

Figs. 3-5 depict the three simplest classes of *closed* non-orientable surfaces, represented as a sphere with one, two and three local cross-caps, respectively. The class shown in Fig. 3 includes the *real projective plane* ($\mathbb{RP}^2$), obtained from $\mathbb{R}^3$ by identifying all points $(\lambda x, \lambda y, \lambda z)$ with fixed (x, y, z) and all real numbers λ; this class is also a Möbius strip with its boundary sewn to a disc. Fig. 4 — a sphere with two cross-caps — is equivalent to a Klein bottle (Fig. 7). This is because a Klein bottle can be constructed by sewing together the boundaries of two Möbius strips — as shown in two different ways in Figs. 8(a) and 8(b). Fig. 8(a) shows how two standard (twisted-strip) representations of Möbius strips are sewn along their boundaries to yield a single Klein bottle. In Fig. 8(b) it is done another way, by identifying the boundaries of two Möbius strips. Each separate Möbius strip is represented as a rectangle with two of its edges identified, as in Fig. 9(c). The final Klein bottle can be represented as a rectangle with its four edges identified pairwise as in Fig. 9(b). The vertices A, A' are identified with each other, as are B and B'; the two single-broken-line arrows are identified with each other (base with base and arrow-tip with arrow-tip), and the two double-solid-line arrows are similarly identified. The two remaining arrow pairs are separately identified *within* each Möbius strip, as in Fig. 9(c) (single-solid-arrows identified with each other, as are the single-dotted-arrows). Proceeding from left to right, the first solid-whit arrow indicates the sewing together of the boundaries of the two Möbius strips. The second solid-white arrow indicates two further operations: a $180°$ twisting of the *right* closed curve ABA to align it with the *left* closed ABA curve, followed by a cut along a curve between the two copies of point A. Neither operation changes the Klein bottle's topology.

Fig. 5 shows a sphere with three cross-caps; this can be shown to be topologically equivalent to a torus with a small disc removed and replaced with a cross-cap (*Dyck's theorem*).

Fig. 6 shows how a Klein bottle can be simultaneously cut along two closed curves while remaining a connected surface: the two cuts open the surface's two cross-caps, converting them into two closed-curve boundaries — the Klein bottle is thus converted into a topological cylinder.

Finally, Figs. 9(a)-9(g) show how to obtain various surfaces — open and closed, orientable and one-sided — by sewing (identifying) the vertices and edges of a single rectangle in various ways.

In general, for a closed orientable surface S of genus g, for which the $\mathbb{R}^2 \to \mathbb{R}^3$ mapping $\boldsymbol{r}(u,v)$ is twice continuously differentiable, the integral curvature *is equal to $\iint_S K\, dA = 4\pi(1-g)$, where K is the* Gaussian curvature *(this follows from the Gauss-Bonnet theorem). For the torus $(g = 1)$ the right-hand side vanishes. And indeed, in terms of the parametric representation of the $\mathbb{R}^3$-embedded torus given above:*

$$K = \frac{\cos v}{b(a + b\cos v)}, \qquad dA = (a + b\cos v) b\, du\, dv$$

and therefore the integral curvature is zero as claimed, as $\int_0^{2\pi} dv \cos v = 0$. Another interesting feature of the torus is that an elliptic function *defines a mapping of a plane into a torus. It arises from the notion that the curve $y^2 = ax^3 + bx^2 + cx + d$ can be parametrized as $x = f(z)$, $y = f'(z)$, where f and f' are elliptic functions (**Jacobi**, 1834).*

II. Topological mappings

Starting with the concept of a set *(such as points in a plane, lines through a point, rotations in 3-D, etc.) one can generalize the idea of a* function *to that of a* map: *A map is a relation between two specified sets that associates a unique element of the second to each element of the first.*

To establish topological equivalence *between sets, one must have a mathematical machinery that is able to transform one of the sets into the other, and this transformation must be a map endowed with various properties.*

There are various methods of mapping one surface (or higher-dimensional manifold space, whether intrinsically defined or embedded) onto another. The most faithful image of a surface is obtained by an isometric, *or* length-preserving, *mapping*[214]. *Here the geodesic distance between any two points is preserved (assuming the surface or space is endowed with a* metric[215]*), all angles remain unchanged, and geodesic lines are mapped into geodesic lines. An isometry also preserves the Gaussian curvature at corresponding points. Hence the only surfaces that can be mapped isometrically into a* part of the plane *are surfaces whose Gaussian curvature is everywhere zero; this excludes, for example, any portion of the sphere. In consequence, no* geographical map *(i.e. map of the earth's surface) can be free of distortions.*

Less accurate, but also less restrictive, are the area-preserving mappings. *They are defined by the condition that the area enclosed by every closed curve be preserved. With the aid of such a mapping portions of the sphere* can *be mapped onto portions of the plane, and this is frequently used in geography. It is achieved in practice by projecting points of the sphere onto the cylinder along the normals of the cylinder. If the cylinder is now slit open along a generator and developed into a plane, the result is an area-preserving image of the sphere in a plane; the distortion increases the further we are from the circle along which the cylinder touches the sphere.*

Another type of mapping, especially useful for navigation, is that of geodesic maps, *where geodesics are preserved. If, for example, a portion of a*

[214] In the Euclidean plane all isometries can be generated by combining a two-parameter translation, a one-parameter rotation, and a single reflection about some fixed axis. In Euclidean $\mathbb{R}^3$, there are 3 translations parameters and 3 rotation-angle parameters, but still only one independent reflection, which can be implemented with the help of a *mirror*. No more than 3 mirrors (i.e. three reflection planes) are needed to generate any isometry in $\mathbb{R}^3$.

[215] In the case of a 2D surface embedded in $\mathbb{R}^3$, the natural surface metric is the one inherited from the Euclidean metric of the "host" $\mathbb{R}^3$ space.

sphere is projected from its center onto a plane, then the great circles are mapped into straight lines of the plane, and the map is therefore geodesic. At the same time, this gives us a (local) geodesic mapping of all surfaces of constant positive Gaussian curvature *into the plane, because all these surfaces can be mapped isometrically into spheres. All surfaces with* constant negative Gaussian curvature *can also be mapped into the plane by a geodesic mapping.*

Yet another type of mapping is that of the conformal*, or* angle-preserving*, mappings, for which the angle at which two curves intersect is preserved. The simplest examples of conformal mapping, apart from the isometric mapping, are* stereographic projections *and the circle-preserving transformations*[216]*. A stereographic projection map, in which a sphere (with its north-pole removed) is placed atop a plane and projected onto it by drawing straight lines from that pole, is also a circle-preserving map.*

It can be shown that very small figures *suffer hardly any distortion at all under general conformal transformations; not only angles are preserved, but the* ratios of distances *(although not the distances themselves) are approximately preserved.* In the small*, the conformal mappings are thus the nearest thing to isometric mappings among all the types of mappings mentioned earlier, for area-preserving and geodesic mappings may bring about arbitrarily great distortions in arbitrarily small figures.*

The most general mappings that are at all comprehensible to visual intuition are continuous invertible mappings (homeomorphisms)*. The only condition here is that the mapping is* one-to-one *and that neighboring points (and* only *such) go over to neighboring points. Thus a homeomorphic mapping may subject any figure to an arbitrary amount of distortion, but it is not permitted to* tear *connected regions apart or to stick separate regions together. Yet, continuous mappings do not always exist that can map (which we refer to here as "continuous" for simplicity's sake) two given surfaces onto each other (Example: the circular disc and the plane annulus bounded by two concentric circles cannot be mapped continuously into each other, even not their boundaries alone!). Clearly, the class of continuous mappings embraces all the types of mapping mentioned so far. The question of when two surfaces can be mapped onto each other by a continuous mapping is one of the problems of topology.*

The simplest type of topological mapping of a surface consists of a continuous mapping (homeomorphism) which is such as to transform the surface

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216 An example of a circle-preserving map is the $\mathbb{R}^2 \to \mathbb{R}^2$ *inversion map* w.r.t. a given circle. If the latter is $x^2 + y^2 = a^2$, this inversion is $x \to \frac{a^2 x}{x^2+y^2}$, $y \to \frac{a^2 y}{x^2+y^2}$. It is a special type of conformal transformation that maps *any* circle into another circle.

as a whole onto itself, and *which is arcwise-connected with the trivial (unity) map (in which each surface point is mapped to itself*[217]*). This type of mapping is called a* deformation. *The* reflection *of a plane about a straight line, on the other hand, is an example of a topological mapping which is not a deformation; for a reflection reverses the sense of traversal (orientation) of every circle, whereas deformations cannot reverse the sense of traversal.*

A point that is mapped onto itself under the mapping is called a fixed point *of the mapping. In the applications of topology to other branches of mathematics, "fixed-point" theorems play an important role. The* theorem *of* ***Brouwer*** *states that every continuous deformation of a circular disc (with the points of the circumference included) onto itself has at least one fixed point. On a sphere, any continuous transformation which carries no point into its diametrically opposite points (e.g. any* small *deformation) has a fixed point.*

Fixed point theorems provide a powerful method for the proof of many mathematical "existence theorems" which at first sight may not seem to be of a geometrical character. Also, topological methods have been applied with great success to the study of the qualitative behavior of dynamical systems. A famous example is a fixed point theorem conjectured by **Poincaré** *(1912), which has an immediate consequence: the existence of an infinite number of* periodic orbits *in the restricted problem of three bodies.*

Apart from the choice of the mapping transformation there is yet another problem that must be resolved; in describing a surface or other manifold, there is the freedom of choice of a suitable coordinate system *(CS). In general we cannot restrict ourselves to manifolds which can be covered by a* single *CS such as is suitable for an n-dimensional Euclidean space $\mathbb{R}^n$; simple examples of various kinds of surfaces embedded in E^3 indicate that, in general, no single CS can exist which covers a given surface completely.*[218]

The simplest example is a 2-dimensional spherical surface in E^3 (the latter having Cartesian coordinates (x_1, x_2, x_3)), which we wish to map onto a planar disc. To obtain a one-to-one correspondence in the mapping, one may choose

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217 Two continuous mappings $f : A \to B$, $g : A \to B$ from a set A to a set B are said to be arcwise-connected (or *continuously deformable* into each other) if there exists an arc of continuous functions $h(s) : A \to B$, $s \in [0, 1]$, such that $h(0) = f$, $h(1) = g$, and h is a continuous map from $[0, 1] \times A$ to B.

218 E^n is the space $\mathbb{R}^n$ with a Euclidean metric (norm). A CS (coordinate system) is a homeomorphism between a region (open subset) of the surface (or higher-dimensional manifold) and $\mathbb{R}^m$, m being the manifold's *dimension* ($m = 2$ for a surface). For a manifold requiring more than one CS, it is assumed that the open subsets *cover* the manifold, and that in the intersection of any two subsets, the two CS maps compose to yield a $\mathbb{R}^m \to \mathbb{R}^m$ homeomorphism.

the hemisphere for which $x_1 > 0$, *which is then continuously mapped onto a disc in the* x_2x_3 *plane. Accordingly, this hemisphere is referred to as a coordinate neighborhood.*

Similarly, 5 other hemispheres corresponding to the respective restrictions $x_1 < 0$; $x_2 > 0$; $x_2 < 0$; $x_3 > 0$; $x_3 < 0$ *can be regarded as coordinate neighborhoods. The totality of these 6 hemispheres covers the sphere completely, and in the overlap of any pair of them, composing the two corresponding maps yields a continuous map of one planar disc onto another. In general, the existence and overlap structure of suitable coordinate neighborhoods depends on the topological properties of the surface taken as a whole. This shows that one must give up on the construction of a unique CS for all points of a space under consideration and use different CS for different parts of the space.*

A surface, however curved and complicated, can be thought of as a set of little curved patches glued together; and topologically (though not geometrically) each patch is just like a patch in the ordinary Euclidean plane. It is not this local *patch-like structure that produces things like the hole in a torus: it is the* global *way all the patches are glued together. Once this is clear, the step to* n *dimensions is easy: one just assembles a space from little patches carved out of* n*-dimensional space instead of a plane. The resulting space is an* n*-dimensional manifold. For example: the motion of three bodies under mutual gravitational forces involves an 18-dimensional phase-space manifold, with 3 position coordinates and 3 velocity coordinates per body.*

III. Algebraic topology

Algebraic topology is the study of the global *properties of spaces by means of abstract algebra. One of the earliest examples is Gauss's linkage formula which tells us whether two closed space curves are linked, and – if so – how many times does any one of them wind around the other. The linkage number remains the same even if we continuously deform the space curves. The central idea here is that continuous geometric phenomena can be understood by the use of integer-valued topological invariants.*

One of the strengths of algebraic topology has always been its wide degree of applicability to other fields. Nowadays that includes fields like theoretical physics, differential geometry, algebraic geometry, and number theory. As an example of this applicability, here is a simple topological proof that every non-constant polynomial $p(z)$ *has a complex zero (root) — a key component in proving the fundamental theorem of algebra.*

Consider a circle of radius R and center at the origin of the complex plane. The polynomial transforms this into another closed curve in the complex plane. If this image curve ever passes through the origin, we have our zero. If not, suppose the radius R is very large. Then the highest power of $p(z)$ dominates and hence $p(z)$ transforms the circle into a curve which winds around the origin the same number of times as the degree of $p(z)$. This is called the winding number *of the curve around the origin. It is always an integer and it is defined for every closed curve which does not pass through the origin. If we deform the curve, the winding number has to vary continuously but, since it is constrained to be an integer, it cannot change and must be a constant unless the curve is deformed through the origin.*

Now deform the image curve by shrinking the radius R to zero and suppose that the image curve never passes through the origin, that is to say, the original circle, in shrinking, never passes through a zero of the polynomial. The image curve gets very small since $p(z)$ is continuous; hence it must have winding number 0 around the origin unless it is shrinking to the origin (which cannot be the case unless $p(0) = 0$). If the image curve is shrinking to the origin, the origin is a zero of $p(z)$. If not, the winding number is 0 which means that the polynomial must have degree 0; in other words, it is a constant.

The winding number of a curve illustrates two important principles of algebraic topology. First, it assigns to a geometric object, the closed curve, a discrete invariant, the winding number which is an integer. Second, when we deform the geometric object, the winding number does not change, hence, it is called an invariant of deformation or, synonymously, an invariant of homotopy*. The field is called* algebraic *topology because an equivalence class of geometric entities possessing the same invariant — e.g. linkage number between curves; winding numbers of curves about points, or of closed surfaces in many-to-one mappings about other closed surfaces; winding numbers of non-shrinkable generator curves on the surface of a surface of nonvanishing genus; et cetera — turn out to form algebraic structures, such as rings and groups, under various geometric operations.*

IV. From curves and knots to manifolds

A simple closed curve (one that does not intersect itself) is drawn in the plane. What property of this figure persists even if the plane is regarded as a sheet of rubber that can be deformed in any way? The length of the curve and the area that it encloses can be changed by a deformation. But there is a

topological property of the configuration which is so simple that it may seem trivial: A simple closed curve C in the plane divides the plane into exactly two domains, an inside and an outside. By this is meant that those points of the plane not on C itself fall into two classes — A, the outside of the curve, and B, the inside — such that any pair of points of the same class can be joined by a curve which does not cross C, while any curve joining a pair of points belonging to different classes must cross C. This statement is obviously true for a circle or an ellipse, but the self-evidence fades a little if one contemplates a complicated curve like the twisted polygon. This problem was first stated by **Camille Jordan** *(1882) in his* Cours d'analyse. *It turned out that the proof given by Jordan was invalid.*

The first rigorous proofs of the theorem were quite complicated and hard to understand, even for many well-trained mathematicians. Only recently have comparatively simple proofs been found[219]*. One reason for the difficulty lies in the generality of the concept of "simple closed curve", which is not restricted to the class of polygons or "smooth" curves, but includes all curves which are topological images of a circle. On the other hand, many concepts such as "inside", "outside", etc., which are so clear to the intuition, must be made precise before a rigorous proof is possible.*

It is of the highest theoretical importance to analyze such concepts in their fullest generality, and much of modern topology is devoted to this task. But one should never forget that in the great majority of cases that arise from the study of concrete geometrical phenomena it is quite beside the point to work with concepts whose extreme generality creates unnecessary difficulties. As a matter of fact, the Jordan curve theorem is quite simple to prove for the reasonably well-behaved curves, such as polygons or curves with continuously turning tangents, which occur in most important problems.

A knot *is formed by first looping and interlacing a piece of string and then joining the ends together. The resulting closed curve represents a geometrical figure the "knotiness" of which remains essentially the same even if it is deformed by pulling or twisting without breaking the string. But how is it possible to give an intrinsic characterization that will distinguish a knotted closed curve in space from an unknotted curve such as the circle? The answer is by no means simple, and still less so is the complete mathematical analysis of the various kinds of knots and the differences between them. Even for the simplest case this has proved to be a daunting task.*

Consider, for example, two knots which are completely symmetric mirror images of one another. The problem arises whether it is possible to deform

[219] A generalization of the *Jordan theorem* to arbitrary surface is used in proving the *surface classification theorem* cited earlier.

one of these knots into the other in a continuous way. The answer is in the negative, but the proof of this fact requires considerable knowledge of the technique of topology and group theory.

Knots are the most immediate topological features of curves in space. Beyond curves come surfaces; beyond surfaces come multidimensional generalizations called manifolds, introduced by ***Riemann****.*

Whereas mathematical analysis and the theory of differential equations deal primarily with "local" properties of a function (only infinitesimally adjacent points are considered), geometry studies the "global" properties of functions (i.e. their properties are analyzed by considering finitely spaced points). This intuitive idea of globality has given rise to the fundamental concept of manifold *as a generalization of the concept of* domain *in Euclidean space.*

A coordinate system describing the positions of points in space is an indispensable tool for studying geometrical objects. Using coordinate systems, we can apply the methods of differential and integral calculus to solve various problems. Therefore, an analysis of spaces which admit such concepts as differentiable or smooth functions, differentiation and integration, has emerged as an independent branch of geometry.

Topologists would like to do for manifolds what they have already done for surfaces and knots. Namely:

(1) Decide when two manifolds are or are not topologically equivalent.

(2) Classify all possible manifolds.

(3) Find all the different ways to embed one manifold in another (e.g. a knotted circle in 3-space).

(4) Decide when two such embeddings are, or are not, the same.

The answer to problems (1) and (2) lies in an area called homotopy theory *which is part of* algebraic topology*. It endeavors to associate various algebraic invariants with topological spaces.* ***Poincaré*** *was one of the fathers of this theory. But problems (1) and (2) have not yet been fully resolved. Problem (3) led topologists to some surprising and counter-intuitive results, as the following example shows. It has been asked: when can two 50-dimensional spheres be* linked *[i.e. embedded such that they cannot be separated by a topology-preserving transformation of the surrounding n-dimensional space]. The answer is:*

cannot link	*for*	$n \geq 102$
can link	*for*	$n = 101, 100, 99, 98$
cannot link	*for*	$n = 97, 96$
can link	*for*	$n = 95, \ldots, 52.$

V. Networks

Graph (or Network) theory had its origin in a paper by Euler (1736) including the famous problem of the bridges of Königsberg. ***Euler*** *saw that the problem could more easily be studied reducing island and banks to points and drawing a network (graph) in which two points are connected by an edge whenever there is a bridge connecting the corresponding two land masses. In this way Euler was able to abstract the problem so that only information essential to solving it was highlighted, and he could dispense with all other aspects of the problem. He could thus rephrase the problem as follows: "Given a connected graph, find a path that traverses each edge of the graph without retracing any edge." Such a path is called a Eulerian traversal or Eulerian path*[220]*. Some experimentation and application of logic lead to the conclusion that in order to have a Eulerian path, it is necessary that for any edge along which the path enters a vertex, there must correspond a distinct edge along which the path leaves it — and that all such edge-pairs be distinct for any given vertex. The only exception occurs for the beginning and ending vertices of the path, if these points are different.*

Networks can be used to solve mazes and guarantee that one can find a path through a maze, if such a path exists, even when no map is explicitly given. Other procedures enable people to retrace their steps to the beginning of a labyrinth. Some of these procedures have applications to problems of computer processing, traffic control, electrical engineering, and many other fields.

During the ten generations elapsed since 1736, mathematicians have developed a new branch of geometry — a geometry of dots and lines, otherwise known as graph theory — that preserves geometrical relations only in their most general outlines. Here lines do not have to be straight, nor are there such things as perpendicular or parallel lines, and it does not make sense to talk about bisecting lines or measuring lengths or angles. The power of graph theory (a sub-field of topology) is that it can be used to model many patterns

[220] *A practical architectural application*: In the hallways of a museum, pictures are hung on one side of each hall. How does one design a tour that will enable a person to see each exhibit exactly once?

in nature — from the branching of rivers to the cracking or brittle of surfaces to subdivision of cellular forms, as well as many abstract concepts. It gives us a way to study spatial structures unencumbered by the details of Euclidean geometry.

A geographical map *shows countries, borders and corners. From such a map we may prepare an abstract* mathematical map *in which countries are* faces (F)*, borders are chains of pairs of adjacent* edges (E) *and corners are* vertices (V)*. In order to study the topology of a map in the technical language of mathematics, we must forget its geographical significance and treat it as merely a network, or graph, being a set of faces, edges and vertices,* $M = \{F, E, V\}$*, with certain* incidence relations *among them (e.g. face* f_1 *has edges* (e_2, e_3)*; edge* e_2 *is shared by faces* $\{f_1, f_3\}$*; vertex* v_1 *is shared by* $\{f_1, f_2, f_3\}$ *and also by* $\{e_2, e_3, e_4\}$*; etc.). In this context the face is represented by some polygon and each edge lies in exactly two faces. Copies of a map formed by placing it on a flexible membrane and stretching the membrane without cutting, are considered identical or* homeomorphic*. Edges and faces thus become distorted but the sets* E*,* F *and* V *and their relational structure (incidence relations) maintain their integrity.*

From a mathematical point of view, maps on a plain and maps on the sphere*, with one point removed, are isomorphic. Since all the enclosed areas, including one additional outer one, are now considered to be faces, and maps are always considered to be in one piece (connected), one can show that* Euler's formula $V - E + F = 2$ *holds for connected planar maps on either a plane or a sphere.*

There is a family of maps for which each vertex, edge or face is like every other vertex, edge or face. They are called regular maps *and are said to have perfect symmetry. Upon finding oneself stranded in a mathematical country defined by such a map, one would experience vistas of sameness in all directions and be hopelessly lost.*

There are only five[221] *such regular maps and they correspond to the five 3-dimensional Platonic Solids. In fact, they are obtained by projecting the edges of a Platonic polyhedron onto a plane from a point directly above the center of one of its faces, and counting the infinite area outside the boundary as an additional face. These are known as* Schlegel diagrams*.*

Visually, this amounts to holding one face of a polyhedron quite close to one's eyes, looking at the structure through the face, and drawing the projection of the structure as seen in this exaggerated perspective. The number of vertices, faces and edges for the Schlegel diagrams then becomes identical to those of the corresponding Platonic Solids.

[221] Except for two *trivial* families, one of which consist of all regular polygons.

Just as there are only five regular maps on the sphere (or plane), there are only three classes of regular maps that can be created on a torus.

For a surface homeomorphic to a sphere with g handles Euler's formula becomes $V - E + F = 2 - 2g$.

On a torus, for example, we have (with $g = 1$) $V - E + F = 0$.

G. R. Kirchhoff *enunciated (1845) laws which allow calculations of currents, voltages and resistances of electrical networks. In the framework of these laws he became interested in the mathematical problem of the number of independent circuit equation in a given network. Considering the electrical network as a geometrical object (map) constructed from points (vertices, V) and lines (edges, E), Kirchhoff proved that, in general, the number of independent circuits[222]is equal to $(E - V + 1)$.*

His paper is quite modern in its approach, and he used various constructions which we now think of as standard in graph theory. But he did not have the algebraic techniques that are needed to extend the results to higher dimensions. However, the basic ideas were latent in Kirchhoff's paper, and it was just those ideas which mathematicians were able to develop in the second half of the 19th century, in order to create what we now call 'algebraic topology'. This development did not happen overnight.

The apparatus of vectors, matrices, and what we call now linear algebra, as well as the abstract algebra of groups, rings, homeomorphisms etc., were not available to Kirchhoff, Listing, and the other mathematicians of the 1840's. However, in the course of time all these ingredients developed into a program which turned some very vague and descriptive ideas about the 'holeyness' of solids into an impressive general theory – an algebraic context within which these ideas can be formulated independently of any intuitive notions. There are many famous names associated with this program. One of them was the Italian mathematician ***Enrico Betti****, who introduced numbers, knows as Betti numbers, which turn out to be a generalization of the Kirchhoff number $(E - V + 1)$.*

But the person who made the greatest advances, in a series of papers published around 1895, was the French mathematician ***Henri Poincaré****. He formulated everything in terms of multi-dimensional objects (complexes), built out of what he called simplexes, and he showed how the rules by which they are fitted together can be described by means of matrices. He also showed how the 'holeyness' of complexes can be described algebraically in terms of properties of these matrices.* ***Veblen*** *(1916) gave a modern treatment of Poincaré's theory.*

[222] This is compatible with *Euler's formula* if we equate the number of independent circuits to $(F - 1)$.

1847 CE **Johann Benedict Listing** (1806–1882, Germany). Mathematician. Started the systematic study of topology as a branch of geometry, and coined the word 'topology'. Some topological problems are found in the works of **Euler**, **Möbius** and **Cantor**, but the subject only came into its own in 1895 with the work of **Poincaré**.

1847–1852 CE **Matthew O'Brien** (1814–1855, England). Mathematician. A forerunner of **Gibbs** and **Heaviside**. Introduced the modern symbols for vector multiplication.

History of the Wave Theory of Sound[223]

The speculation that sound is a wave phenomenon grew out of observations of water waves. The rudimentary notion of a wave *is that of an oscillatory* disturbance *that moves away from some source and transports no discernible amount of matter over large distances of* propagation.

The possibility that sound exhibits analogous behavior was emphasized by the Greek philosopher ***Chrysippos*** *(ca 240 BCE), by the Roman architect*

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[223] For further reading, see:

- Crighton, D.G. et all, *Modern Methods in Analytical Acoustics*, Springer Verlag: Berlin, 1992, 738 pp.
- Pierce, A.D., *Acoustics*, American Institute of Physics, 1989, 678 pp.
- Dowling, A.P. and J.E. Ffowcs Williams, *Sound and Sources of Sound*, Ellis Horwood, 1983, 321 pp.
- Lord Rayleigh, *Theory of Sound*, Vols I-II, Dover: New York, 1945.
- Morse, P.M. and K.U. Ingard, *Theoretical Acoustics*, McGraw-Hill, 1968, 927 pp.

*and engineer **Vitruvius** (ca 35 BCE), and by the Roman writer **Boethius**[224] (ca 475–524).*

*The pertinent experimental result that the air motion generated by a vibrating body (sounding a single musical note) is also vibrating at the same frequency as the body[225], was inferred with reasonable conclusiveness in the early 17^{th} century by **Marin Mersenne** (1636) and **Galileo Galilei** (1638).*

Mersenne's description of the first absolute determination of the frequency of an audible tone (at 84 Hz) implies that he had already demonstrated that the frequency ratio of two vibrating strings, radiating a musical note and its octave, is as $1\colon 2$. The perceived harmony (consonance) of two such notes would be explained if the ratio of the air oscillation frequency is also $1\colon 2$, which in turn is consistent with the source-air motion frequency equivalence hypothesis.

*The analogy with water waves was strengthened by the belief that air motion associated with musical sound is oscillatory and by the observation that sound travels with finite speed. Another matter of common knowledge was that sound bends around corners, which suggested diffraction, a phenomenon often observed in water waves. Also, **Robert Boyle**'s (1660) classic experiment on the sound radiation by a ticking watch in a partially evacuated glass vessel provided evidence that air is necessary, both for the production and transmission of sound.*

The apparent conflict between ray and wave theories played a major role in the history of the sister science of optics, but the theory of sound developed almost from the beginning as a wave theory.

*When ray concepts were used to explain acoustic phenomena (as was done by **Reynolds** and **Rayleigh** in the 19^{th} century), they were regarded, either explicitly or implicitly, as mathematical approximations to a well-developed wave theory.*

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[224] Born into an aristocratic Christian family and became a consul (510). He wrote texts on geometry and arithmetic which were of poor quality but used for many centuries during a time when mathematical achievements in Europe were remarkable low. Boethius fell from favor and was imprisoned and later executed for treason and magic.

[225] The history of this is intertwined with the development of the laws of vibrating strings and the physical interpretations of musical consonances, which goes back to **Pythagoras** (ca 550 BCE) and perhaps earlier. Thus, the dual nature of wave-motion in both time and frequency domains goes back all the way to the ancient Greeks.

The successful incorporation of geometrical optics into a more comprehensive wave theory had demonstrated that viable approximate models of complicated wave phenomena could be expressed in terms of ray concepts. This recognition has strongly influenced 20^{th} *century development in architectural acoustics, underwater acoustics, and noise control.*

The mathematical theory of sound propagation began with ***Isaac Newton*** *(1642–1727), whose* Principia *(1686) included a mechanical interpretation of sound as being* pressure *pulses transmitted through neighboring fluid particles*[226]*. Substantial progress toward the development of a viable theory of sound propagation resting on firmer mathematical and physical concepts was made in* 1759–1816 *by* ***Euler****,* ***d'Alembert****,* ***Lagrange*** *and* ***Laplace****. During this era, continuum physics, or field theory, began to receive a definite mathematical structure. The* wave equation *emerged in a number of contexts, including the propagation of sound in air. The theory ultimately proposed for sound in the* 18^{th} *century was incomplete from many standpoints, but the modern theories of today can be regarded for the most part as refinements of that developed by Euler and his contemporaries.*

The linearized equations of the acoustic field are derived directly from the general equations of fluid motion on the basis that the fluid velocity $\boldsymbol{u}$*, the* change *of pressure* p*, and the* change *of density* ρ *— are all small compared to the sound velocity* c*, average ambient density* ρ_0*, and average background pressure* p_0*, respectively, such that products of the small entities can be neglected in the equations.*

There are three fundamental equations relating the above entities:

(1) Newton's equation of motion (conservation of the fluid linear momentum) relating the pressure gradient to the linear fluid acceleration:

$$\nabla p = -\rho_0 \frac{\partial \boldsymbol{u}}{\partial t};$$

where $\boldsymbol{u}$ *is the fluid velocity vector.*

(2) the equation of continuity *(conservation of mass):*

$$\rho_0 \operatorname{div} \boldsymbol{u} + \frac{\partial \rho}{\partial t} = 0;$$

[226] The fundamental relation $\lambda f = c$ [λ = wavelength; f = frequency; c = phase velocity] appeared explicitly for the first time in Newton's *Principia* (1686). The first measurement of the sound speed in air was evidently made by **Mersenne** (1635, 1644). The time was measured from the visual sighting of a firing of a cannon to the reception of the transient sound pulse at a known distance from the source.

(3) the equation of state, specifying the functional dependence $p = p(\rho)$, *subjected to the expansion*

$$p - p_0 = (\rho - \rho_0)\left(\frac{dp}{d\rho}\right)_{\rho_0} + \cdots \approx (\rho - \rho_0)c^2,$$

where $c^2 = \left.\frac{dp}{d\rho}\right|_{\rho_0}$ *and* c *the ambient velocity of sound.*

Newton *(1686), applying Boyle's law* $p = \rho f(T)$ *[isothermal process], obtained* $c = \sqrt{\frac{p_0}{\rho_0}} = 290\ \frac{m}{sec}$ *at* $T = 293\,^\circ K$, *15% lower than the observed value.*

Laplace *(1816) improved on Newton's result by correctly assuming that sound waves pass too rapidly for a significant exchange of heat to take place. For an adiabatic expansion in a perfect gas, he used* $p\rho^{-\gamma} = p_0\rho_0^{-\gamma}$, *which led him to*

$$c^2 = \frac{dp}{d\rho} = \gamma\frac{p_0}{\rho_0} = \gamma RT$$

with $c = 343\ \frac{m}{sec}$ *at* $T = 293\,^\circ K$,

$$\gamma = c_p/c_v \;=\; \text{ratio of specific heats},$$

and

$$R \;=\; \text{universal gas constant}.$$

[Clearly, the theoretical prediction of the speed of sound in liquids is more difficult than in gases. For example, c in sea water *depends on the pressure, salinity, water temperature and the amount of dissolved and suspended gas.]*

The above equations then imply the approximate relations

$$p = \rho c^2 + const.,$$

$$k\,\mathrm{div}\,\boldsymbol{u} + \frac{\partial p}{\partial t} = 0,$$

where $k = \rho_0 c^2$ *is the* incompressibility. *The combination of the conservation laws for mass and momentum leads to the wave equation*

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$$

for the acoustic pressure changes. The further assumption $\boldsymbol{u} = \mathrm{grad}\,\psi$, *implies that the fluid velocity* $\boldsymbol{u}$ *also obeys the same wave equation. It then follows that all field entities are expressible in terms of the potential* ψ*:*

$$p - p_0 = -\rho_0\frac{\partial\psi}{\partial t};$$

$$\rho - \rho_0 = -\frac{\rho_0}{c^2}\frac{\partial \psi}{\partial t}; \qquad \boldsymbol{u} = \nabla \psi.$$

The wave equation for ψ is

$$\nabla^2 \psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}.$$

For one-dimensional motion,

$$\psi = \psi(x - ct); \qquad u = \psi'$$

implies at once the relations $p = \rho_0 c u$; $\rho = \rho_0 \frac{1}{c} u$. *Certain entities formed of the basic field elements* $\{p, u, c, \rho_0\}$ *are of use in acoustic engineering:*

$$Z = \rho_0 c \equiv \sqrt{\rho_0 k} \qquad \textit{(impedance)};$$

$$W = \frac{1}{2}\rho_0 |u|^2 + \frac{1}{2k}|p|^2 = \rho_0 |u|^2 = \frac{|p|^2}{\rho_0 c^2}$$

(wave energy density = fluid momentum flux);

$$I = pu = Wc$$

(sound intensity = rate at which acoustic energy crosses a unit area per unit time).

The application of the Fourier transform to the pressure wave equation yields the Helmholtz equation (1860):

$$\nabla^2 p + \frac{\omega^2}{c^2} p = 0,$$

where ω is the angular frequency of the harmonic Fourier component.

It is of interest to note that ***Euler****, in his "Continuation of the Researches on the Propagation of Sound" (1759, 1766), already derived the Helmholtz spectral wave equation for the* particle displacement *(or velocity).*

The solution of the Helmholtz equation for a symmetrical point-source *yields the well-known result that the sound intensity falls of as the square of the distance from the source in a free open space. For sources of large area, the approximation does not hold and the sound intensity may at first fall off proportionally to the first power of the distance. Finally, in enclosed regions the sound intensity may decrease very slowly, or not at all, with distance.*

Pressure is measured in units of Pascal, denoted $Pa = \frac{\text{Newton}}{\text{m}^2} = 10\ \frac{\text{dyn}}{\text{cm}^2}$ *[Newton = 10^5 dyn, Joule = Newton×meter; Watt = Newton×meter/sec].*

Another unit is the $bar = 10^5$ *Newton*$/m^2 = 10^6$ *dyn*$/cm^2 \approx$ *Kg*$/cm^2$; $1\mu bar = 10^{-6}$ *bar* $=$ *dyn*$/cm^2$. *Atmospheric pressure* $\approx 10^5 Pa \approx$ *Kg*$/cm^2$.

1847–1856 CE **Jean Frederic Frenet** (1816–1888, France). Mathematician. Contributed to differential geometry of curves and surfaces. Introduced the so-called *Frenet-Serret*[227] *formulae* for the moving-trihedral on a space curve. He was a man of wide erudition and a classical scholar.

Frenet was born at Perigueux and graduated from the École Normale Superior (1840). He was a professor at Toulouse and Lyons.

1847–1861 CE **Ignaz Philipp Semmelweis** (1818–1865, Hungary). Obstetrician. Pioneer of antisepsis[228]. Proved (1847–1849) that *puerperal fever* (childbed fever) is brought to the woman in labor by the hands and instruments of examining physicians and can be eliminated through a thorough cleansing, in a solution of water with chloride of lime, of the hands, instruments, and other items brought in contact with the patient. Published (1861) *Die Aetiologie, der Begrift und die Prophylaxis des Kindbettfiebers*.

Semmelweis was born in Buda to Jewish parents and was educated at the Universities of Pest and Vienna, graduating M.D. in 1844. At the time when he was appointed assistant professor in a maternity ward, the mortality rate from puerperal fever stood at about 20 percent. His antiseptic measures caused this rate to drop to 1.2 percent by May 1847. His superior, Johann Klein, apparently blinded by jealousy and vanity, and supported by other reactionary teachers, drove Semmelweis from Vienna (1849).

Fortunately, in the following year Semmelweis was appointed obstetric physician at Pest in the maternity department, then as terribly afflicted as Klein's clinic had been. In the course of his six years of tenure there he succeeded, by antiseptic methods, in reducing the mortality rate to 0.85 percent. However, constant conflicts with his uncooperating superiors brought

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[227] **Joseph Alfred Serret** (1819–1885, France). Mathematician. Graduated from the Ecole Polytechnique (1840). Professor of celestial mechanics at College de France (1861); Professor of Mathematics at Sorbonne (1863). Succeeded Poinsot in the Academie des Sciences (1860).

[228] In 1854, **Heinrich Schröder** and **Theodor von Dusch** showed that bacteria could be removed from air by filtering through cotton-wool. In 1867, **Joseph Lister** (1827–1912, England) reported his method of antiseptic surgery [son of Joseph Jackson Lister (1786–1869)].

him within the gates of an asylum (1865). He brought with him into this retreat an infected dissection wound which caused his death — a victim of the very disease for the relief of which he had already sacrificed health and fortune.

1847–1894 CE **Hermann Ludwig Ferdinand von Helmholtz** (1821–1894, Germany). One of the foremost scientists of the 19^{th} century. Surgeon, physiologist, physicist, mathematician, chemist, musical scientist and philosopher. Helmholtz was among the last of the universalists: his research spanned almost the entire gamut of science.

In one of the epoch-making papers of the century, he formulated in 1847 the universal law of conservation of energy. Presented (1858) the first mathematical account of *rotational* fluid flow, introducing the important concepts of *vorticity*, *circulation*, *vortex flow*[229] and *vortex lines*. In 1860, Helmholtz

[229] Circulatory flow that is irrotational everywhere (except possibly at $r = 0$) is possible and is known as *circulatory flow without rotation*. In this case if the fluid is also incompressible and the flow *stationary* the velocity field has to satisfy both the matter conservation $(\mathrm{div}(\boldsymbol{V}) = 0)$ and irrotationality $(\boldsymbol{\Omega} = \mathrm{curl}\,\boldsymbol{V} = 0)$ conditions. The simplest solution of this class exhibiting circulatory flow about $r = 0$ has

$$\boldsymbol{V} = u_\theta(r)\boldsymbol{e}_\theta,$$

while irrotationality requires

$$\boldsymbol{\Omega} = \frac{\partial(ru_\theta)}{r\partial r} = 0.$$

It therefore follows that

$$ru_\theta = K = \text{constant}$$

(which is the law of conservation of angular momentum in disguise; the fluid angular-momentum density is $J = \rho u_\theta r$]. Thus

$$\boldsymbol{V} = K\frac{\boldsymbol{e}_\theta}{r}$$

representing irrotational motion except at the point $r = 0$, where the vorticity $\boldsymbol{\Omega}$ and the velocity become infinite (this is obviously an idealization of actual such flows). The circulation along a steamline $r = \text{const.}$ is

$$\Gamma = \oint \boldsymbol{V}\cdot d\boldsymbol{\ell} = 2\pi K$$

and the motion is known as *vortex flow*. It plays an important role in aerodynamics. On the basis of experimental evidence and the theory of viscous flow, one can assume that there is a fluid core or nucleus surrounding the center of

developed the mathematical theory of Huygens' principle for 'monochromatic' steady-state scalar waves. He also showed that an arbitrary continuously differentiable vector-field can be represented at each point as a superposition of the gradient of a scalar potential and a curl of a vector potential.[230]

Helmholtz made a great contribution to our understanding of thermodynamics; he was first to apply minimum principles to thermodynamics, and showed that for reversible processes, the role of the *action* was played by the "*Helmholtz free energy*", F.

In 1854 Helmholtz seized upon the problem of the sun's luminosity. Previously, **Kant** had calculated that if the sun's light came from ordinary combustion, it would have burned up in only 3000 years. Helmholtz then argued that the tremendous weight of the sun's outer layers, pressing radially inward, should cause the sun to gradually contract: Consequently, its interior gases will become compressed, and heat up. Hence gravitational contraction causes the sun's gases to become hot enough to radiate energy into space. He was thus able to boost the theoretical age of the sun to some 20 million years. This in turn meant that the sun extended beyond the earth's orbit only 20 millions years ago, to which geologists could not agree on the basis of the earth's present surface features. **Kelvin** supported and 'improved' Helmholtz's theory and it is known as *Helmholtz-Kelvin contraction*.

In other fields of science, Helmholtz contributed to the subjects of: fermentation, animal heat and electricity, muscular contraction, velocity of nerve

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the flow and that the core rotates approximately like a solid body. Within the core we have circulatory flow with constant angular velocity and outside the core we have circulatory flow without rotation. Inside the core

$$u_\theta \sim r$$

while outside

$$u_\theta \sim \frac{1}{r}.$$

Such a combination is known as an *eddy* or simply a *vortex*. The central core is called the vortex core. The *tornado* and *water spout* (or even the common *bathtub vortex*) are examples of such a flow. The *stability* of the vortex is determined by its Reynolds' number.

If an eddy occurs in a fluid that is otherwise undisturbed, the spatial location of the eddy remains unaltered. However, if a uniform stream is superposed on it, it will *move* with the stream. Such a vortex is known as a *free vortex*.

230 This theorem is now recognized as a special case of a result from Cartan's exterior calculus in an arbitrary, n-dimensional manifold. The more general result relates to algebraic topology through the *de Rham cohomology*.

impulses[231], invention of the ophthalmometer, physiological optics, color vision, physiological acoustics and meteorological physics.

From 1869 to 1871 Helmholtz involved himself in the verification of Maxwell's predictions concerning electromagnetic waves. He entrusted the subject into the hands of his favorite pupil, **Heinrich Hertz**, and the latter finally gave an experimental verification of their existence and velocity.

Helmholtz was born in Potsdam, near Berlin. His father was a high school teacher and his mother was a lineal descendant of the Quaker **William Penn** (founder of the state of Pennsylvania).

As his parents were poor and could not afford to allow him to pursue a purely scientific career, he became a surgeon in the Prussian army. He lived in Berlin from 1842 to 1849, when he became a professor of physiology in Königsberg. In 1855 he removed to assume the chair of physiology in Bonn. In 1858 he became professor of physiology at Heidelberg, and in 1871 he was called to occupy the chair of physics in Berlin.

Helmholtz married twice and had 4 children. He was a man of simple but refined tastes, noble carriage and somewhat austere manner. His life, from first to last, was one of devotion to science.

1848 CE A year of revolutions in almost every European country. It was the natural climax of a process of reaction and revolt which began after the defeat of Napoleon at Waterloo in 1815. Thereafter, Europe entered a period of instability, characterized by a long series of upheavals. The revolution of 1848 was the culmination of the political, economical and social unrest of the time — of the struggle between the aristocracy and the middle classes, the rapid increase of population from 180 million in 1800 to 266 million in 1850, the fact that more and more people now lived in cities, the conflict between the bourgeoisie and the rising proletariat, and the movements for national liberation and reunion. And it confounded all the protagonists, compelling a reappraisal of ideas and a realignment of forces.

In some sense, the French Revolution and its sequel in Napoleonic imperialism, disrupted the historic continuity of European society and shattered most of its traditions.

All the significant problems of the period arose out of these events. This break in continuity engendered a quest for new patterns of interpretation — nationalism, socialism, vast philosophical systems like those of **Marx** and **Hegel**, new conceptions of historical, scientific, literary and artistic ideas.

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[231] He actually measured the speed of nerve impulses (1852).

Table 4.4: Timeline of the Industrial Revolution, 1770–1848

- ca 1770 — Consolidation of the steam engine by **James Watt**
- 1775–1783 — American Revolution
- 1780 — Industrial Revolution under way
- 1789–1794 — The French Revolution
- 1799–1815 — Reign of Napoleon
- 1800–1850 — Romanticism in literature and the arts
- 1815 — The Congress of Vienna and the congress system of European diplomacy
- 1820 — Revolutions in Greece and Spain
- 1830 — Rise of liberalism and nationalism
- 1830, 1848 — Periods of revolution in Europe
- 1832 — Parliamentary reform in Great Britain
- 1848 — **Karl Marx's** 'Communist Manifesto'.

Europe's search for stability after 1815 was marked by a contest between the forces of the past and the forces of the future. For a while it seemed as though the traditional agencies of power — the monarchs, the landed aristocracy and the Church — might once again resume full control. But potent new forces were ready to oppose relapse into the past. With the quickening of industrialization, there was now not only a middle class of growing size and significance but a wholly new class, the urban proletariat. Each class had its own political and economical philosophy — liberalism and socialism respectively — which stood opposed to each other as well as to the traditional conservatism of the old order.

Nationalism as an awareness of belonging to a particular nationality was nothing new. What was new was the *intensity* that this awareness now assumed: for the mass of the people, nationalism became their most ardent emotion, and national unification or independence their most cherished aim.

The Vienna settlement (1815) ignored the stirrings of nationalism and the hopes for democracy that had been awakened by the French Revolution. It was mainly interested in peace and order and the restoration of conditions as they

were before the French Revolution[232]. Indeed, there was no war among the great European powers for 40 years, and no war of world-wide dimensions for a whole century. The Triple Alliance of Austria, Russia and Prussia guaranteed to maintain the territorial *status quo* in Europe and the existing form of government in every European country, i.e. aiding legitimate governments against revolutions.

A first wave of reaction that followed the peace settlements of 1815, manifested itself in the first wave of revolutions (1820–1829) in Spain, Portugal, Italy, Greece and Russia. The second wave of Revolutions swept France, Belgium and Poland during 1830–1833. The third wave (1848–1849) lasted for over a year and affected most of Europe with the exception of England and Russia. In Italy, Germany, Austria, and Hungary, the fundamental grievance was still the lack of national freedom and unity. In Western Europe the chief aim of revolutions was the extension of political power beyond the upper middle class. With the revolutions of 1848, socialism for the first time became an issue of modern politics.

In addition, severe economic crises particularly affected the lower classes: everywhere the small artisan was fighting against the competition of large-scale industry, which threatened to deprive him of his livelihood. At the same time, the industrial workers in the new factories were eking out a miserable existence on a minimum wage. There were also periodic upheavals in agriculture, primarily as a result of crop failures.

The revolutions of 1848 had failed everywhere due to weaknesses in the revolutionary camp (lack of widespread popular support, indecision among their leaders and the lack of well-defined programs) and the continued strength of the forces of reaction. The burden of the revolution fell on the workers whereas the middle class, in most countries, did not really want a revolution. It preferred to achieve its aims through reform, as had been done in England. There was no attempt to coordinate the revolutions in different countries, although the forces of reaction worked together.

Two forces emerged from the revolutions that henceforth were to dominate the history of Europe — nationalism and socialism. These now became, respectively, the main issues in the struggle of nation against nation and class against class.

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[232] In Spain and Naples the returning Bourbons abolished the liberal reforms that had been granted in 1812. In the Papal States, Pope Pius VII got rid of the French legal reforms, re-established the Jesuits, put the Jews back into the ghettos, and forbade vaccination against smallpox!
In Piedmont, Victor Emmanuel I had the French botanical gardens torn up by the roots and the French furniture thrown out of the windows of his palace!

1848–1867 CE **Karl (Heinrich) Marx** (1818–1883, Germany and England). Political economist. A critic of capitalism. Sought a scientific formula for social justice. The most influential social thinker of modern times. Marx approach was philosophical, Hegelian, and his materialist conception of history is basic to his philosophy of economic determinism (*Historical Materialism*). Marx defined *Communism* as the common ownership of the means of production, an ideal system to be achieved by shifting control over economic resources from the capitalists to the proletariat. This transfer of properly rights would result in the permanent abolition of private property, with public ownership of all means of production, including the farms and factories, raw materials, transportation and communication facilities, and the like.

In his major work *Das Kapital* (*Capital*, 1867, 1885, 1894), Marx systematically developed his theory of *surplus value*, which maintains that the worker is exploited in an inequitable distribution of the products of his labor by the owners of the means of production. The surplus is the difference between what he gets in order to subsist and what is totally derived from what he creates[233].

Marx accepted the *Hegelian dialectic*, which states that every thesis contains its own antithesis, its negation, opposite, or contradiction, and that the

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[233] **Ayn Rand** (1905–1982, USA), social philosopher, maintained that most workers in a capitalistic economy earn far more – both in the value of their wages and through the infrastructure made possible by such an economy – than they could ever bring into existence on their own. Thus, according to Rand, the "surplus value" works the *other* way, and is a de facto gift from enterpreneuring individuals to those whom they employ (or whose employment is made possible by the entrepreneurs' inventions and business acumen).

She was born in St. Petersburg, Russia, as Alissa Rosenbaum, to Jewish parents. Emigrated to the USA (1926). Espoused her philosophy of "rational selfishness" (*Objectivism*) in novels, and in non-fiction books such as "*For the New Intellectual*" (1961); "*The Virtue of Selfishness*" (1965); "*Capitalism: The Unknown Ideal*" (1966); and "The New Left" (1971). Rand held that the source of all happiness, progress and justice lies with the *productive individual*, free to pursue his own agenda by relentlessly applying *reason* and by engaging in *voluntary*, *non-coercive* cooperation/competition with other individuals for mutual benefit and satisfaction. She thus staunchly opposed religion and all other forms of *mysticism*, as well as any social order based upon *altruism*; regarded **Aristotle** as the most important philosopher; and taught that the *United States* – the only country founded upon laissez-faire capitalism and the principles of the enlightenment – was (in the 19^{th} century), and potentially still is, the most moral county in the history of mankind.

two conflicting forces merge to produce a synthesis, a new and greater reality. He applied this logical principle to socio-economic history. The two socio-economic classes, the property-owing class (capitalists) and the workers (proletariat), who must sell their labor in order to survive — are antithetical to each other.

Under the influence of **Ludwig Andreas Feuerbach**[234] (1804–1872, Germany), a pupil of Hegel in Berlin, Marx adopted the economic interpretation of history (*Historical Materialism*). It contends that a particular society's mode of economic production determines the nature of its cultural and social structure. Marx traced the relevant cause-effect relationship from ancient to modern times. He noted that the chief mode of production among the ancient Greeks and Romans was replaced by feudalist methods of production during the medieval period. Feudalism and the institution of serfdom upon which it depended yielded to capitalism in the modern period when the mode of production was changed through wider use of machinery and the factory system. Marx concluded that capitalism, by its very nature, is self-destructive and hence must capitulate to Socialism, that owing to the dialectical character of history, each historical period carries within itself the "*germs of its own destruction*" (Hegel's principle of negativity).

Marx held that the victory of the proletariat in their struggle could be predicted with the certainty of a scientific experiment (hence the term '*Scientific socialism*'). However, the history of the world in the 13 decades that elapsed since the appearance of the *Capital* (culminating with the collapse of Communism in Soviet Russia and Eastern Europe) proved that there were many blind spots in Marx' socialist theories: while the rich were getting richer, the poor did not necessarily get poorer.[235] The general standard of living in the world's industrial nations was to reach heights undreamed of by Marx. Man, furthermore, does not seem to be moved exclusively, or even primarily, by economic concerns. Despite Marx' attack on religion, the established churches have continued to play an important role even in the lives of the lower classes.

Another force that increasingly came to command the allegiance of rich and poor alike was *nationalism*. The great wars of the last century have been fought not between the "oppressed" and their "oppressors" but between workers of different nations for the defense or the greater glory of their own country. Marx' fundamental errors arise from an uncritical extrapolation of

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[234] Brother of the mathematician **Karl Wilhelm Feuerbach** (1800–1834, Germany), after which the '9-point circle' is named.

[235] **Paul Samuelson** (Nobel prize in economics, 1970) said: "one may ignore the entire life-work of Karl Marx for the impoverishment of the working class simply did not happen."

what he observed in capitalist societies[236] to all class societies, and from a disregard of the enormous influence which political, national, and moral forces have exerted on the development of capitalism as an economic system.

[236] Much of the political tension of Europe during the first half of the 19^{th} century was a manifestation of underlying economic unrest caused by the gradual transformation of Europe's economy from agriculture to industry. In this new scheme of things, the impact of the railroad was overwhelming. Here was an entirely new industry, answering a universal need, employing thousands of people, offering unprecedental opportunities for investment, and introducing greater speed into all industrial and commercial transactions. As the workers grew in number, they became aware that they constitute a new and separate class whose interests conflicted with those of their employers. This conflict prompted the emergence of *Utopian Socialism* that proposed to distribute the profits of human labor in such a way that every member of society receive an equitable share, economically, socially and politically. ("From each according to his capacity, to each according to his need".)

However, the Utopian Socialists [**Henri de Saint-Simon** (1760–1825), **Charles Fourier** (1772–1837); **Louis Blanc** (1811–1882), **Robert Owen** (1771–1858)] failed because they believed in the natural goodness of man and the perfectibility of the world. A more realistic and more militant type of Socialism was needed that would use the worker's potential economic and political power to wrest concessions from the middle class.

The class-struggle, as enunciated by Karl Marx, is the main doctrine of the theory as well as the means of achieving socialism. Because the economic forces in the modern world are in constant conflict, Marx proclaimed that the working classes, out of historic necessity, must make their bid for power by uniting, bringing about social and political changes and achieving dominance in society through the "dictatorship of the proletariat". Marx's basic error was his failure to appreciate the importance of *noneconomic forces* in society: religion, emotion, prestige, genius, stupidity and such factors as climate and geography. His economic theories themselves based on conceptions of *static economy*: the theory of surplus value did not take sufficient account of the importance of capitalist equipment, administrative ability, initiative, and the willingness to take risks. The capitalist-industrial revolution, far from pressing more and more people into proletarian poverty, increased production to such an extent that it improved the general standard of living of all men. Indeed, in the most highly industrialized countries the proletarian class is rapidly disappearing. The great revolutions inspired by Marxist doctrines have taken place not in industrial societies (where Marx expected them to occur), but in societies still overwhelmingly agricultural, beset by very real economic hardships. In all revolutions inspired by Marxism, the *state* has played a dominant role in the reorganization of society, but nowhere is there any sign of its withering away.

Despite errors and shortcomings in his teachings, however, the contribution of Marx to modern thought have been most fruitful. By bridging the gap between politics and economics, he enriched our understanding of the past. Prior to Marx, the division of society into rich and poor, haves and have-nots, was accepted as a natural, unchangeable fact. It was chiefly due to Marx that we came to realize the importance of economic factors, being jolted out of complacent acceptance of the *status quo*. By predicting far reaching changes, he made people aware that changes were possible. The threat of revolutionary changes, conjured up in Marx' writings, did much to hasten the peaceful evolution that has so markedly improved the condition of the lower class in all industrial societies. Almost every social movement in the 20th century has been influenced by Marxist ideology.

Karl Marx was born in Trier (Trèves), Rhenish Prussia. His paternal grandfather, Meir Levi (later surnamed Marx[237]), was the Rabbi of Trèves and his paternal grandmother, Chaya (neé Lwow), was a descendant of an unbroken chain of Ashkenasi rabbis, at least 8 centuries long[238]. His father, Hirschel (Heinrich) Marx (1782–1838), was a lawyer and judge in Trèves. He married Henriette Pressburg, daughter of the Rabbi of Nijmegen, Holland, and had his entire family baptized as Christian Protestants (1824) for business and social reasons.[239]

Marx went (1835) to the universities of Bonn and Berlin. He studied first law, then history and philosophy and in 1841 took the degree of doctor of philosophy. In Berlin he became acquainted with the philosophy of Hegel and interacted with the 'Young Hegelians'[240]. At 24 he became an editor of a

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[237] Marx was the 12^{th} generation of the famous Jewish medieval scholar **Rabbi Meir Katzenellenbogen**, the MAHARAM of Padua (1482–1565), the great ancestor of Europe's leading Rabbis, Talmudists and heads of the Rabbinic Courts in principal cities and towns for over three centuries. The MAHARAM himself was the 17^{th} generation of RASHI (1040–1105).

[238] **Marcus** is an ancient Roman name and means: "belonging to the god Mars". Jews with Hebrew name of **Moshe** or **Mordecai** often selected Marcus or Mark as the non-Hebrew name. This became the family name Marks or Marx.

[239] Although a trained lawyer, he could not practice law as a Jew in Trier, Prussia.

[240] **Karl Marx** was a product of this school of thought. Unlike his fellow renegades **Ludwig Feuerbach**, **Bruno Bauer** [*Die Judenfrage*, 1843], and **G. F. Daumer** who became virulent racist antisemites, Marx himself stopped short of full-fledged antisemitism, but in his own way reinforced the negative stereotype of the Jew as the personification of modern capitalism, which would later be adopted by the Nazis and their imitators. Thus, the implementation of Marx's vision ["As soon as society succeeds in destroying the empirical essence of Judaism, the Jew will become *impossible*... The total emancipation of Jewry, is

paper in Cologne, Germany, but his radical ideas soon got him into censorship trouble and he had to flee to Paris to escape arrest. With him went his young wife, Jenny von Westphalen, whom he had married (1842) in spite of both families misgivings. [She was a most faithful companion to Marx during all the vicissitudes of his career and died in 1881 after bearing him 6 daughters, 3 of whom reached marriageable age and 2 of whom outlived him.]

In Paris (1844) Marx met **Friedrich Engels** (1820–1895), a son of a German textile manufacturer, whose ideas were in complete accord with Marx' and who collaborated with him in writing. This was the beginning of a close friendship and an uninterrupted collaboration and exchange of ideas which lasted for nearly 40 years. He also befriended the poet **Heinrich Heine**, who contributed some of his poems to Marx' radical magazine. Following his expulsion from Paris (1845), Marx lived for a time in Brussels, Belgium. He later returned to Paris (1848) but only to be expelled again (1849).

Marx then went to England and made his home in London for the next 34 years. He lived in wretched poverty (3 of his children died through the lack of medicines). Sometimes Marx could not go out because his clothes were at the pawnbroker. He spent day after day in the British Museum library, his bills being paid by Engels.[241]

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the emancipation of society from Judaism", 1843] of Communism in the USSR in the name of 'human emancipation' would cause untold suffering to Jews and other national or religious minorities. His writings were used in the Soviet Union to justify the most vulgar antisemitic propaganda.

At the same time, the fact that the founder of Communism was himself born a Jew, made him the arch-symbol of Jewish revolutionary subversion for the conservative and radical Right all over the world! Modern antisemitism seized on the prominent role which 'non-Jewish Jews' like Marx played in Socialist, Communist and other radical movements to construct a new myth of the Jew as the 'ruthless cosmopolitan' enemy of all national values, religious traditions, social cohesion, and bourgeois morality.

[241] Marx great intellectual talents could only be matched by his tenacity and perseverance: he would be found at his writing desk from nine in the morning until three o'clock the next morning — with time off only for meals and bedtime stories. When he worked at the British Museum he would arrive when the library opened at nine, and leave only when it closed at seven: to a penniless refugee, the great domed reading room offered the advantages of dependable central heating and comfortable chairs.

He would encourage his disciples to study harder. "Learn, learn," was the categorical imperative which he would shout often at them, though the message was already conveyed in the example he set up and of what they could see of

While in Brussels, Marx and Engels had written the epochal revolutionary pamphlet '*Manifesto of the Communist Party*' (1848). It contains the simplest expression of Marx' beliefs (see Table 4.4). The ideas in it were later developed at length in the three volumes of his major work '*Das Kapital*'.

Some historians claimed that the ideas expounded in the *Manifesto* were directly taken by Karl Marx from the writing of **Adam Weishaupt**[242] (1778), the founder of the *Order of the Illuminati* in Bavaria.

Only eight people were present to hear Engels' funeral oration in Highgate Cemetary on March 15, 1883. Marx' descendants, the Longuet family, live today in France.

Marx never had a steady income. No one knew anything about him outside a small circle of German exiles and a few intellectuals (only in 1917, with the rise of the communists in Russia, the works of Marx became known in Europe). Marx's economic theories made no immediate impact on the debate inside the worker's movement or on other thinkers except after his death (1883). This is true of his theories on *value* and *surplus value*, *accumulation*, *exploitation*, *crisis and appropriation*, *class struggle* and *revolution*. But by the end of the 19th century, several such theories were hotly discussed with the worker's movement.

Marx was an apostate Jew, he had no Jewish education and never sought to acquire any. He tried to shut Judaism showing the smallest concern for any of the injustice inflicted on Jews throughout his lifetime. But his suppressed

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his own tremendous labors.

[242] **Weishaupt** (1748–1811, or 1830) was born in Ingolstadt, Bavaria of orthodox Jewish parents who had converted to Catholicism in 1748. When his father died (1754) young Adam was turned over to be raised by the Jesuits. He graduated from the University of Ingolstadt (1768) and was made a professor of Law (1772), after his conversion to Protestantism. He was initiated as a Freemason (1774) and then founded (1776) the *Order of the Illuminati*, some proto-communist organization dedicated to bringing about a proletarian revolution. In 1784 the Illuminati attempted a coup against the Habsburgs, but the plot was revealed by police spies that had infiltrated the order. This led to the total ban of all secret societies in Bavaria and membership was punished by death. Weishaupt was forced to flee to a neighboring province (1785). The *French Revolution* of 1789 has been widely attributed to the machinations of the Illuminati. Although this statement is an exaggeration, it cannot be denied that several persons who were intensively involved in the revolution were active members, among others the **Comte de Mirabeau** (1749–1791). After the rise of **Napoleon Bonaparte**, the Illuminati were utterly crushed.

self-hatred manifested itself through venomous attacks on friends, benefactors and especially Jews.

However, despite his ignorance of Judaism as such, there can be no doubt about his Jewishness: his notion of progress was profoundly influenced by Hegel, but his sense of history as a positive and dynamic force in human society, governed by iron laws, is profoundly Jewish. His Communist vision is deeply rooted in Jewish apocalyptic thought and messianism. His methodology too, was wholly rabbinical: all his conclusions were derived solely from books, his temperament was religious, and he was quite incapable of conducting objective, empirical research. Marx's theory of how history, class and production operate, and will develop, is not a scientific theory at all but a Kabbalistic theory of the Messianic Age.

The roots of Marx's anti-Semitism went deep. He was not merely a Jewish thinker, but also an anti-Jewish thinker. Therein lies the paradox, which has a tragically important bearing both on the history of Marxist development and on its consummation in the Soviet Union and its progeny. Marx's personal anti-Semitism, however disagreeable in itself, might have played no great part in his lifework had it not been part of a systematic theory in which Marx profoundly believed. In fact it is true to say that Marx's theory of Communism was the end-product of his theoretical anti-Semitism.

Worldview XXII: Marx

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"Die Philosophen haben die Welt nur verschieden interpretiert; es kommt darauf an, sie zu verändern."
"The philosophers have only interpreted *the world in various ways; the point, however, is to* change *it."*

(Epitaph on his tomb in Highgate Cemetery, London)

* * *

"Darwin's book is very important and serves me as a basis in natural science for the class-struggle in history."

* * *

"Natural science will in time include the science of man, as the science of man will include natural science; there will be one science."

* * *

"Lucretius is the truly Roman heroic poet; his heroes are the atoms, indestructible, impenetrable, well-armed, lacking all qualities but these; a war of all against all, the stubborn form of eternal substance. Nature without gods, gods without a world."

* * *

"At the entrance to science, as at the entrance to Hell, there should be posted the demand: 'Here the spirit should be firm. Here the promptings of fear should be heeded'."

* * *

"The more of himself man attributes to God, the less he has left in himself."

* * *

"Religion is the opiate of the people."

* * *

"Social revolution never occur because of the weakness of the strong; for that you need the strength of the weak."

Philosophies of Social Criticism

By the middle of the nineteenth century, the coming of the Industrial Age, with its revolutionary political, social, and economic effects, had made itself felt over most of Europe and it also extended to other parts of the world. For the next two decades, people's attention in Europe and the United States became absorbed by momentous political developments. Once the political situation had become stabilized, however, shortly after 1870, a second wave of economic development swept over Europe and the world, a wave of such magnitude that it is often referred to as a "Second Industrial Revolution."

Marxian socialism, in its ultimate effects on society, turned out to be the most important attack on the capitalist philosophy of laissez faire. There were other critics of this philosophy, however, who tried in various ways to awaken their contemporaries to the social problems created by the industrialization of society.

Humanitarianism

Writers like ***Honoré de Balzac*** *in France and* ***Charles Dickens*** *in England, by dwelling in their novels on the more sordid aspects of the new industrialism, played on human sympathy in the hope of creating a climate favorable to reform. The historian* ***Thomas Carlyle****, in his* Past and Present *(1843), showed deep concern over the growing division between the working classes on the one hand and the wealthy classes on the other. He turned against the "mammonism" and the "mechanism" of his age and admonished the new captains of industry to be aware of their responsibilities as successors to the old aristocracy. Like Carlyle,* ***Benjamin Disraeli****, one of the rising young Tories, in his social novel* Sybil *(1845), deplored the wide gap that industrialization had opened between the rich and the poor. It was, he said, as though England had split into two nations "between whom there was no intercourse and no sympathy."*

Anarchism

*One other form of social protest, the effects of which were not felt until later in the 19th century. It is the belief that every form of regulation or government is immoral, and that restraint of one person by another is an evil which must be destroyed (*Anarchism *comes from Greek word meaning* without

government). Anarchism dates back to ancient times. It also appeared among early Christian groups.

Anarchism, like socialism, hoped to overthrow capitalism. But while the socialists were ready to use the state as a stepping stone for the realization of their aims, the anarchists were deeply opposed to any kind of governmental authority and organization. One of the earliest theorists of anarchism and the first to use the word anarchy, *was the French social philosopher* ***Pierre-Joseph Proudhon*** *(1809–1865). In his pamphlet First Memoir on Property (1840) he asked the question, "What is property?" and replied with the well-known slogan, "Property is theft!" This seeming opposition to private property appeared to align Proudhon with communism and endeared him to Marx. The latter's admiration cooled, however, when he discovered that Proudhon was less interested in overthrowing the middle class than in raising the worker to the level of that class. Proudhon was against any kind of government, be it by one man, a party, or a democratic majority. "Society," he wrote, "finds its highest perfection in the union of order with anarchy."*

Thus Proudhon, often called the father of anarchism, became the first to make anarchism a mass movement. His philosophical, *or* individualistic, *anarchism urged the willing cooperation of free men without any regulation (law) or government.*

Terroristic anarchism began under the leadership of ***Mikhail Bakunin*** *(1814–1876, Russia). A theorist of anarchism, he also practiced what he preached. Bakunin was involved in several revolutions, was three times condemned to death, and spent long years in prison and Siberian exile. Most of the evils of his day Bakunin attributed to two agencies — the state and the Church. His ideal society was a loose federation of local communities, each with a maximum of autonomy. In each of these communities the means of production were to be held in common. The way to achieve this governmentless state of affairs, Bakunin held, was not by waiting patiently for the state to wither away, as Marx had held, but by helping matters along, if necessary by means of terrorism, assassination, and insurrection. The years 1881–1912 witnessed a whole series of assassinations attributed to anarchists (Czar Alexander II of Russia, 1881; President Carnot of France, 1894; Prime Minister of Spain Antonio Canovas del Castilio, 1897; Empress Elizabeth of Austria, 1898; King Humbert of Italy, 1900; President William McKinley of the United States, 1901; Prime Minister of Spain José Canalejas y Mendez, 1912).*

Anarchism under the leadership of the Russian physical geographer and political philosopher Prince ***Peter Alekseievitch Kropotkin*** *(1842–1921), during the late 1800's assumed a more rigid communistic form. Kropotkin*

rejected the terroristic methods of Bakunin, but he also opposed the authoritative type of communism.

Under such kind of anarchism, the state would be eliminated and the future society would be built on the communes, *or* village communities, *which had existed in feudal Russian society. Each commune would be a self-sufficient group.*

He advanced a theory of mutual aid *(1904) as a rationale for eliminating all forms of government except true self-government. He based this theory on the evidence of various studies of animal behavior, and his main source for inspiration was* **Darwin's** *theory of evolution through the survival of the fittest. Kropotkin insisted that* mutual aid *is as important a principle of nature as* mutual hate; *that* mutual hate, *and the struggle for existence, as Darwin had shown, exists only with respect to different species; that among the same species there is a spirit of mutual* cooperation *for existence; and that since man is a single species, all men should cooperate and help each other to survive. Thus, "the aspirations of man are at one with nature," and "mutual aid, therefore, is the predominant fact of nature."*

Born into the Russian nobility, Kropotkin entered the Imperial army in 1862 and served until 1867. Visiting Switzerland in 1872 he became a convinced anarchist, under the influence of Bakunin's teaching. Back in Russian he began active propaganda for the movement, and in 1874 was arrested and imprisoned. Two years later he escaped and fled from Russia, beginning an exile, mainly spent in London, ended only by the revolution of 1917. He saw anarchist communism very much as the next natural stage of social revolution, and part of the wider evolutionary process. In contrast to Darwin's social disciples, he believed that mutuality and cooperation were features of the animal world and already significant forces in society, however masked by coercive government. With **Marx** *he believed that modern productive techniques opened up possibilities of good living conditions for all; capitalism with its wage system must be replaced by communism. The Soviet state which emerged from the Bolshevik revolution did not have his sympathies. He did not reject the use of force, and supported the Allies against Germany in WWI. His* Mutual Aid: a Factor of Evolution *(London, 1902) is one of the continuing classics of anarchist thought.*

During the 1870's some of the nihilists fell under the influence of Mikhail Bakunin and his philosophy of anarchism. In 1879 this terrorist faction formed a secret society, "The Will of the People," whose aim was to overthrow the government by direct action and assassination.

Frightened by these manifestations of radicalism, Alexander II reverted to a policy of renewed reaction. Yet by reverting to repression, he merely helped to strengthen the revolutionary forces he hoped to combat. This fact was

brought home to him in several attempts on his life, and in 1880 he tried once again to return to his initial policy of reform. But by then it was too late.

Anarchists tried to mobilize working-class support behind the Russian General Strike which was a central feature of the Russian Revolutions of 1905 and 1917. But anarchism never developed into a well-defined movement, partly because of Bakunin's death in 1876, partly because of the impracticable nature of its doctrine.

The strength and influence of anarchism declined throughout the world in the 1900's after the rise of totalitarian states elsewhere. They were active in the Spanish Civil War of 1936–1939, and in the latter half of the 20th century anarchism attracted urban terrorists.

1848–1851 CE **Armand Hippolyte Louis Fizeau** (1819–1896, France). Outstanding experimental physicist. In an experiment (1851) of great historical value[243], he showed that the Galilean transformation of velocities does not apply to the velocity of light in moving media. This result constituted a strong motivating factor for Einstein in his development of relativity theory.

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243 Consider a medium in which the velocity of light is $u = \frac{c}{n}$ when the medium is at rest w.r.t. an observer. If the medium itself moves with velocity v w.r.t. that observer, the STR predicts that the combined velocity of light relative to the observer is

$$U = \left(\frac{c}{n} + v\right) / \left(1 + \frac{c}{n}\frac{v}{c^2}\right) = \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right)\left(1 + \frac{v}{nc}\right)^{-1} \approx \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right),$$

keeping only terms of order $\left(\frac{v}{c}\right)$. This is in agreement with Fizeau's experiment, but contrary to the prediction

$$U = \frac{c}{n} + v$$

which is obtained by using the Galilean, rather then the Lorentz transformation. In pre-relativistic times, the extra term was attributed to the dragging along of the ether by the moving body which, in turn, was accounted for by ad-hoc theories of **A.J. Fresnel** (1788–1827) and **G.G. Stokes** (1819–1903). After the ether theory was demised by the Michelson-Morley experiment, Fizeau's experiment remained without a plausible explanation until 1905.

In 1849, Fizeau devised a laboratory experiment to measure the velocity of light in air and in water. In 1850, he measured the velocity of propagation of the electromagnetic field in matter: his values ranged from $\frac{1}{3}c$ in iron wires to $\frac{2}{3}c$ for copper wires. In 1848, he established experimentally the existence of the *Doppler effect* for light waves and discovered the so-called '*red shift*'.

1848–1872 CE **Heinrich Eduard Heine**[244] (1821–1881, Germany). Mathematician. He was born in Berlin, Germany, the eighth of nine children. His father was a banker. Eduard studied at Berlin and Göttingen and from 1848 was a professor at the University of Halle. He was still teaching there when Georg Cantor joined the faculty in 1874.

Influenced by Weierstrass' lectures at Göttingen, Heine introduced the ϵ–δ definition of limits. He published about 50 papers, most of them dealing with special functions, but his name is best known for its association with the so-called Heine-Borel theorem. Borel's name is associated with this result due to his recognition of its importance and his use of it as a separate theorem in 1895.

1848–1878 CE **Karl Theodor Wilhelm Weierstrass** (1815–1897, Germany). One of the greatest mathematical analysts of the 19^{th} century. Sought to separate analysis, especially the calculus, from geometry and reduce the principles of analysis to real number concepts (a program known as '*arithmetization of analysis*').

Weierstrass began his mathematical career with papers on Abelian functions, but his most important contribution to mathematics is his construction of the *theory of complex functions* by means of power series. He showed special interest in entire functions and functions defined by infinite products. He rigorized the concept of *uniform convergence*[245] (1854) and exhibited a class of continuous non-differentiable functions[246].

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[244] A member of the Jewish banking family founded by **Solomon Heine**, and therefore a relative of the poet **Heinrich Heine**.

[245] Discovered independently at about the same time by **Cauchy** (1853) and **G.G. Stokes** (1847) and by **P.L.V. Seidel** (1821–1896, Germany) in 1848.

[246] In 1876 Weierstrass stated that the function

$$\sum_{n=0}^{\infty} b^n \cos(\pi a^n x), \qquad a > 1, \qquad \frac{1}{a} < b < 1,$$

is continuous but nowhere differentiable. Earlier, in 1872, he published the result only for $a =$ odd integer and $ab > 1 + \frac{3\pi}{2}$. Weierstrass was not the first to produce functions of this kind. Riemann had asserted already in 1861

In algebra, he gave a postulational definition of a determinant and contributed to the theory of bilinear and quadratic forms. One of the important contributions of Weierstrass to analysis is known as *analytic continuation.* He has shown that the infinite power series representation of a function $f(z)$, about a point z_1 in the complex plane, converges at all points within a circle C_1 whose center is z_1 and which passes through the nearest singularity. If now, one expands the same function about a second point z_2 within C_1, $z_2 \neq z_1$, this series will be convergent within a circle C_2 having z_2 as center and passing through the singularity nearest to z_2. This circle may include points outside C_1, hence one has expanded the area of the plane within which $f(z)$ is defined analytically by power series. The process can be continued with still other circles. Weierstrass thus defined an analytic function as one power series together with all those that are obtainable from it by analytic continuation. The impact of this idea is felt particularly in mathematical physics, in which solutions of differential equations are rarely found in any form other than as an infinite series.

In his drive to arithmetize the calculus, Weierstrass contributed to the definition of a real number and provided an improved definition of the limit concept. He is the herald of the *age of rigor*, replacing older heuristic devices and intuitive views by critical, logical precision. In todays textbooks the definitions of a limit of a function are in essence those introduced by Weierstrass and Heinrich Eduard Heine (1821–1881, Germany) a century ago, and the so-called delta-and-epsilon proofs, or *epsilontics*, are now part of the mathematicians stock-in-trade.

Weierstrass was born at Ostenfelde in the district of Münster, Germany. His father was a customs officer in the pay of the French (who at the time

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(without proof) that the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2}$$

was nowhere differentiable. A deeper insight into this function was, however, only achieved in 1916, when **Hardy** proved that $f(x)$ has no finite derivative at any point $\pi\xi$, where ξ is either irrational or rational of the form $\frac{2A}{4B+1}$ or $\frac{2A+1}{2B}$ (A and B integers). **J. Gerver** (1970) continued Hardy's efforts and showed that for rational ξ of the form $\frac{2A+1}{2B+1}$ the Riemann function has, on the contrary, the finite derivative $\left(-\frac{1}{2}\right)$. Thus Gerver succeeded to show that the Riemann function does not in fact belong to the class of continuous, nowhere differentiable functions. In 1972 **A. Smith** gave a proof that $f(x)$ has no finite derivative *at any point other* than those of the Gerver form.

dominated Europe). His mother died when he was 11 years old, and his stepmother contributed very little, to say the least, to Karl's education.

Weierstrass not only did extremely well at school, but at the age of 15 was able to secure a job as an accountant. After analyzing Karl's qualities, his father concluded that he should prepare for public service: the youngster was sent to study law at the University of Bonn. After four years at Bonn, Karl returned home an expert in drinking and fencing, but without the law degree. He was then sent by his family to embark on a secondary-school teaching career, at the Academy of Münster, for which he was ready in 1841. At Münster Weierstrass became fascinated by the lectures of his mathematics professor, **Christoph Gudermann**[247], who was an enthusiast of elliptic functions. Gudermann's idea was to base everything on the power series representation of a function. This idea was the main tool for the greatest part of Weierstrass' work. Karl spent the next 15 years in the capacity of an unknown school teacher in an obscure village. However, the publication of his memoir on Abelian functions in the Crelle Journal in 1854 brought him at once into the limelights of the mathematical world, and he moved to Berlin in 1856. But only in 1864 was he awarded full professorship at the University of Berlin and could finally devote all his time to advanced mathematics.

Due to occasional spells of vertigo, he never trusted himself to write his own formulae on the blackboard, but dictated to an assistant who wrote them for him. Among the most important of his students were **Hermann Amandus Schwarz** (1843–1921), **Sonja Kovalevsky** (1850–1891), **Georg Cantor** (1845–1918), **Magnus Gösta Mittag-Leffler** (1846–1927) and **David Hilbert** (1862–1943).

1849 CE **Paul Julius von Reuter** (1816–1899, Germany). Journalist. Founded in Aachen (1849) a central telegraphic and pigeon-post bureau for collecting and transmitting news, forerunner of Reuter's News Agency with headquarters in London (from 1851); removed to England (1851) and became naturalized British subject; created baron (1871) by Duke of Saxe-Coburg-Gotha.

Born at Cassel, Germany as **Israel Beer Josaphat**, he was baptized (1844), when he assumed the name Reuter. At the age of 13 he became a

[247] **A. Cayley** called the function $\phi(u) = \sin^{-1}\{\tanh u\}$ the *Gudermannian* of u, and denoted it by $gd\, u$. Then

$$u = gd^{-1}\phi = \log\{\tan\phi + \sec\,\phi\} \equiv \log\tan\left[\frac{\pi}{2} + \frac{\phi}{2}\right]$$

clerk in his uncle's bank at Göttingen, where he chanced to make the acquaintance (1829) of **Carl Friedrich Gauss** (1777–1855), whose experiments in telegraphy were then attracting some attention. Reuter's mind was thus directed to the value of the speedy transmission of information, and in 1849, on the completion of the first telegraph lines in Germany and France he found an opportunity of turning his ideas to account.

There was a gap between the termination of the German line at Aix-la-Chapelle and that of the French and Belgian line at Verviers. Reuter organized a news-collecting agency at each of these places, his wife being in charge of one, himself at the other, and bridged the interval by pigeon-post. On the establishment of through telegraphic communications, Reuter endeavored to start a news agency in Paris, but finding that the French government restrictions would render the scheme unworkable, removed to England (1851).

The first submarine cable (between Dover and Calais) had just been laid, and Reuter opened an office in London for the transmission of intelligence between England and the Continent. In 1859 Reuter extended his sphere of operations all over the world. In 1866 he laid down a special cable from Cork to Crookhaven, which enabled him to circulate news of the American Civil War several hours before the steamer could reach Liverpool.

1849–1859 CE **Arthur Cayley** (1821–1895, England). One of the principal mathematicians of the 19^{th} century and one of the most prolific mathematicians of all times, rivaled in productivity only by Euler and Cauchy [the number of his papers and memoirs exceeded 800]. Cayley wrote upon every subject of pure mathematics and also upon theoretical dynamics, physical astronomy and physical geography. He was primarily an algebraist, and his main achievements are as follows:

(1) Created the theory of matrices[248] and developed it as a pure algebra

[248] Among the key theorems discovered by him is the *Cayley-Hamilton Theorem*: Every matrix satisfies its own characteristic equation, i.e. if $f(\lambda)$ is the *characteristic polynomial* of $\boldsymbol{A}$, then $f(\boldsymbol{A}) = 0$. This theorem was established by **Hamilton** (1853) for a special class of matrices. **Cayley** (1858) enunciated the general result without proof.
The manipulation of matrices is often generally facilitated by the Cayley-Hamilton theorem, which provides an easy method for expressing any polynomial in $\boldsymbol{A}$ as a polynomial of degree not exceeding $n-1$, when n is the degree of $f(\lambda)$. Thus, if $\boldsymbol{A}$ satisfies the equation

$$\boldsymbol{A}^4 - \pi^2\boldsymbol{A}^2 = 0,$$

we have

$$\sin \boldsymbol{A} = \boldsymbol{A} - \pi^{-2}\boldsymbol{A}^3.$$

(1857); showed that quaternions can be represented as matrices of special form. [Matrices, considered as arrays of coefficients in homogeneous linear transformations, were tacitly in existence long before Cayley, but their properties were not studied for their own sake.] The importance of matrix theory in the mathematical machinery of modern physics echo the prophetic statement made by P.G. Tait: "*Cayley is forging the weapons for future generations of physicists*[249]".

(2) Initiated the ordinary analytic geometry of n-dimensional space (1843), using determinants as the essential tool and introducing the modern notation for determinants. [Simultaneous work on the same subject was done by **Grassmann** and **Ludwig Schläfli** (1814–1895, Switzerland, 1852).]

(3) Introduced the notion of an abstract finite group and what we call today '*finite group algebra*'.

(4) Contributed to the theory of algebraic invariants, which later proved to be essential to tensor algebra.

(5) Research on the singularities of curves and surfaces[250].

Similarly, it can be shown that if

$$\boldsymbol{S} = \begin{bmatrix} 0 & \nu & -\mu \\ -\nu & 0 & \lambda \\ \mu & -\lambda & 0 \end{bmatrix},$$

then

$$e^{\boldsymbol{S}} = \boldsymbol{I} + \frac{\sin\omega}{\omega}\boldsymbol{S} + \frac{1-\cos\omega}{\omega^2}\boldsymbol{S}^2,$$

where

$$\omega^2 = \lambda^2 + \mu^2 + \nu^2.$$

The latter result is immediately applicable to the general finite three-dimensional rotation matrix about an arbitrary axis.

[249] His work on matrices served as a primary mathematical tool for the theory of quantum mechanics as developed by **Heisenberg** (1925)

[250] In a memoir "on Contour and Slope Lines" (*Philosophical Magazine 18* p.264 1859) Cayley introduced the first elements of *physical geography* such as topological *contour-lines* and its application to geological surveying, as well as to mathematical topology. He discovered the relations $S = P + 1$ ($S =$ number of summits, $P =$ number of passes) and $I = B + 1$ ($I =$ number of bottoms, $B =$ number of bars), deducing it from the theory of maxima and minima of continuous functions of two variables.

Unaware of this contribution, **J. C. Maxwell** (*Philosophical Magazine* 1870) rederived most of Cayley's results.

(6) By developing algebras satisfying structural laws different than those obeyed by common algebra, he opened [together with **Hamilton** and **Grassmann**] the floodgates of modern abstract algebra to an enormous variety of systems. Some of these are known as: groupoids, quasigroups, semigroups, monoids, rings, lattices, fields, vector spaces and Boolean algebras[251].

(7) Showed in 1885 that three-dimensional as well as four-dimensional rotations can be represented by quaternions. Similar results were obtained by **Felix Christian Klein** (1849–1925); hence the '*Cayley-Klein rotation parameters*'. These were systematically used by Felix Klein and **Sommerfeld** (1868–1951, Germany) in their classical book "*Über die Theorie des Kreisels*" (1897).

Cayley was born at Richmond in Surrey, of Russian origin on his mother's side. At age 14 he arrived at King's College school, London. He soon showed remarkable mathematical ability and entered Trinity College, Cambridge. In 1842 he graduated Senior Wrangler. In 1846 Cayley decided to adopt the law as a profession and indeed practiced law during 1849–1863. Then he was elected to the new Sadlerian chair of pure mathematics in Cambridge, which he held thereafter.

251 As early as 1849 Cayley wrote a paper linking his ideas on permutations with Cauchy's. In 1854 Cayley wrote two papers which are remarkable for the insight they have of abstract groups. At that time the only known groups were groups of permutations and even this was a radically new area, yet Cayley defined an abstract group and gave a table to display the group multiplication. He realized that matrices and quaternions formed groups.

Matrix Algebra[252] *– A powerful Mathematical Tool*

It is clear from the historical survey shown in Table 4.5 that the earliest notions of determinants arose 23 centuries ago in connection with the simplest algebraic structures then known to mathematicians, namely the solution of linear systems of equations.

The subject of matrices, too, was well developed before it was 'officially' created by ***A. Cayley*** *(1858). Logically, the idea of a matrix precedes that of a determinant but historically the order was the reverse and this is why the basic properties of matrices were already clear by the time matrices were introduced: just because the use of matrices were well-established, it occurred to Cayley to introduce them as distinct entities.*

Determinants and matrices arose in connection with elimination theory, transformation of coordinates, change of variables in multiple integrals, solution of systems of differential equations arising in planetary motion and reduction of quadratic forms in 3 or more variables (geometrical surfaces) to standard form.

In themselves matrices and determinants say nothing directly that is not already stated in the equations or the transformations themselves. Neither determinants nor matrices have influenced deeply the course of mathematics despite their utility as compact expressions. Nevertheless, both concepts have proved to be highly useful tools and are now part of the apparatus of mathematics.

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252 For further reading, see:

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- Turnbull, H.W., *The Theory of Determinants, Matrices and Invariants*, Dover Publications: New York, 1960, 374 pp.
- Barnett, S., *Matrices*, Oxford University Press, 1990, 450 pp.
- Heading, J., *Matrix Theory for Physicists*, Longmans, Green and Company: London, 1958, 241 pp.
- Gantmacher, F.R., *The Theory of Matrices*, Vols I-II, Chelsea Publishing Company: New York, 1960, 366+276 pp.
- Lay, D.C., *Linear Algebra and Its Applications*, Addison Wesley, 2003, 492 pp.
- Hohn, Franz E., *Elementary Matrix Algebra*, Dover: New York, 2002, 522 pp.
- Stephenson, G., *An Introduction to Matrices, Sets and Groups for Science Students*, Dover: New York, 1965, 164 pp.

Table 4.5: Milestones in the history of matrices and determinants

Year(s)	Mathematicians and their achievements
ca 300 BCE	*Babylonians studied problems which lead to simultaneous linear equations.*
200–100 BCE	*In the text 'Nine Chapters of the Mathematical Art', written during the Han Dynasty, Chinese mathematicians gave the first known example of matrix methods: the author set up coefficients of a system of 3 linear equations in 3 unknowns as a table on a 'counting board' and then proceeded to solve the system by a method now known as the 'Gaussian elimination' (early 19th century).*
1545 CE	***Cardano** (in his 'Arts Magna') gave a rule for solving a system of two linear equations, now known as 'Cramer's Rule'.*
ca 1658 CE	***De Witt** (in his 'Elements of Curves') showed how a transformation of the axes reduces a given equation of a conic to canonical form. This amounts to diagonalizing a symmetric matrix.*
1683 CE	***Leibniz** (Germany) and **Seki Kowa** (Japan) independently introduced determinants and gave methods for calculating them. Seki knew that a determinant of n^{th} order, when expanded, has $n!$ terms and that rows and columns are interchangeable. Leibniz, on the other hand, knew that the solubility condition for an homogeneous linear system of equations is that the coefficient matrix has determinant zero. He also proved what is essentially Cramer's Rule and that a determinant could be expanded using any column (now called the 'Laplace expansion'). Leibniz also studied quadratic forms which led naturally towards matrix theory.*

Table 4.5: (Cont.)

Year(s)	Mathematicians and their achievements
ca 1735 CE	**Maclaurin** *(in his Treatise of Algebra) proved Cramer's Rule* for 2×2 and 3×3 *systems of equations.*
1747 CE	**d'Alembert** *introduced the concept of an eigenvalue while studying the motion of a string with masses attached to it at various points.*
1750 CE	**Cramer** *gave the general rule for a* $n \times n$ *system of equations. It arose out of a desire to find the equation of a plane curve passing through a number of given points.*
1764–1771 CE	**Bezout** and **Vandermonde** *gave methods for calculating determinants.*
1772 CE	**Laplace** *gave an expansion of a determinant (he called it 'resolvent') which now bears his name. It arose in connection with his studies of the orbits of the inner planets.*
1773 CE	**Lagrange**, *solving a problem in mechanics, showed that the volume of a tetrahedron formed by 4 points (0,0,0),* (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) *is expressible by the determinant* $$\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$
1801–1809 CE	**Gauss** *introduced the term 'determinant' while studying quadratic forms. In this connection he described matrix multiplication and the inverse of a matrix. In his work on the orbit of the asteroid Pallas,*

Table 4.5: (Cont.)

Year(s)	Mathematicians and their achievements
	Gauss obtained a system of 6 linear equations in 6 unknowns. He gave a systematic method for solving such equations which is now known as the Gaussian elimination method.
1812–1841 CE	***Cauchy** expounded the first systematic work on determinants, introducing the concept of minors and adjoints. He proved the multiplication theorem for* $n \times n$ *determinants,* $c_{ij} = \sum_n a_{ik} b_{kj}$*, (1841). In the context of quadratic forms, Cauchy used the term 'tableau' for the matrix of coefficients. He calculated the eigenvalues and gave results on diagonalization of a matrix in the context of converting a form to a sum of squares (1826). He also introduced the idea of similar matrices (1826), showed that if two matrices are similar they have the same characteristic equation and proved (again, in the context of quadratic forms) that every real symmetric matrix can be diagonalized (1826).*
1829 CE	***Sturm** (Switzerland) defined the concept of the eigenvalue in the context of solving an ordinary differential equation. However, neither d'Alembert nor Sturm realized the generality of the idea they were introducing and saw them only in the specific context in which they were working.*
1841 CE	***Jacobi** generalized the determinant concept to include elements that are functions. **Cayley** used two vertical lines on either side of the array to denote a determinant.*
1850–1851 CE	***Sylvester** introduced the term matrix. Defined equivalence of two matrices (1851).*

Table 4.5: (Cont.)

YEAR(S)	MATHEMATICIANS AND THEIR ACHIEVEMENTS
1853 CE	***Hermite** introduced Hermitian matrices (matrix equal to its transpose conjugate) and showed that its eigenvalues are real. In 1854 he was first to use orthogonal matrices.*
1858 CE	***Cayley** created the theory of matrices, singling out the matrix for its own sake, giving it an abstract definition and establishing matrix algebra (addition, multiplication, scalar multiplication and inverses). He gave an explicit construction of the inverse of a matrix and also proved that a* 2×2 *matrix satisfies its own characteristic equation.*
1870 CE	***Jordan** defined the canonical or normal form of a matrix.*
1874 CE	***Kronecker** defined the direct matrix product.*
1878–1879 CE	***Frobenius** defined the minimum polynomial of a matrix as the polynomial of the lowest degree which the matrix satisfies, formed from the factors of the characteristic polynomial. He also defined the rank of a matrix (1879) as the least* r*-rowed minor whose determinant is not zero. Gave a formal definition to orthogonal matrices(equal to the inverse of its transpose). Defined congruent matrices. Proved the Cayley–Hamilton theorem for* $n \times n$ *matrices.*
1885 CE	***A. Buchheim** (1859–1888) proved that the eigenvalues of a real symmetric matrix are real (**Cauchy** proved it for determinants).*
1892 CE	***W.H. Metzler** introduced transcendental functions of a matrix, writing each as a power series in a matrix. He established series for* e^A, e^{-A}, lnA, $sinA$, $sin^{-1}A$ *for matrices* A.

Table 4.5: (Cont.)

Year(s)	Mathematicians and their achievements
1904 CE	***K. Hensel*** *proved that the minimal polynomial of a matrix divides any other polynomial satisfied by the matrix.*
1907 CE	*The textbook 'Introduction to Higher Algebra' by* ***M. Bôcher*** *brought matrices into their proper place within mathematics.*
1908 CE	***H. Minkowski*** *gave covariant formulation of relativistic electrodynamics in terms of matrices.*
1925 CE	***W. Heisenberg***[253] *formulated quantum mechanics in terms of matrices, establishing matrix mechanics.*

A. Determinants

Systematic treatments of determinants began with ***Cauchy*** *(1812–1840) and continued throughout the* 19^{th} *century by* ***Jacobi*** *(1832–1846),* ***Catalan*** *(1839–1846),* ***Bertrand*** *(1850),* ***Hermite*** *(1854–1856),* ***Cayley*** *(1855),* ***Cremona*** *(1856),* ***Bellavitis*** *(1857),* ***Souillart*** *(1858),* ***Weierstrass*** *(1858),* ***H.J. Smith*** *(1861),* ***R.F. Scott*** *(1878) and* ***Hadamard*** *(1892).*

The determinant

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{1}$$

stands for the number

253 Because matrix algebra was not taught in the curriculum of graduate physics in German universities, neither Heisenberg nor Born knew what to make of the appearance of matrices in the concept of the atom. **David Hilbert** had to tell then to look for differential equations with the same eigenvalues.

$$a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{12}a_{21}a_{33}. \quad (2)$$

Each term in (2) comprises of a product of three elements of the determinant, such that no two elements are from the same row or the same column; e.g. the circled elements, corresponding to the last term in (2). The first index of the element indicates its row and the second index stands for its column. Note that the indices of each term in (2) can be considered as a permutation *of 1, 2, 3. Thus $a_{11}a_{22}a_{33}$ corresponds to the identity permutation*

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},$$

while $a_{13}a_{22}a_{31}$ corresponds to the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

and $a_{12}a_{21}a_{33}$ to the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

In general, the determinant D_n of the array

$$D_n = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}, \quad (3)$$

is the number

$$D_n = \sum \pm a_{1i}a_{2j}...a_{np} = \sum \pm a_{i1}a_{j2}...a_{pn}, \quad (4)$$

where the summation is over all permutations $(i, j, ..., p)$ of $(1, 2, ..., n)$ and the sign accords with the parity of the permutation. In each term of the sum there is one element from each row and one from each column but no two elements have their row or column in common.

***Cauchy** (1826) encountered determinants in his study of the quadratic form*

$$f(x_1, x_2, x_3) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3. \quad (5)$$

This function can be associated with the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{31} \end{vmatrix}, \qquad a_{ij} = a_{ji},$$

where the roots of the characteristic equation $|a_{ij} - \lambda\delta_{ij}| = 0$ *determine the principal axes.*

Sylvester *(1840) came across a determinant in his studies of the theory of equations. He asked: what is the condition under which the two equations*

$$f(x) = ax^3 + bx^2 + cx + d = 0;$$

$$g(x) = px^2 + qx + r = 0$$

possess a common root? Now , if such a root exist, it is also a common root of the system of the 5 equations

$$xf(x) = 0, \quad f(x) = 0, \quad x^2g(x) = 0, \quad xg(x) = 0, \quad g(x) = 0,$$

namely

$$\begin{aligned} ax^4 + bx^3 + cx^2 + dx \quad &= 0,\\ ax^3 + bx^2 + cx + d &= 0,\\ px^2 + qx + r &= 0,\\ px^3 + qx^2 + rx \quad &= 0,\\ px^4 + qx^3 + rx^2 \qquad &= 0. \end{aligned}$$

If we treat these as 5 linear equations, homogeneous in $\{x^4, x^3, x^2, x, 1\}$*, the condition for their consistency is, by the theory of linear equations,*

$$\begin{vmatrix} a & b & c & d & 0 \\ 0 & a & b & c & d \\ 0 & 0 & p & q & r \\ 0 & p & q & r & 0 \\ p & q & r & 0 & 0 \end{vmatrix} = 0. \tag{6}$$

This determinant is Sylvester's eliminant and comprises the relation that the 7 parameters $(a, b, c, d; p, q, r)$ *must obey.*

Certain classes of determinants gained importance in numerical analysis and mathematical physics:

- Continuants

  A continuant is a determinant all of whose elements are zero except those in the main diagonal and in the two adjacent diagonal lines parallel to and on either side of the main diagonal, i.e. $D_{ij} = 0$ *for* $|i - j| > 1$*.*

A general continuant has the form

$$D_{n,1} = \begin{vmatrix} a_1 & b_1 & \cdots & & & & \\ c_1 & a_2 & b_2 & & & & \\ & c_2 & a_3 & b_3 & & & \\ \vdots & & & \ddots & \ddots & \ddots & \\ & & & & c_{n-2} & a_{n-1} & b_{n-1} \\ & & & & & c_{n-1} & a_n \end{vmatrix}. \tag{7}$$

Expansion in terms of the elements of the first column, leads to the difference equation $D_{n,1} = a_1 D_{n,2} - b_1 c_1 D_{n,3}$, *where* $D_{n,j}$ *means that the first element of the main diagonal is* a_j *and the last is* a_n. ***Sylvester*** *(1855) showed that continuants are linked to continued fractions in the following way:*

$$\frac{D_{n,1}}{D_{n,2}} = a_1 - \frac{b_1 c_1}{D_{n,2}/D_{n,3}}$$

develops into

$$\frac{D_{n,1}}{D_{n,2}} = a_1 - \cfrac{b_1 c_1}{a_2 - \cfrac{b_2 c_2}{a_3 - \cfrac{b_3 c_3}{\ddots \quad a_{n-1} - \cfrac{b_{n-1} c_{n-1}}{a_n}}}}. \tag{8}$$

The case $a_n = a, \quad b_n = b, \quad c_n = c$ *was investigated by* ***R.F. Scott*** *(1878)*

$$D_n = \begin{vmatrix} a & b & & & 0 \\ c & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & c & a & b \\ 0 & & & c & a \end{vmatrix}_n \tag{9}$$

The $n \times n$ *determinant obeys the difference equation*

$$D_{n+2} - aD_{n+1} + bcD_n = 0 \qquad n \geq 1,$$

the solution of which is

$$D_n = \frac{(a+\Delta)^{n+1} - (a-\Delta)^{n+1}}{2^{n+1}\Delta}, \qquad \Delta = \sqrt{a^2 - 4bc}.$$

For $a = 2\cos\theta, \quad c = b = 1, \qquad D_n = \frac{\sin(n+1)\theta}{\sin\theta} = U_n(\cos\theta).$

Considering D_n *as an* $n \times n$ *matrix where* $a,\ b,\ c$ *are real and* $bc > 0$, *the eigenvalues of* D_n *are given by*

$$\lambda_s = a + 2\sqrt{bc}\cos\frac{s\pi}{n+1}, \qquad s = 1,\ 2, ..., n.$$

In the 1950's, investigations of the stability of numerical solutions of the heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad u = u(x,t), \quad u(0,t) = u(1,t) = 0, \quad t > 0$$

led to the explicit finite-difference scheme

$$u_{i,j+1} = ru_{i-1,j} + (1-2r)u_{i,j} + ru_{i+1,j}.$$

This, in turn, is manifested through the behavior of the eigenvalues of the above tridiagonal matrix with $a = 1-2r, \quad b = c = r$. *It was found that the scheme is stable for* $r \leq \frac{1}{2}$, *where* $r = \frac{k}{h^2}, \quad h =$ *spatial mesh-size,* $k =$ *temporal mesh size.*

Another case of interest is $\quad b_n = 1, \quad c_n = -1.$ *Here, the continued fraction has the form:*

$$\frac{P_n}{Q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_{n-1}}}}. \tag{10}$$

where

$$P_n = \left| \begin{matrix} a_0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & a_1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-2} & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & a_{n-1} \end{matrix} \right|_{n=1,\ 2,...}$$

$$Q_n = \begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & a_2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-2} & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & a_{n-1} \end{vmatrix}_{n=2,\ 3,\ldots} \qquad (Q_1 = 1),$$

and where the sequential subdeterminants are

$$\begin{array}{ll} P_1 = a_0 & Q_1 = 1 \quad \textit{(defined)} \\ P_2 = a_0 a_1 + 1 & Q_2 = a_1 \\ P_3 = a_0 a_1 a_2 + a_0 + a_2 & Q_3 = a_1 a_2 + 1 \\ P_4 = a_0 a_1 a_2 a_3 + a_0 a_3 + a_2 a_3 + 1 & Q_4 = a_1 a_2 a_3 + a_1 + a_3 \end{array}$$

An example of a practical application of the above theory was given by ***Muir*** *(1889): a rapidly converging series for the extraction of a square root. It was based on previous work done on continued fractions by* ***Lagrange****, who proved that any quadratic number has a continued fraction expansion which is periodic from some point onward. So if* $N > 0$ *is an integer which is not a perfect square,*

$$\sqrt{N} = a_1 + \frac{1}{a_2 +}\frac{1}{a_3 +} \cdots \frac{1}{2a_j +}\frac{1}{a_{j+1} +} \cdots \tag{11}$$

for some $j \geq 1$. *For example*

$$\sqrt{41} = 6 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{12 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{12 + \cdots}}}}}} \tag{12}$$

In general, it can be shown that

$$\sqrt{N} = a_1 + \frac{Q_n(a_2, ..., a_n)}{P_n(a_1, ..., a_n)} + \frac{(-)^n}{2P_n(a_1, ..., a_n)Q_n(a_2, ..., a_n;\ a_1)} - \cdots \tag{13}$$

For the above example, this yields $\sqrt{41} = 6.403\ 124\ 237$, *with an error*

less than 5×10^{-9}. *The next approximant improves the accuracy to 25 decimal places.*

Another, more complicated example of a continuant, applicable to the theory of musical scales, is the algorithm

$$\log_{a_0} a_1 = \frac{\log a_1}{\log a_0} = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cdots}}}, \tag{14}$$

where $\{n_1,\ n_2,\ n_3, ...\}$ *is a sequence of positive integers, and* $a_0 > a_1 > 1$. *The* n_i *are determined by the relations* $a_i^{n_i} < a_{i-1} < a_i^{n_i+1}$ *with the sequence* a_i *recursively defined by* $a_{i+1} = a_{i-1}/a_i^{n_i}$, $i = 0,\ 1,\ 2...$ *Thus, for* $a_0 = 3$, $a_1 = 2$ *the* n_i *sequence is* $[1, 1, 1, 2, 2, 3, 1...]$; *already the first 5 terms yield an excellent approximation for the musical fifth:* $\log 2/\log 3 \approx 12/19$, *from which the Greek result,* $\left(\frac{3}{2}\right)^{12} \approx 2^7$, *follows!*

Another result obtained through the above algorithm is $\log 3/\log 5 \approx 13/19$.

Note that $\frac{p_k}{q_k}$, *the* k^{th} *convergent to* $\frac{\log a_1}{\log a_0}$ *is given by* $p_k = n_k p_{k-1} + p_{k-2}$, $q_k = n_k q_{k-1} + q_{k-2}$ $(n_0 = 0,\ p_{-2} = 0,\ p_{-1} = 1,\ p_0 = 0,\ p_1 = 1;\ q_{-2} = 0,\ q_{-1} = 0)$, *where* $\{p_k,\ q_k\}$ *are the respective continuants:*

$$p_k = \begin{vmatrix} n_2 & 1 & & & & & \\ -1 & n_3 & 1 & & & & \\ & -1 & n_4 & 1 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & \cdot & -1 & n_{k-1} & 1 \\ & & & & & -1 & n_k \end{vmatrix}, \quad k = 2, 3, ...; \tag{15}$$

$$q_k = \begin{vmatrix} n_1 & 1 & & & & & \\ -1 & n_2 & 1 & & & & \\ & -1 & n_3 & 1 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & \cdot & -1 & n_{k-1} & 1 \\ & & & & & -1 & n_k \end{vmatrix}, \quad k = 1, 2, 3, ... \tag{16}$$

The study of continuants began with ***Jacobi*** *(1850) and* ***Sylvester*** *(1853).*

- ALTERNANTS

When the elements of the first row of a determinant are all functions of one variable, the elements of the second row are the same respective functions of a second variable, and so on, the determinant is called an alternant (and similarly for columns): for example

$$\begin{vmatrix} \sin x & \cos x & 1 \\ \sin y & \cos y & 1 \\ \sin z & \cos z & 1 \end{vmatrix} \tag{17}$$

A well-known alternant is due to ***A.T. Vandermonde****(1772),*

$$V_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix} \tag{18}$$

It can be shown that

$$V_n = \prod_{n \geq j > i \geq 1} (\lambda_j - \lambda_i).$$

The corresponding Vandermonde matrix is thus non-singular iff all the λ's are different from each other. This matrix finds application in numerical analysis, where the coefficients of an interpolating polynomial are determined from the data: it is required to determine the polynomial

$$y = a_0 + a_1 x + a_2 x^2 + ... + a_{n-1} z^{n-1}$$

so that it passes through n given points (x_1, y_1), $(x_2, y_2), ..., (x_n, y_n)$. The coefficients are determined by the matrix equation $aV = y$, where

$a = [a_0, a_1, ..., a_{n-1}]$, $\quad y = [y_1, ..., y_n]$ and V is the Vandermonde matrix with $\lambda_i = x_i$.

The study of alternants began with ***Cauchy*** *(1812) and* ***Jacobi*** *(1841).*

- Recurrents

Determinants associated with polynomials, ratios of polynomials, binary quartics and ratios of infinite power series are known as recurrents. For example

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_ny^n = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n \\ y & x & 0 & \cdots & 0 & 0 \\ 0 & y & x & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & y & x \end{vmatrix}. \tag{19}$$

Another example comes from the algebra of infinite series. If

$$\frac{a_0 + a_1\frac{x}{1!} + a_2\frac{x^2}{2!} + \cdots}{b_0 + b_1\frac{x}{1!} + b_2\frac{x^2}{2!} + \cdots} = c_0 + c_1\frac{x}{1!} + c_2\frac{x^2}{2!} + \cdots, \tag{20}$$

then c_n is given by the determinant

$$c_n = \frac{1}{b_0^{n+1}} \begin{vmatrix} b_0 & 0 & 0 & \cdots & a_0 \\ b_1 & b_0 & 0 & \cdots & a_1 \\ b_2 & 2b_1 & b_0 & \cdots & a_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b_n & C_1^n b_{n-1} & C_2^n b_{n-2} & \cdots & a_n \end{vmatrix}. \tag{21}$$

If however, we take the ratio of two polynomials

$$\phi(x) = c_0x^m + c_1x^{m-1} + \cdots + c_m,$$
$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

its formal Laurent series is

$$\frac{\phi(x)}{f(x)} = A_0 x^{m-n} + A_1 x^{m-n-1} + A_2 x^{m-n-2} + \cdots ,$$

and have, upon equating coefficients on both sides of the equation $f(x)\frac{\phi(x)}{f(x)} = \phi(x)$

$$c_0 = a_0 A_0, \quad c_1 = A_0 a_1 + A_1 a_0, \quad c_2 = A_0 a_2 + A_1 a_1 + A_2 a_0, ...$$

Solving for A_r*, we get (**Hagen**, 1883)*

$$A_r = \frac{(-)^r}{a_0^{r-1}} \begin{vmatrix} c_0 & a_0 & 0 & 0 & \cdots & 0 \\ c_1 & a_1 & a_0 & 0 & \cdots & 0 \\ c_2 & a_2 & a_1 & a_0 & \cdots & 0 \\ . & . & . & . & \cdots & . \\ c_{r-1} & a_{r-1} & a_{r-2} & a_{r-3} & \cdots & a_0 \\ c_r & a_r & a_{r-1} & a_{r-2} & \cdots & a_1 \end{vmatrix}_{r+1} . \tag{22}$$

*A famous recurrent is associated with **Laplace**, who produced an explicit expression for the values of the Bernoullian numbers (Johann Bernoulli, 1713), defined as the coefficients of the power series expansion*

$$\frac{t}{e^t - t} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \qquad |t| < 2\pi.$$

Laplace gave the formula

$$B_n = (-)^n n! \begin{vmatrix} \frac{1}{2!} & 1 & 0 & \cdots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \cdots & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \ddots & 0 \\ . & . & . & . & 1 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & \frac{1}{2!} \end{vmatrix} . \tag{23}$$

The numbers B_n figure in the power expansion of the functions $\tan t$, $\tanh t$, $t\cot t$, $t\coth t$, *in the Euler-Maclaurin summation formula, and in the asymptotic form of Euler's gamma function. Some Bernoullian numbers are:*

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30},$$

$$B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}, \quad B_{14} = \frac{7}{6},$$

$$B_{2n+1} = 0, \quad n = 1,\ 2,\ 3, \ldots$$

The first 62 Bernoullian numbers were computed by **Adams** *(1877).*

- Circulants

A determinant such that any row is obtained from the preceding one by passing the last element to the first place, or the first element to the last place, is called a circulant. Thus:

$$c(a_1, a_2, \ldots, a_n) = \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \cdot & \cdot & \cdots & \cdot \\ a_2 & a_3 & \cdots & a_1 \end{vmatrix};$$

$$c'(a_1, a_2, \ldots, a_n) = \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_1 \\ \cdot & \cdot & \cdots & \cdot \\ a_n & a_1 & \cdots & a_{n-1} \end{vmatrix} \tag{24}$$

By transposition of rows it appears that

$$c'(a_1, a_2, \ldots, a_n) = (-)^{\frac{1}{2}(n-1)(n-2)} c(a_1, a_2, \ldots, a_n),$$

where c' belongs to the class of symmetric determinants.

Certain circulants may form a group under determinant multiplication,

e.g.

$$\begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix} \begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix} = \begin{vmatrix} A & C & B \\ B & A & C \\ C & B & A \end{vmatrix} \tag{25}$$

Explicitly

$$(a^3+b^3+c^3-3abc)(x^3+y^3+z^3-3xyz)=(A^3+B^3+C^3-3ABC),$$

where

$$A=ax+by+cz,\quad B=bx+az+cy,\quad C=cx+bz+ay$$

or

$$A=ax+bz+cy,\quad B=bx+ay+cz,\quad C=cx+az+by.$$

If $a=x,\ b=y,\ c=z$, *we obtain by induction*

$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}^{2^n} = \begin{vmatrix} A_n & B_n & C_n \\ C_n & A_n & B_n \\ B_n & C_n & A_n \end{vmatrix}. \tag{26}$$

Explicitly:

$$(x^3+y^3+z^3-3xyz)^{2^n}=A_n^3+B_n^3+C_n^3-3A_nB_nC_n,$$

with $A_n=A_{n-1}^2+2B_{n-1}C_{n-1}$ *etc.*

Likewise, for $n=4$

$$\begin{vmatrix} x & y & z & u \\ u & x & y & z \\ z & u & x & y \\ y & z & u & x \end{vmatrix} = (x^2+z^2-2yu)^2-(u^2+y^2-2zx)^2$$

$$\begin{vmatrix} x & y & z & u \\ u & x & y & z \\ z & u & x & y \\ y & z & u & x \end{vmatrix} \begin{vmatrix} X & Y & Z & U \\ U & X & Y & Z \\ Z & U & X & Y \\ Y & Z & U & X \end{vmatrix} = \begin{vmatrix} A & B & C & D \\ D & A & B & C \\ C & D & A & B \\ B & C & D & A \end{vmatrix}$$

where

$$\begin{aligned} A &= xX+yY+zZ+uU \\ B &= uX+xY+yZ+zU \\ C &= zX+uY+xZ+yU \\ D &= yX+zY+uZ+xU \end{aligned}$$

Circulants were introduced by ***Catalan*** *(1846) and further investigated by* ***Bertrand*** *(1850),* ***Sylvester*** *(1855),* ***Bellavitis*** *(1857) and* ***Souillart*** *(1858).*

- Pfaffians

The determinant of a skew-symmetric matrix can always be written as the square of a polynomial in the matrix elements. This polynomial is called the Pfaffian of the matrix. The Pfaffian is nonvanishing only for $2n \times 2n$ *skew-symmetric matrices, in which case it is a polynomial of degree* n. *For example*

$$Pf\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = a; \qquad Pf\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} = af - be + dc$$

B. Matrices

Matrices entered mathematics with Cayley in connection with linear transformations *of the type*

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \tag{27}$$

(where $a,\ b,\ c,\ d$ *are real numbers), which may be thought of as* mapping *the point* (x, y) *into the point* (x', y'). *Since the above transformation (or map) is completely determined by the four coefficients* $a,\ b,\ c,\ d$ *it can be symbolized by the square array*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

which is called a (square) matrix (of order 2).

If the transformation given above is followed by a second transformation

$$\begin{aligned} x'' &= ex' + fy', \\ y'' &= gx' + hy', \end{aligned} \tag{28}$$

the combined (composition) map can be shown to be the transformation

$$\begin{aligned} x'' &= (ea + fc)x + (eb + fd)y, \\ y'' &= (ga + hc)x + (gb + hd)y. \end{aligned} \tag{29}$$

This leads to the following definition for the product of two matrices,

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{bmatrix}. \tag{30}$$

For brevity we shall state Cayley's definition for 2 by 2 and 3×3 matrices though the definitions apply to $n \times n$ matrices. Two matrices are equal iff their corresponding elements are equal. Cayley defined the zero matrix *and the* unit matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Addition of matrices is defined by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a + e & b + f \\ c + g & d + h \end{bmatrix}, \tag{31}$$

and if λ is any real number

$$\lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \lambda = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}. \tag{32}$$

In the resulting algebra of matrices, it may be easily shown that addition is both commutative and associative and that multiplication is associative and distributive over addition. But multiplication is not commutative, *as is shown by the simple example*

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{33}$$

This example also demonstrates that the product of two matrices may be zero without either being zero.

A special class of a linear transformations in two dimensions is that of rotations of a plane through an angle θ_1 in a counterclockwise sense,while the

coordinate axes remain fixed (active rotation). If the point (x, y) is carried into position (x', y'), then

$$x' = x\cos\theta_1 - y\sin\theta_1,$$
$$y' = x\sin\theta_1 + y\cos\theta_1. \tag{34}$$

In matrix notation, this takes the form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{35}$$

Planar rotation through the angle θ_2 followed by a second rotation through θ_1 is written as

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{36}$$

Applying the law of matrix multiplication to this product, we obtain, through the use of certain trigonometric identities,

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{37}$$

which just states the expected result that two consecutive rotations by angles θ_1 and θ_2 are equivalent to a single rotation by the sum of the angles $(\theta_1+\theta_2)$.

The set of rotation matrices in (37) constitute a multiplicative group *of matrices known as the* orthogonal group *of $\mathbb{R}^2$, or $O(2)$; any matrix in $O(2)$ is called* orthogonal, *defined as having the property*

$$A^T A = AA^T = I, \tag{38}$$

where A^T is the transpose of A and I is the unit matrix. Explicitly, these relations imply

$$\sum_{k=1}^{n} a_{kr}a_{ks} = \delta_{rs}, \qquad (r,\ s = 1, ..., n)$$
$$\sum_{k=1}^{n} a_{rk}a_{sk} = \delta_{rs}, \qquad (r,\ s = 1, ..., n) \tag{39}$$

While either one of the relations (39) implies (38), Eq. (38) is also equivalent to the property that the vectors $A \cdot \begin{bmatrix} x \\ y \end{bmatrix}$, $A \cdot \begin{bmatrix} u \\ v \end{bmatrix}$ are orthogonal iff $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} u \\ v \end{bmatrix}$ are (hence the adjective "orthogonal" for matrices in $O(2)$).

This property, in turn, is equivalent to $\boldsymbol{u}' \cdot \boldsymbol{v}' = \boldsymbol{u} \cdot \boldsymbol{v}$ *for any two vectors* $\boldsymbol{u} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$, *where* $\boldsymbol{u}' = A\boldsymbol{u}$, $\boldsymbol{v}' = A\boldsymbol{v}$, *and* $\boldsymbol{u} \cdot \boldsymbol{v} = \sum_{i=1}^{n} u_i v_i$ *is the scalar product of two vectors.*

Equations (38), (39) and the preservation of scalar products, all generalize to orthogonal matrices in $n = 3$ *and higher dimensions, where the group of orthogonal matrices is denoted* $O(n)$; *but (36), (37) imply* $O(2)$ *is a commutative group, which does not hold for* $O(n)$ *with* $n \geq 3$.

Another interesting outcome of the law of matrix multiplication is the result

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{40}$$

If we denote

$$X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

Eq. (40) can be written as

$$X^2 = -I. \tag{41}$$

As I *plays the role for matrices that 1 plays for numbers, this suggests that we should think of the matrix* X, *in some sense, as a square root of minus one. Note that since* X *is obtained from the rotation matrix*

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

by inserting $\theta = \pi/2$, *the interpretation of (41) is that the symbol* $a + ib$ *stands for the matrix* $Ia + ib$, *namely*

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{42}$$

Given two $n \times n$ *matrices* A *and* B, *three types of products find applications in linear mathematical physics (linear vector spaces):*

- *The ordinary matrix product* $(A \cdot B)_{ik} = \sum_{j=1}^{n} A_{ij} B_{jk}, \quad (i, k = 1, 2, ...n)$
- *The scalar product* $A : B = \sum_{i,k=1}^{n} A_{ik} B_{ik}$
- *The Kronecker (direct) product* $(A \otimes B)_{ik,jl} = (A)_{ij} (B)_{kl}$

An example of the Kronecker product is

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix} = \left[\begin{array}{cc|cc} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ \hline a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{array}\right], \quad (43)$$

where

$$trace(A \otimes B) = (trace A)(trace B) \quad (44)$$

since (using the summation convention for repeated indices) $(A \otimes B)_{ik,ik} = A_{ii}B_{kk}$.

The Kronecker product arises in the following way: Let $\overrightarrow{x}$, $\overrightarrow{y}$ *be two vectors in* n *dimensions,* $\overrightarrow{x} = x_i\overrightarrow{e_i}$, $\quad \overrightarrow{y} = y_k\overrightarrow{e_k}$; *then*

$$\overrightarrow{x} \otimes \overrightarrow{y} = x_i y_k (\overrightarrow{e_i} \otimes \overrightarrow{e_k}).$$

Explicitly $\overrightarrow{x} \otimes \overrightarrow{y}$ *is a column vector with components* $(i = 1, ..., n;$ *k=1,...,n)*

$$\overrightarrow{x} \otimes \overrightarrow{y} = x_i y_k = x_1y_1, ...\ x_1y_n ...\ x_ny_1, ...\ x_ny_n.$$

Next apply the linear transformation of the coordinates

$$\overrightarrow{x'} = A\overrightarrow{x} \quad i.e. \quad x'_i = A_{ij}x_j$$

$$\overrightarrow{y'} = B\overrightarrow{y} \quad i.e. \quad y'_k = A_{kl}y_l$$

Then, the Kronecker product transforms according to the law

$$\overrightarrow{x'} \otimes \overrightarrow{y'} = (A \otimes B)(\overrightarrow{x} \otimes \overrightarrow{y})$$

It remains to express $(A \otimes B)$ *explicitly in terms of* A *and* B. *But since*

$$(\overrightarrow{x'} \otimes \overrightarrow{y'}) = x'_i y'_k = A_{ij}x_jB_{kl}y_l = (A \otimes B)_{ik,jl}x_jy_l$$

we have

$$(A \otimes B)_{ik,jl} = (A)_{ij}(B)_{kl}.$$

- Inverse of a matrix

A linear system of n equations in n unknowns can be written in concise matrix notation as a matrix product

$$A\overrightarrow{X} = \overrightarrow{b} \tag{45}$$

where $\overrightarrow{X}$ is the column vector

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and $\overrightarrow{b}$ is the column vector

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Its solution, when A is non-singular ($|A| \neq 0$) is written as

$$\overrightarrow{X} = A^{-1}\overrightarrow{b} \tag{46}$$

Here the matrix A^{-1} is the inverse *of the matrix A, given explicitly by*

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ldots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}, \tag{47}$$

where A_{jk} is the cofactor *of a_{jk} in $|A|$. We note that in A^{-1} the cofactor A_{jk} occupies the same place as a_{kj} (not a_{jk}) does in A.*

Thus, for example, for

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \qquad ad \neq bc,$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \tag{48}$$

A Hermitian matrix H, is such that its transpose is equal to its complex conjugate, i.e., $H^T = H^*$. An example is

$$H = \begin{bmatrix} 1 & 2-i & 4i \\ 2+i & 3 & -1-i \\ -4i & -1+i & 4 \end{bmatrix}, \qquad i = \sqrt{-1}. \tag{49}$$

For real matrices the concept of Hermitian matrix reduces to that of a symmetric matrix. A skew Hermitian *matrix satisfies $H^T = -H^*$, which for real matrices reduces to* skew-symmetry.

A unitary matrix *U is such that its inverse equal its conjugate transpose, i.e., $U^{-1} = (U^*)^T$ or $UU^{*T} = U^{*T}U$. Two examples are:*

$$U_2 = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}; \qquad U_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} \tag{50}$$

A real unitary matrix is simply an orthogonal matrix. Note that U_3 may arise in the following way: consider a vector in a spherical coordinate system

$$\boldsymbol{u} = \boldsymbol{e_r} u_r + \boldsymbol{e_\vartheta} u_\vartheta + \boldsymbol{e_\varphi} u_\varphi .$$

We can also recast this in the form

$$\boldsymbol{u} = \boldsymbol{e_0} u^0 + \boldsymbol{e_-} u^- + \boldsymbol{e_+} u^+$$

where we use complex basis vectors and complex components:

$$\begin{aligned} \boldsymbol{e_-} &= \tfrac{1}{\sqrt{2}}(\boldsymbol{e_\vartheta} - i\boldsymbol{e_\varphi}) & u^- &= \tfrac{1}{\sqrt{2}}(u_\vartheta + iu_\varphi) \\ \boldsymbol{e_0} &= \boldsymbol{e_r} & u^0 &= u_r \\ \boldsymbol{e_+} &= \tfrac{1}{\sqrt{2}}(-\boldsymbol{e_\vartheta} - i\boldsymbol{e_\varphi}) & u^+ &= \tfrac{1}{\sqrt{2}}(-u_\vartheta + iu_\varphi). \end{aligned} \tag{51}$$

Then

$$\begin{bmatrix} u^0 \\ u^- \\ u^+ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} u_r \\ u_\vartheta \\ u_\varphi \end{bmatrix} \tag{52}$$

Just as an orthogonal *$n \times n$ matrix preserves the $\mathbb{R}^n$ scalar product of vectors, $\boldsymbol{u} \cdot \boldsymbol{v}$, so a* unitary *$n \times n$ matrix can easily be shown to preserve the scalar product $\boldsymbol{u}^* \cdot \boldsymbol{v} = u_j^* v_j$, where $\boldsymbol{u}, \boldsymbol{v}$ belong to $\mathbb{C}^n$*

(vector space of complex n-tuples). The set of $n \times n$ unitary matrices is again a multiplicative group, denoted $U(n)$ and called the ($n \times n$) unitary group.

- Transformation of matrices

There are 4 fundamental relations possible between two given square matrices:

- *Equivalence* $B = PAQ$ (**H.J.S. Smith**, *1861*)
- *Similarity* $B = P^{-1}AP$ (**Frobenius**, *1878*)
- *Congruence* $B = P^{T}AP$ (**Frobenius**, *1878*)
- *Hermitian Congruence* $B = P^{*T}AP$ (**Hermite**, *1854*)

If $P^{-1} = P^{T}$ we have orthogonal similarity; if $P^{-1} = P^{*}$ we have unitary similarity, and if $P^{T} = P^{*}$ the transformation is real.

- Eigenvalues and eigenvectors

Let $A = [a_{jk}]$ be a given $n \times n$ matrix and consider the vector equation

$$Ax = \lambda x \tag{53}$$

where λ is a number (scalar) and x is a vector.

It is clear that the zero vector $x = 0$ is a solution of (53) for any value of λ. A value of λ for which (53) has a solution $x \neq 0$ is called an eigenvalue or characteristic value (or latent root) of the matrix A. The corresponding solutions $x \neq 0$ of (53) are called eigenvectors or characteristic vectors of A corresponding to that eigenvalue λ. The set of the eigenvalues is called the spectrum of A. The largest of the absolute values of the eigenvalues of A is called the spectral radius of A.

The problem of determining the eigenvalues and eigenvectors of a matrix is called an eigenvalue problem. Problems of this type occur in connection with physical and technical applications.

Let us consider (53). If x is any vector, then the vectors x and Ax will, in general, be linearly independent. If x is an eigenvector, then x and

Ax are linearly dependent; corresponding components of x and Ax are then proportional, the factor of proportionality being the eigenvalue λ.

Any $n \times n$ square matrix has at least 1 and at most n distinct (real or complex) eigenvalues.

To see this we write (53) out:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= \lambda x_1, \\ a_{21}x_1 + \cdots + a_{2n}x_n &= \lambda x_2, \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= \lambda x_n. \end{aligned}$$

By transferring the terms on the right-hand side to the left-hand side we obtain

$$\begin{array}{ccccccc} (a_{11}-\lambda)x_1 & + & a_{12}x_2 & +\cdots+ & a_{1n}x_n & = 0 \\ a_{21}x_1 & + & (a_{22}-\lambda)x_2 & +\cdots+ & a_{2n}x_n & = 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1}x_1 & + & a_{n2}x_2 & +\cdots+ & (a_{nn}-\lambda)x_n & = 0 \end{array}$$

In matrix notation,

$$(A - \lambda I)x = 0.$$

By Cramer's theorem, this homogeneous system of linear equations has a nontrivial solution if and only if the corresponding determinant of the coefficients is zero:

$$D(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0. \quad (54)$$

$D(\lambda)$ is called the characteristic determinant, and (54) is called the characteristic equation corresponding to the matrix A. *By developing*

$D(\lambda)$ we obtain a polynomial of n^{th} degree in λ. This is called the characteristic polynomial *corresponding to A.*

Once the eigenvalues have been determined, corresponding eigenvectors can be determined. Since the system is homogeneous, it is clear that if x is an eigenvector of A, then kx, where k is any constant, not zero, is also an eigenvector of A corresponding to the same eigenvalue.

Let us, for example, determine the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}.$$

The characteristic equation

$$D(\lambda) = \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 6 = 0$$

has the roots $\lambda_1 = 6$ and $\lambda_2 = 1$. For $\lambda = \lambda_1$ the system assumes the form

$$-x_1 + 4x_2 = 0$$

$$x_1 - 4x_2 = 0.$$

Thus $x_1 = 4x_2$, and

$$x_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

is an eigenvector of A corresponding to the eigenvalue λ_1. In the same way we find that an eigenvector of A corresponding to λ_2 is

$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

where x_1 and x_2 are linearly independent vectors.

- Application to a simple mechanical system

Matrices find many application to geometry, mechanics, electromagnetic theory, relativistic electrodynamics, quantum mechanics, probability theory and game theory.

Historically, the concept of vectors and matrices were automatically derived from the application of Newton's laws to simple physical configurations governed by a system of linear ordinary differential equations with constant coefficients.

A mechanical system is composed of a linear array of n equal masses m interconnected by linear springs of equal stiffness μ. In addition, each mass is connected to a common support force $F(t)$ by means of a spring of stiffness ν. Taking into account frictional attenuation and 'next-neighbor interaction', the displacement of the i-th mass at distance $x_i(t)$ from equilibrium is given by the differential equation

$$m\frac{d^2x_i}{dt^2} = \mu_i(x_{i+1} - x_i) - \mu_{i-1}(x_i - x_{i-1}) - \nu_i x_i - 2hm\frac{dx_i}{dt} + mF(t)$$

or

$$\ddot{x}_i = \frac{\mu}{m}(x_{i+1} - 2x_i + x_{i-1}) - \frac{\nu}{m}x_i - 2h\dot{x}_i + F(t). \tag{55}$$

In abbreviated notation, (55) takes on the form

$$\ddot{\boldsymbol{X}} + 2h\dot{\boldsymbol{X}} - [K_n]\,\boldsymbol{X} = F(t)\,[I_n] \tag{56}$$

where

$$[K_n] = -\frac{\nu}{m}\,[I_n] + \frac{\mu}{m}\,[T_n]\,, \tag{57}$$

$$\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad [T_n] = \begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix} \tag{58}$$

and $[I_n]$ is the n-dimensional unit matrix.

To obtain a formal solution of (56) we fall back for a moment on the one-dimensional equation of motion of a single attenuated harmonic oscillator driven by a time-dependent force

$$\ddot{X} + 2h\dot{X} - \frac{k}{m}X = F(t).$$

The solution of this equation is known to be

$$X(t) = \int_0^t F(\tau)G(t-\tau)d\tau + X(0)G_1(t) + \dot{X}(0)G(t) \tag{59}$$

where

$$G(t) = \frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_1 - \alpha_2}; \qquad G_1(t) = \frac{\alpha_2 e^{\alpha_1 t} - \alpha_1 e^{\alpha_2 t}}{\alpha_2 - \alpha_1};$$

$$\alpha_{1,2} = -h \pm \sigma; \qquad \sigma = \sqrt{h^2 + k/m}. \tag{60}$$

Explicitly, for $F(t) = qU(t)$, $X(0) = 0$, $\dot{X}(0) = 0$, *with* $U(t)$ *standing for the Heaviside unit step-function*

$$X(t) = e^{-ht}\left[a_0 \cosh(\sigma t) + b_0 \frac{\sinh(\sigma t)}{\sigma}\right]U(t) - a_0 U(t), \tag{61}$$

$$a_0 = mq/k; \quad b_0 = mhq/k; \quad q = \textit{force per unit mass}$$

In a similar way, we may write the solution of (56)

$$\boldsymbol{X}(t) = \left\{e^{-ht}\left[\cosh(\sigma t) + h\frac{\sinh(\sigma t)}{\sigma}\right] - I_n\right\} K_n^{-1}\boldsymbol{q}U(t) \tag{62}$$

where $\cosh(\sigma t)$, $\frac{\sinh(\sigma t)}{\sigma}$, K_n *and* K_n^{-1} *are* $(n \times n)$ *matrices,* $\boldsymbol{q}$ *a column* n*-vector and*

$$\sigma^2 = h^2 I + K_n. \tag{63}$$

Since the eigenvalues of σ^2 are[254]

$$\lambda_s = h^2 - \nu - 4\mu \sin^2 \frac{\pi s}{2(n+1)}, \qquad s = 1, 2, ..., n, \tag{64}$$

the explicit forms of the matrices participating in (62) are

$$\cosh \sigma t = \sum_{s=1}^{n} B_n^{(s)} \cosh\left(t\sqrt{\lambda_s}\right), \qquad \frac{\sinh \sigma t}{\sigma} = \sum_{s=1}^{n} B_n^{(s)} \frac{\sinh\left(t\sqrt{\lambda_s}\right)}{\sqrt{\lambda_s}}$$

$$\frac{1}{K_n} = \sum_{s=1}^{n} \frac{1}{\hat{\lambda}_s} B_n^{(s)} \qquad \hat{\lambda}_s = -\nu - 4\mu \sin^2 \frac{\pi s}{2(n+1)} \tag{65}$$

$$\left\{B_n^{(s)}\right\}_{ij} = \frac{2}{n+1} M_{is}^{(n)} M_{sj}^{(n)} (\textit{no summation over } s),$$

where M_n is the symmetric modal matrix whose columns are the eigenvectors of K_n and its $(sk)^{th}$ term is $\sin\left(\frac{\pi sk}{n+1}\right)$

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[254] The *eigenvalues* of $[T_N]$ are known to be equal to

$$\left\{-4\sin^2 \frac{\pi n}{2(N+1)}\right\}, \quad n = 1, 2, ...N$$

The corresponding *eigenvectors* are

$$\left[\sin \frac{\pi n}{N+1},\ \sin \frac{2\pi n}{N+1}, ..., \sin \frac{\pi n N}{N+1}\right].$$

In general, the eigenvalues of the 3-diagonal matrix

$$\begin{bmatrix} a & b & & & & \\ c & a & b & & & \\ & c & a & b & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & a & b \\ & & & & c & a \end{bmatrix}$$

are

$$\lambda_n = a + 2\sqrt{bc}\cos\left(\frac{\pi n}{N+1}\right), \qquad n = 1, 2, ..., N.$$

If $a = 1 - 2r, \quad b = c = r$, then $\lambda_n = 1 - 4r\sin^2\left(\frac{\pi n}{2(N+1)}\right)$.

$$M_n = \begin{bmatrix} \sin\dfrac{\pi}{n+1} & \sin\dfrac{2\pi}{n+1} & \cdots & \sin\dfrac{n\pi}{n+1} \\ \sin\dfrac{2\pi}{n+1} & \sin\dfrac{4\pi}{n+1} & \cdots & \sin\dfrac{2n\pi}{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ \sin\dfrac{n\pi}{n+1} & \sin\dfrac{2n\pi}{n+1} & \cdots & \sin\dfrac{n^2\pi}{n+1} \end{bmatrix}, \qquad M^{-1} = \frac{2}{n+1}M. \tag{66}$$

Defining

$$Q(\lambda_s) = 1 - \left(\cosh t\sqrt{\lambda_s} + h\frac{\sinh t\sqrt{\lambda_s}}{\sqrt{\lambda_s}}\right)e^{-ht}, \qquad s = 1, 2, ..., n$$

we may present the solution as

$$\boldsymbol{X}(t) = \left[\sum_{s=1}^{n} \frac{Q(\lambda_s)}{(-\overset{\wedge}{\lambda}_s)} B_n^{(s)}\right] \boldsymbol{q}U(t). \tag{67}$$

- Triangular Matrices

A square $n \times n$ *matrix* L *is called lower triangular matrix if all elements of* L *above the principal diagonal are zero*

$$L = \begin{bmatrix} l_{1,1} & & & & 0 \\ l_{2,1} & l_{2,2} & & & \\ l_{3,1} & l_{3,2} & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ l_{n,1} & l_{n,2} & \ldots & l_{n,n-1} & l_{n,n} \end{bmatrix}.$$

Analogously, a matrix of the form

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ & u_{2,2} & u_{2,3} & \cdots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ 0 & & & & u_{n,n} \end{bmatrix}$$

is called upper triangular matrix.

If the entries on the main diagonal are 1*, the matrix is termed normed (or unit) upper/lower triangular. Because matrix equations with triangular matrices are easy to solve they are very important in numerical analysis.*

A special type of a normed triangular matrix is one in which all the off-diagonal entries are zero except for entries in one column. Such a matrix is called a Gauss matrix, and its inverse is again a Gauss matrix.

$$L_i = \begin{bmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & l_{i+1,i} & \ddots & & \\ & & \vdots & & \ddots & \\ 0 & & l_{n,i} & & & 1 \end{bmatrix},$$

$$L_i^{-1} = \begin{bmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & -l_{i+1,i} & \ddots & & \\ & & \vdots & & \ddots & \\ 0 & & -l_{n,i} & & & 1 \end{bmatrix}.$$

Here the off-diagonal entries are replaced by their opposites.

Note that:

- *A matrix which is simultaneously upper and lower triangular is diagonal. The identity matrix is the only matrix which is both normed upper and lower triangular.*
- *A matrix which is simultaneously triangular and normal, is also diagonal. This can be seen by looking at the diagonal entries of* A^*A *and* AA^**, where* A *is a normal, triangular matrix.*
- *The transpose of an upper triangular matrix is a lower triangular matrix and vice versa. The determinant of a triangular matrix equals the product of the diagonal entries, and the eigenvalues of a triangular matrix are the diagonal entries.*
- *The product of two upper triangular matrices is upper triangular, so the set of upper triangular matrices forms an algebra.*

– *A matrix equation in the form* $Lx = b$ *is very easy to solve. It can be written as a system of linear equations*

$$\begin{array}{ccccccccc} l_{1,1}x_1 & & & & & & & = & b_1 \\ l_{2,1}x_1 & + & l_{2,2}x_2 & & & & & = & b_2 \\ \vdots & & \vdots & & \ddots & & & & \vdots \\ l_{m,1}x_1 & + & l_{m,2}x_2 & + & \cdots & + & l_{m,m}x_m & = & b_m \end{array}$$

which can be solved by the following recursive relation

$$\begin{array}{ccl} x_1 & = & \frac{b_1}{l_{1,1}}, \\ x_2 & = & \frac{b_2 - l_{2,1}x_1}{l_{2,2}}, \\ & \vdots & \\ x_m & = & \frac{b_m - \sum_{i=1}^{m-1} l_{m,i}x_i}{l_{m,m}}. \end{array}$$

A matrix equation $Ux = b$ *with an upper triangular matrix* U *can be solved in an analogous way.*

Let A *be an invertible square matrix. An 'LU decomposition'*[255] *gives an algorithm to decompose* A *into normed lower triangular matrix* L *and an upper triangular matrix* U *in the form* $A = LU$, *where* L *and* U *are of the same size as* A. *For a* 3×3 *matrix this becomes*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & (l_{21}u_{12} + u_{22}) & (l_{21}u_{13} + u_{23}) \\ l_{31}u_{11} & (l_{31}u_{12} + l_{32}u_{22}) & (l_{31}u_{13} + l_{32}u_{23} + u_{33}) \end{bmatrix}$$

It yields 9 equations in 9 unknowns $\{l_{21}, l_{31}, l_{32}, u_{11}, u_{12}, u_{13}, u_{22}, u_{23}, u_{33}\}$.

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[255] For further reading, see:

- Press, W.H. et al, *Numerical Recipes in C*, Cambridge University Press, 1988, 735 pp.
- Lay, D.C., *Linear Algebra and its Applications*, Addison-Wesley, 2003, 492 pp.
- Horn, R.A. and C.R.Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

In general there are n^2 equations in n^2 unknowns. To solve the matrix equation

$$Ax = LUx = L(Ux) = b,$$

we first solve $Ly = b$ *for y and then solve* $Ux = y$ *for x.*

Other decompositions by means of triangular matrices are:

LDU decomposition

$A = LDU$, *where D is a diagonal matrix, and L, U are normed triangular matrices.*

PLU decomposition

$A = PLU$, *where P is a permutation matrix (i.e., a matrix of zeros and ones that has exactly one entry in each row and column).*

PLUQ decomposition

$A = PLUQ$, *where P and Q are permutation matrices.*

It can be shown that:

(1) *An invertible matrix admits an LU factorization if and only if all its principle minors are non-zero. The factorization is unique if we require that the diagonal of L (or U) consists of ones. The matrix has a unique LDU factorization under the same conditions.*

(2) *If the matrix is singular, then an LU factorization may still exist. In fact, a square matrix of rank k has an LU factorization if the first k principal minors are non-zero.*

(3) *Every invertible matrix admits PLU factorization. Finally, every square matrix A has a $PLUQ$ factorization.*

(4) *The matrices L and U can be used to calculate the matrix inverse.*

Cholesky[256] Decomposition *(1905)*

Every real symmetric and positive definite matrix A (i.e. $x^T Ax$ is positive for every non-zero vector x) can be expressed in the Cholesky decomposition

$$A = LL^T = U^T U,$$

[256] **Andre-Louis Cholesky** (1875–1918, France). Mathematician. His novel method of solving simultaneous algebraic linear equations was discovered by him in 1905, published posthumously in 1924, and became widely known through A.M. Turing in 1948. Cholesky was born near Bordeaux, France. He graduated from the Ecole Polytechnique (1897) under Camille Jordan and the Army Artillery School (1899). He then served in the Army's Geodetic Section in Tunisia, Algeria and Crete (1902–1912). He was killed in action in 1918 in North of France.

where U is an invertible upper triangular matrix whose diagonal entries are positive, and L is a lower triangular matrix with positive diagonal elements. Thus L can be seen as the "square root" of A. To solve $Ax = b$, one solves first $Ly = b$ for y, and then $L^T x = y$ for x. Cholesky decomposition is often used to solve the normal equations in linear least squares *problems; they give $A^T Ax = A^T b$, in which $A^T A$ is symmetric and positive definite.*

If A is Hermitian *and positive definite, then we can arrange matters so that U is the conjugate transpose of L. In this case*

$$A = LL^*.$$

The Cholesky decomposition always exists and is unique.

To derive $A = LL^T$, we simply equate coefficients on both sides of the equation:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ l_{31} & l_{32} & \vdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & \cdots & l_{n1} \\ 0 & l_{22} & \cdots & l_{n2} \\ 0 & 0 & \vdots & l_{n3} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & l_{nn} \end{bmatrix}$$

to obtain:

$$\begin{array}{lll} a_{11} = l_{11}^2 & \rightarrow & l_{11} = \sqrt{a_{11}} \\ a_{21} = l_{21} l_{11} & \rightarrow & l_{21} = a_{21}/l_{11} \\ a_{22} = l_{21}^2 + l_{22}^2 & \rightarrow & l_{22} = \sqrt{(a_{22} - l_{21}^2)} \\ a_{32} = l_{31} l_{21} + l_{32} l_{22} & \rightarrow & l_{32} = (a_{32} - l_{31} l_{21})/l_{22}, \text{ etc.} \end{array}$$

In general, for $i = 1, \ldots, n$ and $j = i + 1, \ldots, n$:

$$\begin{aligned} l_{ii} &= \sqrt{\left(a_{ii} - \textstyle\sum_{k=1}^{i-1} l_{ik}^2\right)} \\ l_{ji} &= \left(a_{ji} - \textstyle\sum_{k=1}^{i-1} l_{jk} l_{ik}\right) / l_{ii}. \end{aligned}$$

Because A is symmetric and positive definite, the expression under the square root is always positive, and all l_{ij} are real.

- Application to Modern Control Theory

Consider a set of linear differential equations

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x} + bu, \tag{68}$$

where $\boldsymbol{x} = [x_1, x_2, \ldots, x_n]^T$ is a vector of variables describing the state of a system, $\frac{d\boldsymbol{x}}{dt} = [\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt}]^T$, A is a given $n \times n$ matrix, b is a given constant column n-vector, and u is a scalar control variable which can be manipulated. If there are m control variables, the set (68) is replaced by $\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x} + Bu$, where B is a $n \times m$ matrix and u is a m-vector.

It is frequently convenient to seek an approximate solution to a problem governed by a differential equation *by first obtaining a* difference equation *which approximately simulates that equation and then satisfying the new equation at a certain discrete mesh of points by direct algebraic methods, with the expectation that the solution of the simulating problem will indeed simulate the solution of the true problem at these points. In modern digitally controlled systems, state variables measurements and actuator commands, occur, in any case at discretely spaced times.*

Thus, if (68) is subjected to the prescribed initial conditions $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, one uses a Taylor expansion to approximate the derivative $\frac{d\boldsymbol{x}}{dt}$ by the divided difference $\frac{\Delta\boldsymbol{x}}{h}$, where h is a conveniently chosen spacing, and hence rewrite (68) in the form

$$\boldsymbol{x}(t+h) = \boldsymbol{x}(t) + h[A\boldsymbol{x}(t) + Bu] + O(h^2). \tag{69}$$

We then denote by $\boldsymbol{y}(t)$ the solution of the difference equation which results from ignoring the terms of order h^2, and require that the resultant equation holds when $t = t_0, t_1 = t_0 + h, \ldots, t_k = t_0 + kh$. With the usual abbreviation $\boldsymbol{y}_k = \boldsymbol{y}(t_k)$, the equation determining the approximation $\boldsymbol{y}_k$ then takes the form

$$\boldsymbol{y}_{k+1} = A\boldsymbol{y}_k + B\boldsymbol{u}_k, \qquad k = 0, 1, 2, \ldots \tag{70}$$

where the associated truncation error accordingly is of order h^2 when h is small. Eq. (70) can be treated as a recurrence formula*. In general, the controlled plant dynamics (68)–(70) may involve explicit time dependencies in A and B.*

Eqs (68) and (70) (as well as non linear and other variants) are applicable to control problems*, which gained importance in the last three*

decades of the 20th century as a discipline for engineers, mathematicians, scientists and other researchers. Examples of control problems include landing a vehicle on the moon, controlling a power-plant car engine or the macroeconomy of a nation, designing robots, and controlling the spread of an epidemic.

In our example, the time-discretized uncontrolled physical system is governed by the homogeneous difference equation $\boldsymbol{y}_{k+1} = A\boldsymbol{y}_k$, *where* A *is an (assumed known)* $n \times n$ *matrix. To control this system (i.e. to induce it to behave in a predetermined fashion), we introduce into it a forcing term, or a control,* $\boldsymbol{u}_k$. *Thus, the controlled system is the inhomogeneous system* $\boldsymbol{y}_{k+1} = A\boldsymbol{y}_k + \boldsymbol{u}_k$. *In realizing this system, it is assumed that the control can be applied to directly affect each of the state variables* $y_{1,k}, y_{2,k}, \ldots, y_{n,k}$ *of the system, at each timestep* t_k. *In most applications, however, this assumption is unrealistic*[257]. *Thus, a more realistic model for the controlled system is*

$$\boldsymbol{y}_{k+1} = A\boldsymbol{y}_k + B\boldsymbol{u}_k, \tag{71}$$

where B *is a* $(n \times m)$ *matrix, and* $\boldsymbol{u}_k$ *is an* $(m+1)$ *vector, with* m *indicating the number of control variables* $u_1(k), u_2(k), \ldots, u_m(k)$, *where* $m \leq n$.

In control problems, two basic questions need to be answered in deciding whether or not a control solution exists. These questions may be posed thus:

(i) Can we transfer the system from any initial state to any other desired state – or make it follow a desired trajectory – to prespecified accuracy and over a given time interval, by application of a suitable control force? (controllability and stability)

(ii) Knowing the vector of output (sensed, measured) variables for a finite length of time, can we determine the initial state of the system? (observability)

The answers to these questions were conceptualized (1960) by ***R.E. Kalman***[258].

[257] Thus for example economists do not know how economic growth and rate of inflation can be controlled, but can affect them by altering some or all of the following variables: taxes, the money supply, prime bank lending rate, etc.

[258] **Rudolf Emil Kalman** (b. 1930, USA). Mathematical system theorist and electrical engineer. Co-invented the *Kalman Filter*, a mathematical technique widely used in control systems and avionics to extract a signal from a series of incomplete and noisy measurements by a succession of optimized updates of

More precise definitions of controllability and observability are as follows:

- *A system is said to be completely state controllable if it is possible to transfer the system state from any initial state $\boldsymbol{x}(t_0)$ to any other desired state $\boldsymbol{x}(t_0)$ in specified finite time by a control trajectory $\boldsymbol{u}(t)$.*
- *A system is said to be completely observable, if every state $\boldsymbol{x}(t_0)$ can be completely identified by measurements of the output $\boldsymbol{z}(t)$ over a finite time interval.*

If a system is not completely observable, this implies that some of its state variables are shielded from observation. As an example consider the system governed by the discretized equations

$$\begin{aligned} y_1(k+1) &= a_{11}y_1(k) + a_{12}y_2(k) + bu(k) \\ y_2(k+1) &= a_{22}y_2(k) \end{aligned} \tag{72}$$

Here

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

Clearly, this system is not completely controllable, since by inspection, $u(k)$ has no influence on $y_2(k)$. Moreover, $y_2(k)$ is entirely determined by the second equation and is given by $y_2(k) = (a_{22})^k y_2(0)$.

In general, Kalman proved[259] *that the system (71) is completely con-*

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feedback gains and controls. Kalman filters were first used during the *Apollo program* of NASA.

Kalman was born in Budapest, Hungary. Obtained his M.Sc. degree from MIT (1954) and his doctorate from Columbia University (1957). Professor of Stanford University (1964–1971), University of Florida (1971–1992) and ETH, Zurich (1973–1992).

[259] The proof hinges on the fact that the explicit solution of the difference equation $y(k+1) = Ay(k) + Bu(k)$ for constant matrices A and B is

$$y(k) = A^n y(0) + W\bar{u}(k),$$

where we define a new $m \times k$ vector:

$$\bar{u}(k) = \begin{bmatrix} u(k-1) \\ u(k-2) \\ \vdots \\ u(0) \end{bmatrix}.$$

trollable if and only if $\operatorname{rank} W = n$*, where*

$$W = [B, AB, A^2B, \dots, A^{n-1}B] \tag{73}$$

is a matrix of n *rows and* mn *columns.*

Consider the system

$$\begin{aligned} y_1(k+1) &= ay_1(k) + by_2(k) \\ y_2(k+1) &= cy_1(k) + dy_2(k) + u(k), \end{aligned}$$

where $ad - bc \neq 0$*. Here* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *and* $u(k)$ *is a scalar control sequence. Now*

$$W = [B, AB] = \begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix}$$

has rank 2 *iff* $b \neq 0$*. Thus the system is completely controllable iff* $b \neq 0$*. If, however, we consider the control system* $\boldsymbol{y}_{k+1} = A\boldsymbol{y}_k + Bu_k$ *with* $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$*, subjected to* $\boldsymbol{y}(0) = \boldsymbol{y}_0 = \begin{bmatrix} y_{01} \\ y_{02} \end{bmatrix}$*, we have* $\boldsymbol{y}(1) = A\boldsymbol{y}_0 + B\boldsymbol{u}(0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_{01} \\ y_{02} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_0 = \begin{bmatrix} y_{02} \\ 0 \end{bmatrix} + \begin{bmatrix} u_0 \\ 0 \end{bmatrix}$*. So, if we pick* $u_0 = -y_{02}$*, then we will have* $y(1) = 0$*. Therefore the system is controllable to zero. But since* $\operatorname{rank}[B, AB] = \operatorname{rank} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 < 2$*, the system is not completely controllable.*

In the previous theory it was assumed that the observed discretized output of the control system is the same as that of the state of the system $\boldsymbol{y}(k)$*. In practice, however, one may not be able to observe the state of the system* $\boldsymbol{y}(k)$ *but rather an output* $\boldsymbol{z}(k)$ *that is related to* $\boldsymbol{y}(k)$ *in a specific manner. The mathematical model of this type of system is given by*

$$\boldsymbol{y}(k+1) = A\boldsymbol{y}(k) + B\boldsymbol{u}(k); \quad \boldsymbol{z}(k) = C\boldsymbol{y}(k) \tag{74}$$

where A *is an* $n \times n$ *matrix,* B *is a* $n \times m$ *matrix,* $\boldsymbol{u}(k)$ *is an* m*-dimensional column vector, and* C *is* $r \times n$ *matrix. The control* $\boldsymbol{u}(k)$ *is the input of the system, and* $\boldsymbol{z}(k)$ *is its output (Fig. 11).*

Roughly speaking, observability *means that it is possible to determine the state of the system* $\boldsymbol{y}(k)$ *by measuring only the output* $\boldsymbol{z}(k)$*. Hence it is useful in solving the problem of reconstructing unmeasurable state*

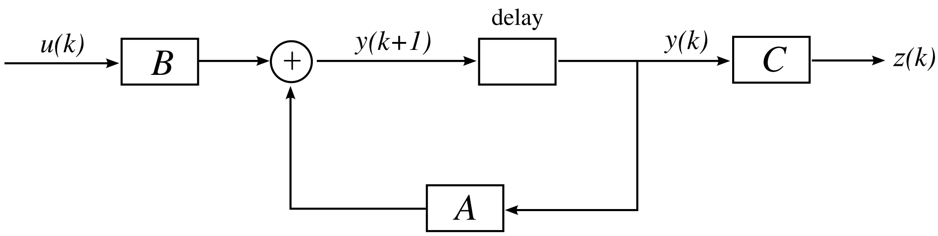

Fig. 11: Flow diagram of a control system

variables from measurable ones. The input-output system (74) is completely observable if for any integer $k_0 \geq 0$, there exists $N > k_0$ such that the knowledge of $\boldsymbol{u}(k)$ and $\boldsymbol{z}(k)$ for $k_0 \leq k \leq N$ suffices to determine $y(k_0) = y_0$.

For constant matrices A, B, C, the exact solution of (74) for $k \geq k_0$ is

$$\boldsymbol{z}(k) = CA^{k-k_0}y_0 + \sum_{j=k_0}^{k-1} CA^{k-j-1}B\boldsymbol{u}(j). \tag{75}$$

Since the second term on the r.h.s. of (75) is known, it may be subtracted from the observed value of $\boldsymbol{z}(k)$. Hence, for investigating a necessary and sufficient condition for complete observability it suffices to consider the case where $\boldsymbol{u}(k) = 0$.

It can then be shown that the system (74) is completely observable iff $\operatorname{rank} V = n$*, where*

$$V = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix}.$$

Returning to the continuous time model, let the initial system state be $\boldsymbol{x}(0)$ and the final state be $\boldsymbol{x}(t_f)$. We say that the system (68) is controllable if it is possible to construct a control signal which, in finite time interval $0 < t \leq t_f$, will transfer the system state from $\boldsymbol{x}(0)$ to $\boldsymbol{x}(t_f)$.

For simplicity we restrict attention to the case of a single component (control input) variable. Let us first assume that the eigenvalues of the matrix A are all distinct, so that the ODE system (68) can be

transformed into the canonical state variable form

$$\begin{bmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} + \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_n \end{bmatrix} u.$$

This equation can be written in component form as

$$\dot{z}_i = \lambda_i z_i + \tilde{b}_i u, \quad i = 1, 2, \ldots, n,$$

which has the solution

$$z_i(t) = e^{\lambda_i t} z_i(0) + e^{\lambda_i t} \int_0^t e^{-\lambda_i \tau} \tilde{b}_i u(\tau) d\tau.$$

The system described by Eq. (68) is then completely controllable if the state variable z_i can be transferred from any initial state $z_i(0)$ to any final state $z_i(t_f)$ in a finite time t_f. In other words, the system is controllable if it is possible to construct a control signal $u(t)$ such that the following equation is satisfied

$$\frac{z_i(t_f) - e^{\lambda_i t_f} z_i(0)}{e^{\lambda_i t_f}} = \int_0^{t_f} e^{-\lambda_i \tau} \tilde{b}_i u(\tau) d\tau.$$

This inverse problem is easily solvable. In fact there are an infinity of functions $u(t)$ on the interval $(0, t_f)$, which solve it, provided $\tilde{b}_i \neq 0$, because otherwise the link between input and the corresponding state variable gets broken and hence it is no longer possible to control that particular state variable.

It therefore follows that the necessary condition of complete controllability is simply that the vector $\tilde{\boldsymbol{b}}$ should not have any zero elements. If any element of this vector is zero, then the corresponding state variable is not controllable. It can be further shown that the condition stated here is in fact both necessary and sufficient.

The result just obtained can be extended to the case where the control variable $\boldsymbol{u}$ is an m-dimensional vector. For the system described by

$$\dot{\boldsymbol{z}} = \boldsymbol{\Lambda} \boldsymbol{z} + \tilde{\boldsymbol{B}} \boldsymbol{u}$$

where $\boldsymbol{\Lambda} = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, and

$$\tilde{\boldsymbol{B}} = \begin{bmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \ldots & \tilde{b}_{1m} \\ \tilde{b}_{21} & \tilde{b}_{22} & \ldots & \tilde{b}_{2m} \\ \vdots & & & \\ \tilde{b}_{n1} & \tilde{b}_{n2} & \ldots & \tilde{b}_{nm} \end{bmatrix}$$

the necessary and sufficient condition for controllability is that the rank of the matrix $\tilde{\boldsymbol{B}}$ must be n_0. It is observed from the above equation that otherwise, it is not possible to influence (all) state variables by the control forces and hence the system is not fully controllable.

Consider the state model of an n-th order single-input, single-output linear time-invariant system,

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u \\ y &= \boldsymbol{c}^T\boldsymbol{x}\end{aligned}$$

The state equation may be transformed to the canonical form by the linear transformation $\boldsymbol{x} = \boldsymbol{M}\boldsymbol{z}$. The resulting state and output equations are

$$\dot{\boldsymbol{z}} = \boldsymbol{\Lambda}\boldsymbol{z} + \tilde{\boldsymbol{b}}u \tag{76}$$

$$\begin{aligned}y &= \tilde{\boldsymbol{c}}^T\boldsymbol{z} \\ &= \tilde{c}_1 z_1 + \tilde{c}_2 z_2 + \cdots + \tilde{c}_n z_n\end{aligned} \tag{77}$$

Since diagonalization decouples the state variables, no z-component now contains any information regarding any other component, i.e., each state must be independently observable. It therefore follows that for a state to be observed through the output y, its corresponding coefficient in Eq. (77) should be nonzero. If any particular $\tilde{c}_i$ is zero, the corresponding z_i can have any value without its effect showing up in the output y. Thus the necessary (and also sufficient) condition for complete state observability is that none of the $\tilde{c}_i$'s (i.e., none of the elements of $\boldsymbol{c}^T\boldsymbol{M}$) should be zero.

The result may be extended to the case of multi-input, multi-output systems where the output vector, after canonical transformation, is given by

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \cdots & \tilde{c}_{1n} \\ \tilde{c}_{21} & \tilde{c}_{22} & \cdots & \tilde{c}_{2n} \\ \vdots & & & \\ \tilde{c}_{p1} & \tilde{c}_{p2} & \cdots & \tilde{c}_{pn} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

or

$$\boldsymbol{y} = \tilde{\boldsymbol{C}}\boldsymbol{z}.$$

The necessary condition for complete observability is that none of the columns of the matrix $\tilde{\boldsymbol{C}}$ be zero.

Kalman's test of observability is as follows. A general n-th order multi-input, multi-output linear time-invariant system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$$

is completely observable if and only if the rank of the composite matrix

$$\boldsymbol{Q}_0 = [\ \boldsymbol{C}^T \vdots \boldsymbol{A}^T\boldsymbol{C}^T \vdots \dots \ (\boldsymbol{A}^T)^{n-1}\boldsymbol{C}^T\ ] \tag{78}$$

is n.

Example *Let us examine the observability of the system given below*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u \tag{79}$$

$$y = \begin{bmatrix} 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boldsymbol{c}^T\boldsymbol{x} \tag{80}$$

The characteristic equation is

$$\boldsymbol{A} - \lambda\boldsymbol{I} = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -2 & -3-\lambda \end{vmatrix} = 0$$

or

$$\lambda(\lambda+1)(\lambda+2) = 0.$$

Therefore the eigenvalues of matrix $\boldsymbol{A}$ are

$$\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = -2.$$

The diagonalized matrix is then

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$$

Under the linear transformation $\boldsymbol{x} = \boldsymbol{M}\boldsymbol{z}$, the output is given by

$$\boldsymbol{y} = \boldsymbol{c}^T\boldsymbol{M}\boldsymbol{z} = \begin{bmatrix} 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

It is found that the system is not completely observable, since the state variable z_2 is hidden from observation.

Let us apply the Kalman's test to the same system. From Eqs. (79) *and* (80)

$$\boldsymbol{A}^T(\boldsymbol{c}^T)^T = \boldsymbol{A}^T\boldsymbol{c} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix};$$

$$(\boldsymbol{A}^T)^2(\boldsymbol{c}^T)^T = (\boldsymbol{A}^T)^2\boldsymbol{c} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}.$$

Therefore the composite matrix in Eq. (78) *is given by*

$$\boldsymbol{Q}_0 = [\ \boldsymbol{c} \vdots \boldsymbol{A}^T\boldsymbol{c} \vdots (\boldsymbol{A}^T)^2\ ] = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 1 & -2 \\ 1 & 1 & -2 \end{bmatrix}.$$

Since

$$\begin{vmatrix} 3 & 0 \\ 4 & 1 \end{vmatrix} \neq 0 \ \textit{and} \ \begin{vmatrix} 3 & 0 & 0 \\ 4 & 1 & -2 \\ 1 & 1 & -2 \end{vmatrix} = 0$$

the rank of the matrix $\boldsymbol{Q}_0$ is $r = 2$, while $n = 3$. Hence one of the state variables is unobservable.

The concepts of controllability and observability found important applications in signal processing and control theory[260]*, where it is sometimes desired to "track" (i.e. maintain an estimate of) a time-varying signal in the presence of noise. If the signal is known to be characterized by some number of parameters that vary only slowly, then the formalism of Kalman filtering tells how the incoming, raw measurements of the signal should be processed to produce best parameter estimate as a function of time. For example, if the signal is a frequency-modulated*

[260] For further reading, see:

- Kalman, R.E., *A New Approach to Linear Filtering and Prediction Problems*, Transaction of the ASME **82**, 35–45, 1960.
- LaSalle, J.P., *The Stability and Control of Discrete Processes*, Springer-Verlag, New York, 1986.
- Barnett, S. and R.G. Cameron, *Introduction to Mathematical Control Theory*, Oxford University Press, 1985.
- Franklin, G.F. et al, *Feedback Control of Dynamic Systems*, Prentice-Hall, New York, 2001.

sine wave, then the slowly varying parameter might be the instantaneous frequency. The Kalman filter for this case is called a phase-locked loop *and is implemented in the circuitry of good radio receivers.*

In control theory a system is said to be controllable *if it is possible to manipulate the control variables in such a way that the system starts out from any initial state and finishes up in any desired state — for example, transferring a spacecraft from an orbit round the earth to a specified orbit round the Moon, or to a 'soft landing' on the Moon, by suitably controlling the rocket motors.*

Indeed, when a deep space probe is launched, corrections may be necessary to place the probe on a precisely precalculated trajectory. Radio telemetry provides a stream of discretized-time observed state vectors, $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k$*, giving information at different times about how the probe's position compares with its planned trajectory.*

Similarly, space shuttle[261] *control systems are absolutely critical for flight. Because the shuttle is an unstable air-frame, it requires constant computer monitoring during atmospheric flight. The flight control system sends a stream of commands to aerodynamic control surfaces and many small thruster jets.*

Modern control theory involves besides input and output variables also feedback and feedforward loops. These feed information on the sensed and desired trajectories into the input. Adaptive control schemes estimate unknown parameters such as inertia or weights of a neural net estimator of unknown dynamics. These parameters are then involved alongside the normal state vector. A control scheme must guarantee suitable levels of stability and controllability in the presence of nonlinearities, delays and noise.

[261] The *Columbia* (12 stories high and weighing 75 tons, launched in April 1981) was the first U.S. space shuttle, a triumph of control systems engineering design, involving many branches of engineering — aeronautical, chemical, electrical, hydraulic, and mechanical.

1849–1857 CE **Antonio Santi Guiseppe Meucci** (1808–1889, Italy and USA). Made pioneering experiments of transmission of the human voice by electricity. Designed and constructed a prototype of an *electromagnetic telephone* many years before Alexander Graham Bell (1876). However, the vagaries of history and the patent office have determined that Meucci will be recognized only in Italy as the true inventor of the telephone.

Meucci was born in Florence, Italy. During 1821–1827 he studied chemistry and mechanics in the '*Accademia di Belle Arti*'. In 1833–1834 he was involved in the conspiracies for the liberation of Italy, being jailed with other patriots. In 1835 he fled the violence of the civil insurrections which raged throughout Italy and reached Havana, Cuba and stayed there as chief engineer of the local Opera House. In 1850, the Meucci's moved to Clifton (Staten Island) NY where he begun to conduct experiments on telephony communications over distances of a few km.

Today in Bensonhurst, New York, there is an iron-fenced triangle of land with fresh sod, some trees, and a small monument that reads: "ANTONIO MEUCCI, 1808–1889, FATHER OF THE TELEPHONE. FIRST US PATENT CAVEAT 3335". He was also memorialized in an Italian postal stamp.

1849–1877 CE **Emil Heinrich Du Bois-Reymond**[262] (1818–1896, Germany). Physiologist. Showed the existence of electrical currents in nerves, correctly arguing that it would be possible to transmit nerve impulses chemically. His experimental techniques proved the basis for almost all future work on electrophysiology.

He was born and educated in Berlin. He studied a wide range of subjects for two years before he finally chose a medical training. Graduating in 1843, he plunged into research on animal electricity and especially on electric fishes. By 1849 he developed a delicate instrument for measuring nerve currents which enabled him to detect an electric current in ordinary muscle tissues, notably contracting muscles. Du Bois-Reymond denounced the *vitalistic* doctrines that were in vague among German scientists and denied that nature contained mystical life forces independent of matter.

He became a professor of physiology at the Berlin university (1858) and was appointed the head of the new Physiological Institute which first opened in Berlin (1877).

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[262] His brothel Paul (1831–1889) was a mathematician, who made contributions to the theory of functions. Their father was a Swiss teacher who settled in Berlin. The family was French-speaking.

1850–1866 CE **James Young** (1811–1883, Scotland). Industrial chemist. Started commercial production of paraffin from crude oil made from heated coal. The crude oil was distilled into its components (or fractions), in containers heated by steam. Thus Young established the basis for *oil refining*[263].

Young was born in Glasgow and studied chemistry under **Thomas Graham** at University College, London. He worked as a chemist in Lancashire (1839) and after 1850 directed his efforts to the establishment of the Scottish mineral-oil industry — for the production of lubricating oils, illuminating oils and paraffin wax.

1850 CE Advent of large sanitary municipal improvement in Western Europe. Before this date the practice of *bathing* was not a general one, and was entirely confined to river and sea baths.

[263] Traditionally, oil was brought to the surface in buckets by workmen who lowered themselves into hand-dug wells.

Science in the Age of Nationalism[264] *(1850–1890)*

The Germans and Italians were the pioneers of modern science, reaching their first peak achievement with the works of **Kepler** *and* **Galileo** *respectively in the early decades of the* 17^{th} *century. But they did not sustain this effort, and almost* 200 *years were to elapse before they produced men of science who were at all comparable.*

The great geographical discoveries opened up opportunities which were the more effectively exploited by England, France and Holland, and these lands became the main centers of European endeavor. In science, as in other fields, England and France retained their leadership right down to the mid or the late 19^{th} *century [their activities were somewhat complementary: the French were inclined toward theoretical interpretation of nature, while the British leaned more to empirical investigation]. In the early decades of the* 19^{th} *century French scholars were the leaders in the world of science, but during* 1850–1870*, the British rose to the forefront once more.*

Meanwhile, the Germans and the Italians adhered to traditions which had been laid down in the 16^{th} *century. Politically they remained divided up into a number of petty principalities (in contrast to the unified states of Britain and France), whilst in science they retained an active interest, but produced little that was novel during most of the* 18^{th} *century.*

It is noteworthy that, of the 90 *or so scientific journals founded before* 1815*,* 53 *were German,* 8 *were Italian,* 15 *were French and* 11 *were English, whilst America, Sweden and Holland, had one each. For such a number of scientific journals to be founded, there must have been a considerable interest in science amongst the Italians and Germans, but it seems that this interest was not active enough to produce markedly novel advances.*

In general, the second half of the 19^{th} *century is marked with an overriding interest and deep belief in science, to a degree that a veritable 'cult of science' developed. Science inspired a positive alternative to the seemingly futile Idealism and Romanticism of the early* 19^{th} *century.*

Scientific research, formerly the domain of a few scientists and gentleman scholars, now became the concern of large numbers of people, especially as the application of science to industry gave an incentive to new inventions. "Pure"

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264 For further reading, see:

- Rich, N., *The Age of Nationalism and Reform* (*1850–1890*), W.W. Norton and Company: New York, 1977, 270 pp.

science continued to be of fundamental importance, but "applied" science — the fusion of science and technology — now took precedence in the minds of most people. A virtually endless series of scientific inventions seemed to provide tangible evidence of man's ability to unlock the secrets of nature.

*By the end of the 19^{th} century, Germany has outstripped both England and France and held the leading position in the physical sciences and mathematics, which climaxed in the 'second scientific revolution' of **Planck** and **Einstein** during 1900–1905.*

1850–1857 CE **Rudolf Julius Emanuel Clausius** (1822–1888, Germany). Theoretical physicist who laid the foundations to thermodynamics and the kinetic theory of gases.

Based on the theoretical results of **James Joule** (1818–1889, England, 1847) and the former theory of heat engines of **Sadi Carnot** (1796–1832, France, 1824), Clausius stated (1850–1865) the first and second laws of thermodynamic[265]and introduced the concept of *entropy*. He formulated (1854–1857) the kinetic theory of gases, defining the concept of *mean free-path*. He assumed different molecular velocities, but the statistical velocity distribution function is due to **Maxwell** (1859). In 1870 Clausius applied to the theory of gases a theorem in mechanics due to the astronomer and mathematician **Yvon Villarceau** (1813–1883, France), known today as the *scalar virial theorem*; the *virial* is the integral of the moments of the molecular forces, partaking in the equation of conservation of mechanical energy. This theorem leads directly to the Van der Waals equation of state.

The scalar virial theorem: although not as important as the conservation of *angular momentum* under central force, on the conservation of *energy* under a conservative force, assumes considerable importance in the kinetic theory of gases, and in its applications to galactic dynamics[266].

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[265] In a paper of 1865 he stated these in the following form:

1. The energy of the universe is constant.

2. The entropy of the universe tends to a maximum.

[266] Consider a general system of mass points with position vectors $\boldsymbol{r}_i$ and applied forces $\boldsymbol{F}_i$ (including any forces of constraint). Starting with the equation of

motion of a single particle

$$\frac{d}{dt}(m_i\boldsymbol{V}_i) = \dot{\boldsymbol{p}}_i = \boldsymbol{F}_i,$$

we derive from it the vector identity

$$m_i\boldsymbol{V}_i^2 + \boldsymbol{F}_i \cdot \boldsymbol{r}_i \equiv \frac{d}{dt}(m_i\boldsymbol{V}_i \cdot \boldsymbol{r}_i).$$

Summing over all particles and time-averaging this equation over a time interval τ, we obtain

$$2\overline{T} + \overline{\sum_i \boldsymbol{F}_i \cdot \boldsymbol{r}_i} = \frac{1}{\tau}\left[G(\tau) - G(0)\right],$$

where

$$G = \sum_i m_i\boldsymbol{V}_i \cdot \boldsymbol{r}_i,$$

$$\overline{T} = \frac{1}{\tau}\int_0^\tau \sum_i \left(\frac{1}{2}m_iV_i^2\right) d\tau.$$

If the motion is periodic, or if coordinates and velocities of all particles remain finite (such that there is an upper bound for G), or if the forces are derived from a potential — then the entity

$$\frac{1}{\tau}\left[G(\tau) - G(0)\right]$$

vanishes or can be made as small as desired. The ensuing result

$$2\overline{T} + \overline{\sum_i \boldsymbol{F}_i \cdot \boldsymbol{r}_i} = 0,$$

is known as the *virial theorem*.

The quantity $-\frac{1}{2}\sum(\boldsymbol{F}_i \cdot \boldsymbol{r}_i)$ is called the *virial* of the system. For a single *particle* moving under a *conservative central force* the theorem reduces to $\overline{T} = \frac{1}{2}\frac{r\partial V(r)}{\partial r}$, where $\boldsymbol{F} = -\boldsymbol{e}_r\frac{\partial V}{\partial r}$. The virial theorem differs in character from mechanical conservation laws in being *statistical* in nature, i.e., it is concerned with time averages of various mechanical quantities.

In general, $\boldsymbol{F}_i$ can be separated into external ($\boldsymbol{f}_i$) and internal ($\boldsymbol{f}_{ij}$) forces. Then,

$$2\overline{T} = -\left[\sum_{\substack{\text{all}\\ \text{particles}}} \boldsymbol{f}_i \cdot \boldsymbol{r}_i + \sum_{\substack{\text{all pairs}\\ \text{of particles}}} \boldsymbol{f}_{ij} \cdot \boldsymbol{r}_{ij}\right], \qquad \boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j.$$

One of the most interesting applications of the virial theorem is the derivation of the *equation of state of a gas*, which describes the relations between the macroscopic quantities such as pressure, volume, and temperature. **Clausius**

Clausius was born at Köslin, in Pomerania. He studied at Stettin, Berlin and Halle. In 1855 he was appointed professor of physics at Zürich. He then held appointments at the universities of Würzburg (1867), and Bonn (1869).

During the Franco-German war he was at the head of an ambulance corps composed of Bonn students.

1850–1871 CE **Adhemar-Jean-Claude Barre de Saint-Venant** (1797–1886, France). Applied mathematician. Contributed to mechanics, elasticity, hydrostatics and hydrodynamics.

Derived solutions for the torsion of noncircular cylinders (1850). Extended Navier's work on the bending of beams (1864). Derived the equations for nonsteady flow in open channels. Also developed a vector calculus similar to that of Grassmann. Stated (1855) the *Saint-Venant principle*[267].

(1870) has shown that for N gas molecules enclosed in a volume V at absolute temperature T and pressure p, the virial theorem leads to the result

$$pV = NkT + \frac{1}{3}\left(\sum_{\substack{\text{all}\\ \text{pairs}}} \boldsymbol{f}_{ij} \cdot \boldsymbol{r}_{ij}\right)_{\text{average}} .$$

For an *ideal gas*, the intermolecular forces are considered zero, and this result reduces to the classical result $pV = NkT$ ($k =$ Boltzmann's constant). In all other cases, it is a good approximation except when the molecules are closely packed or the temperatures are very low. Clausius also generalized Clapeyron's equation expressing the relation between the pressure and temperature at which two phases of a substance are in equilibrium (*Clausius-Clapeyron equation*).

267 *Saint-Venant principle* (as formulated by **Boussinesq** (1889)): In elastostatics, if the boundary tractions on a part S_1 of the boundary S are replaced by a *statically equivalent* traction distribution, the effect on the stress distribution in the body are negligible at points where distance from S_1 is large compared to the maximum distance between points of S_1. The principle has been widely accepted on *empirical grounds*, and a precisely stated version of it was proved by **S. Sternberg** (1954). The principle is of great importance in applied elasticity, where it is frequently invoked to justify solutions in long slender structural members where the end traction boundary conditions are satisfied only in an average sense, so that the correct stress resultant acts on the ends. In such solutions, the *actual* stress distribution near the ends may differ considerably from the *calculated* stress distribution. The *exact* solutions in such cases require elaborate calculations.

The principle is not limited to linear elastic solid or infinitesimal displacements.

Saint-Venant was a student of **Liouville** (1839–1840). He taught mathematics at the Ecole de Ponts et Chausses when he succeeded **Coriolis**.

1850–1871 CE Six European wars established a new balance of power, out of which came the unifications of Italy and Germany and the unprecedented economic growth and scientific development of Europe.

The *Crimean War* (1854–1856) arose out the conflict between Russia and the Western powers over economical interests in the Near East, caused by the slow disintegration of Turkish rule in the Balkans. The defeat of Russia curtailed its influence over the area adjacent to the Ottoman Empire. More than 500,000 people lost their lives in the war. The cost of the war (both sides) have been about $310 million[268] (1903).

In the war of 1859 Austria lost to France and Italian forces, and was consequently driven out of Lombardy. In 1860, Sicily and Southern Italy were liberated from French rule by a small expedition force of ca 1000 men under the leadership of **Giuseppe Garibaldi**, who defeated an army twenty time its size. The kingdom of Italy[269] was proclaimed in 1861.

Although no precise proof is available, **Goodier** (1937) has argued on the basis of energy as follows: Let p be the order of magnitude of the *surface forces*, and a the order of magnitude of the linear dimension representative of the surface ΔS upon which the forces act. Then the components of the *stress tensor* will be of order (pa^2), the components of the *strain tensor* of order $\frac{p}{E}$, the components of the *displacement* of order $\frac{pa}{E}$, the total *work* done by the applied forces of order $\frac{p^2a^3}{E}$ and the *energy density* is of order $\frac{p^2}{E}$. The work done by the applied forces is just sufficient to affect a volume whose magnitude is of order a^3; outside this volume there can be no deformation and one can therefore assume that the region affected will be the immediate vicinity of the surface ΔS upon which the surface forces act.

The concept of Saint-Venant principle does not apply to problems in *elastodynamics*, where the governing equations are hyperbolic, and we know that any fine structure of the surface pressure distribution is propagated all the way to infinity, or at least to the far-field [e.g. the field of a line load traveling at supersonic speeds over the free surface].

[268] On the night of November 14, 1854 a violent storm in the Black Sea wrecked the entire British supply fleet; nearly 30 vessels with their cargo were sunk. Nobody bothered to read the *barometer*!

[269] The independence and unification of Italy was conducted under the joint efforts of **Giuseppe Mazzini** (1805–1872), **Camillo di Cavour** (1810–1861) and **Giuseppe Garibaldi** (1807–1882). These men were, respectively, the '*soul*', the '*brain*' and the '*sword*' of the independence movement.

In the Austro-Prussian war of 1866 (battle of Sadowa[270]), the Austrians were defeated and surrendered Venice to Italy.

Previously, in 1863, Denmark had suffered a crushing defeat at the hands of Prussia and Austria and had to surrender Schleswig and Holstein to the victors.

Finally in 1870, the battle of *Sedan* decided the outcome of the Franco-German war[271].

Thus, in 1871, Germany emerged as a great power — both militarily and culturally. From 1871 to 1945, the influence of that belatedly unified nation made itself felt in every major international crisis and in the history of every country. Compared to German unification, the unification of Italy seems of minor importance today, though it did not appear so at the time. Of much greater consequence was the tragic fate of the Second French Empire. Its defeat at the hand of Prussia sowed some of the seeds that brought forth the great wars of the 20^{th} century. At the time however, it seemed as though the Continent had at long last found the stability that statesmen before 1850 had tried so hard to achieve. The future was to show the precariousness of the new order in Europe.

1850–1881 CE **Ferdinand Julius Cohn** (1828–1898, Germany). Botanist and bacteriologist. Defined and named the term *bacterium* and founded the study of the *bacteriology.* First to treat bacteria systematically by dividing them into genera and species (1872). Assisted **Robert Koch** in his work on *anthrax* (1876). Helped disprove the notion of spontaneous generation. Showed that plant and animal protoplasm are one and the same substance.

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[270] The battle of *Sadowa.* Austria was deeply divided and poorly prepared. She was further handicapped by having to fight on two fronts. The Prussians were in excellent military form and led by a master-strategist, **Helmut von Moltke**.

[271] The main cause of the Franco-Prussian War of 1870–1871 was French resentment of the growing power of Prussia. Relations between the two countries grew worse when Prussia appeared to support the claim of a German prince to the throne of Spain. The final spark occurred when **Otto von Bismarck** (1815–1898), chief minister of Prussia, made public a telegram from King William which he had altered to appear insulting to the French. Bismarck hoped war with France would unite Germany behind Prussia. France at once declared war, although not ready to fight. In six weeks its main army, with the Emperor Napoleon III, had surrendered, and in January 1871 Paris capitulated after 132-day siege. Under the Treaty of Frankfurt, which ended the war in 1871, France gave Prussia the provinces of Alsace and Lorraine, and paid an indemnity of 5000 million Franc.

Cohn was born in Breslau (then Wroclaw, Poland) to Jewish parents. He was educated at Breslau and Berlin[272] (PhD: 1847, at the age of 19), and became an associate professor at Breslau already at the age of 31. At an early age he exhibited astonishing ability with the microscope, which he did much to improve. Although his early researches were especially on algae, he soon widened the scope of his interest to fungi and bacteria and other lower life-forms. He had also a clear perception of the important bearings of mycology and bacteriology in infective diseases. Cohn founded the first institute of plant physiology (1866), the world's first institute specializing in plant physiology.

Bacterial studies outside medicine remained superficial until Cohn. He distinguished four groups on the basis of external form and specific fermentive activity. He recognized that bacteria take *nitrogen* from simple ammonia compounds, elucidated their life-cycles, identified spores and suggested that bacteria were motile cells devoid of walls. Cohn is regarded as the father of bacteriology in that he was the first to account it a separate science and define bacteria. He observed sexual formation of spores in the fungal genera *Sphaeroplea Pilobolus.*

1851 CE **John Gorrie** (1803–1855, USA). Physician and inventor. Invented the first machine for mechanical refrigeration, based on principles of present-day mechanical refrigerators: a steam-engine driven piston compressed the air in a cylinder. When the piston withdrew, the air expanded, absorbing heat from a bath of brine in which the cylinder was immersed.

Gorrie spent most of his life practicing medicine in Apalachicola, Fla. There he investigated the artificial cooling of sickrooms and hospitals. In 1851, he patented an ice-making machine but he lacked funds to manufacture it.

1851 CE **Heinrich (Henri) Daniel Ruhmkorff** (1803–1877, Germany and France). Mechanic, manufacturer of instruments of physics and electrical researcher. Invented (1851) the *Ruhmkorff induction-coil* which could produce

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[272] He went to the University of Breslau (1842) in order to study philosophy and soon became interested in botany, but because he was a Jew he was unable to obtain a degree there. This prompted his transfer to the University of Berlin. Upon his return to Breslau (1849) he had to wait ten years before he was made associate professor and another thirteen years to become a full (ordinary) professor of botany — the first non-assimilated Jew in Prussia to obtain this rank. Indeed, **Robert Remak** (1815–1865) was not allowed to hold a senior teaching position in any German university. **Julius Sachs** (1832–1897), however, was appointed full professor at Breslau already in 1868, but he had been fully assimilated and relinquished his religion long before.

sparks more than 30 cm in length. He thus improved on the two-winding induction spark-coils of **Callan** (1836), on the basis of the research conducted by **Mason** and **Breguet** (1842).

The Ruhmkorff coils, which produced high-voltage current within a second armature winding, were used for operation of *Geissler and Crooks tubes*, in the first *radio transmitters*, for detonation devices as well as in other primitive electrical and electronic devices.

The induction-coil is built as follows: upon an iron core is wound a primary coil consisting of a relatively small number of turns of thick wire, and over this (generally in several layers insulated from one another) a secondary coil consisting of a large number of turns of thin wire. The necessary variations of the magnetic field of the primary current (supplied by a voltage source in its circuit) are produced by making and breaking this current at a rapid speed. A condenser is usually connected in parallel with the make-and-break. It consists of a large number of sheets of tin-foil insulated from each other by means of paraffined paper or sheets of ebonite. Alternate tin-foil sheets are connected together so that the capacity of the whole is very great.

The action of the induction-coil is as follows: When the primary circuit is "made", the magnetic flux through the coil increases. When the primary circuit is suddenly "broken", the magnetic field disappears rapidly and the corresponding field energy, previously stored up chiefly in the air, becomes available. It cannot give rise to a current in the primary coil, for this is now open. The whole of the field energy therefore goes to produce current in the secondary circuit, provided that this is closed. If the secondary circuit is also open, then in consequence of the rapid decrease of magnetic induction and the large number of turns in the secondary coil, a high voltage is produced between the secondary terminals. This tends to cause a spark or arc to pass, with consequent of *closing* of the secondary circuit and consumption of the greater part of the field energy.

The object of the condenser is to reduce the voltage between the contacts of the make-and-break at the moment of breaking the primary circuit, thus preventing sparking and arcing at this point. Since it is connected in *parallel* with the make-and-break, the condenser acts as a shunt. Hence, the introduction of the condenser not only protects the contacts of the make-and-break against damage by arcing, but also increases the efficiency of the induction coil.

Ruhmkorff was born in Hannover, Germany. After apprenticeship to a German mechanic, he worked in England with **Joseph Brahmah**, inventor of the hydraulic press. In 1855 he opened his own shop in Paris, which became widely known for the production of high-quality electrical apparatus. The induction-coil awarded him (1858) a 50,000-franc prize by the Emperor

Napoleon III as the most important discovery in the application of electricity. Ruhmkorff coil was popular for energizing discharge tubes and in particular for *generating X-rays* (discovered in 1895 by Roentgen). His doubly wound induction-coil later evolved into the *alternating-current transformer*.

1851–1859 CE **Georg Friedrich Bernhard Riemann** (1826–1866, Germany). A profound mathematician who greatly influenced the mathematics of the 20^{th} century. His ideas concerning geometry of space had a major effect on the development of modern mathematical physics and provided the concepts and methods used later in General Relativity Theory. He was an original thinker, and a host of methods, theorems and concepts are named after him [*Riemann surface*; *Riemann integral*; *Riemann hypothesis* etc.]. Obtained his doctoral degree in Göttingen (1851) under **Gauss**. His work can be classified according to the following topics:

(1) *Theory of functions of complex variable*, based upon the Cauchy-Riemann relations. Introduced geometrical representation of multi-valued functions (Riemann surfaces), rendering geometric interpretation to the hidden analytical properties of functions. In this he paved the road to modern topology.

Riemann is considered, with **Cauchy** and **Weierstrass**, as one of the three founders of complex function theory. The *method of steepest descent* (also known as the *saddle-point method*) occurs in a posthumously-published fragment of Riemann (Gesammelte Werke, 1892). It was rediscovered in 1910 by **Peter Debye** (1884–1966, Holland).

(2) *Theory of functions of real variable* (1854). Developed the concept of the *Riemann integral*. Generalized Dirichlet's criteria for the validity of Fourier expansions. This inspired Cantor's theory of sets and then led to the concept of the *Lebesgue integral*.

(3) *Differential equations*. Aimed to characterize all linear differential equations whose solutions are expressible in terms of Gauss' hypergeometric function, and to achieve systematic classification of all linear differential equations with rational coefficients according to the number and nature of their singularities.

(4) *Differential geometry* (1854). Followed up the work of his teacher **Gauss** on curved surfaces and took the final step in a far-reaching generalization of differential geometry.

While Gauss' theory is a direct descendant of cartography and geodesy, Riemann, in one of the most prolific contributions ever made to geometry, passed immediately to the general quadratic differential form in n variables, with variable coefficients. He introduced space as a topological

> manifold in an arbitrary number of dimensions. A *metric* was defined in such a manifold by means of a quadratic differential form:
>
> $$ds^2 = \sum_{i,j=1}^{n} g_{ij} dx_i dx_j,$$
>
> where the g_{ij}'s are suitable functions of $(x_1, x_2, \dots, x_n)$; different systems of g_{ij}'s define different Riemannian geometries on the manifold under discussion.

He introduced the *Riemannian curvature tensor*, which reduces to the Gaussian curvature when $n = 2$, and whose vanishing he showed to be necessary and sufficient for the given quadratic metric to be equivalent (isometric) to the Euclidean metric. From this point of view, the curvature tensor measures the deviation of the Riemannian geometry from Euclidean geometry. The physical significance of geodesics appears in its simplest form as a consequence of Hamilton's principle in the calculus of variations.

In a general Riemannian space, g_{ij} is a symmetric non-singular $(\det g_{ij} \neq 0)$ covariant second-rank tensor field. The dependence of g_{ij} on the coordinates x^j is *arbitrary* except that its partial derivatives will be assumed to exist and be continuous to any required order.

A special case of a Riemann space in which a global Cartesian system can be set up is known as a *Euclidean space*. This condition imposes certain restrictions on the metric tensor g_{ij}, namely that the independent components of the *Riemann-Christoffel tensor* $R^{r}_{\cdot ijk}$ vanish everywhere.[273]There are thus $\frac{n(n-1)}{2}$ conditions on the $\frac{1}{2}n(n+1)$ independent metric coefficients of an Euclidean space, in any coordinate system.

The Cartesian coordinate system has the advantage that the distance ds between two neighboring points $\boldsymbol{x}$ and $\boldsymbol{x} + d\boldsymbol{x}$ is given by the Pythagorean theorem $ds^2 = dx^i dx^i$ and therefore $g_{ij} = \delta_{ij}$. If $(\boldsymbol{x}', \ \boldsymbol{x}' + d\boldsymbol{x}')$ are the coordinate of the *same* point in another Cartesian frame, then $ds^2 = d\overline{x}^i d\overline{x}^i$ and it follows that ds^2 is invariant w.r.t. a transformation of the coordinates from one rectangular frame to another. However, in an Euclidean space it is often convenient to employ a coordinate frame which is not Cartesian, and this is achieved by a curvilinear transformation of the coordinates. Thus, *curvilinear orthogonal coordinate frames* are generated with metric tensors g_{ij} represented by diagonal matrices. (If the curvilinear coordinates are non-orthogonal, off-diagonal elements of g_{ij} will appear.)

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[273] $R^{r}_{\cdot ijk}$ and g_{lm} together obey, identically, certain algebraic symmetry conditions and differential equations (the latter are the Bianchi identities). This reduces the number of functional conditions on g_{ij} to $\frac{1}{2}n(n-1)$.

In another class of Riemannian spaces, no admissible global transformation exists which reduces $ds^2 = g_{ij}dx^i dx^i$ to the Pythagorean form $ds^2 = dy^i dy^i$, i.e. no global Cartesian coordinate system can be found. These spaces are *non-Euclidean*, e.g. the 2-dimensional surface of a sphere. In this example we can always find a *local* Cartesian frame in which $ds^2 = du^2 + dv^2$ at a single *point* or even along a local *curve*, but never in any local *two dimensional* region, let alone globally (in contradistinction, the surfaces of the right circular cylinder and cone are locally, though not globally, Euclidean.) Clearly, one may *approximate* any sufficiently small region by a *flat* (Euclidean) *space* provided that the region taken is small enough.

Consider a 2-dimensional Riemann surface with metric equation $ds^2 = g_{11}dv^2 + 2g_{12}dvdw + g_{22}dw^2$ where (v, w) are some Gaussian coordinates and $g_{11} > 0$. The coefficients g_{11}, g_{12}, and g_{22} are functions of position and contain all the information about the geometry of the surface. A point P on the surface is selected and local coordinates (x, y) are found for which the metric is *locally Euclidean* at P. A general definition of the new coordinates is

$$dv = A(x,y)dx + B(x,y)dy,$$
$$dw = C(x,y)dx + D(x,y)dy,$$

where

$$A = \frac{\partial v}{\partial x}, \quad B = \frac{\partial v}{\partial y}, \quad C = \frac{\partial w}{\partial x}, \quad D = \frac{\partial w}{\partial y}.$$

Then

$$ds^2 = g'_{11}dx^2 + 2g'_{12}dxdy + g'_{22}dy^2,$$

where

$$g'_{11} = A^2 g_{11} + 2ACg_{12} + C^2 g_{22},$$
$$g'_{12} = ABg_{11} + (AD + BC)g_{12} + CDg_{22}$$

and

$$g'_{22} = B^2 g_{11} + 2BDg_{12} + D^2 g_{22}.$$

We are free to choose not only the values of A, B, C, D at P, but also the values of their first derivatives at P, provided the two compatibility conditions ($\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$, $\frac{\partial C}{\partial y} = \frac{\partial D}{\partial x}$) are obeyed. Thus, the values of A, B, C, D, and their 6 independent first derivatives provide 10 free variables, enough flexibility to arrange at P:

$$g'_{12} = 0, \quad g'_{11} = g'_{22} = +1,$$
$$\frac{\partial g'_{11}}{\partial x} = \frac{\partial g'_{12}}{\partial x} = \frac{\partial g'_{22}}{\partial x} = \frac{\partial g'_{11}}{\partial y} = \frac{\partial g'_{12}}{\partial y} = \frac{\partial g'_{22}}{\partial y} = 0.$$

It follows that a Euclidean surface with metric equation $ds^2 = dx^2 + dy^2$ will match the current surface locally at P, up to deviations quadratic in $x - x_p$, $y - y_p$. In other words, a *plane* can always be drawn so as to pass through any arbitrary point on a 2-dimensional Riemann surface so that it is *locally tangential* to the surface. Furthermore, this plane can be deformed such that it remains intrinsically un-curved and its (previously) straight lines match all of the manifold's geodesics at P in both direction and curvature. (Notice that the conditions on the metric components and derivatives only make up 9 equations, whereas there are 10 degrees of freedom. The residual degree of freedom amounts to the choice of *orientation* of the x and y axes on the plane.)

It is even possible to choose A, B, C, D such that g'_{ij}, $\partial g'_{ij}/\partial x$, $\partial g'_{ij}/\partial y$ vanish along a *finite curve* on the manifold which passes through P.

A similar procedure can be followed in higher dimensional spaces; some coordinate transformation can always be found which converts the metric coefficients locally (or in the vicinity of a curve section) to a sum of squares, up to corrections quadratic in the geodetic distance from the given point or curve. Riemann spaces are thus said to be locally flat (or locally Euclidean) in this restricted sense. It is *not* possible, however, to arrange that the 2^{nd} derivatives as well as the first derivatives of the coefficients g_{11}, g_{12}, and g_{22} all simultaneously vanish.

If $g_{11}g_{22} - g_{12}^2 < 0$, the quadratic form obtained in the locally-flat coordinate system is $ds^2 = dx^2 - dy^2$ rather than $x^2 + dy^2$. The space involved is still locally flat (in the above sense). It is referred to as a *pseudo-Riemannian space*. The space-time of STR is pseudo-Euclidean (*Minkowski space*), that is, its metric assumes the form $ds^2 = c^2dt^2 - d\boldsymbol{r}^2$ *everywhere*.

A work published after Riemann's death contains what is now known as the *Riemann-Christoffel tensor* in the general-relativistic theory of gravitation (GTR). Riemann made the remarkable conjecture that his new metrics would reduce questions concerning the material universe and the "binding forces" holding it together, to problems in *pure geometry.*

His unifying principle enabled him to classify all existing forms of geometry and allowed the creation of any number of new types of abstract spaces, many of which have since found a useful place in geometry and modern physical theories.

Einstein conceived the geometry of spacetime as a pseudo-Riemannian geometry (locally pseudo-Euclidean) in which the curvature and geodesics are determined by the distribution of matter. In this curved space,

planets move in their orbits around the sun by simply coasting along geodesics, instead of being pulled into curved paths by a mysterious force of gravity whose nature no one had ever really understood.

(5) *Analytic number theory* was founded by Riemann in his path-breaking paper of 1859, devoted to the *Prime Number Theorem*. It launched a tidal wave in several branches of pure mathematics, and its influence will probably still be felt for many years to come. He generalized Euler's identity to complex values of s. The resulting function is known as the *Riemann zeta-function*:

$$\zeta(s) = 1 + 2^{-s} + 3^{-s} + \cdots; \qquad s = \sigma + iy.$$

He made six conjectures with regard to the nature of this function[274].

[274] This Dirichlet series is convergent for $\sigma > 1$, and uniformly convergent in any finite region in which $\sigma \geq 1 + \delta$, $\delta > 0$. It therefore defines an analytic function $\zeta(s)$, regular for $\sigma > 1$.

An equivalent definition of the zeta-function is

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where p runs through all the primes. This is known as *Euler's product* and is absolutely convergent for $\sigma > 1$. Euler considered it for particular values of s only, and it was **Riemann** who first considered $\zeta(s)$ as an analytic function of a complex variable. Since a convergent infinite product of non-zero factors is not zero, $\zeta(s)$ *has no zeros for* $\sigma > 1$.

The analytic function $\zeta(s)$ can be *continued* beyond the half-plane $\sigma > 1$. One such extension is through the relation

$$(1 - 2^{1-s})\zeta(s) = 1 - 2^{-s} + 3^{-s} - 4^{-s} + \cdots, \qquad \sigma > 0.$$

Riemann has extended the definition of the zeta-function for *all* complex values of s through a line integral in the complex s plane. He has also shown that the zeta function satisfies the remarkable functional equation:

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-\left(\frac{1-s}{2}\right)}\zeta(1-s).$$

Some special values and relations involving $\zeta(s)$ are:

$\zeta(0) = -\frac{1}{2}$; $\zeta(1) = \infty$;

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \lim_{n\to 1}\left[(1 - 2^{1-n})\zeta(n)\right] = \ln 2$;

$\zeta(2) = \frac{\pi^2}{6}$; $\zeta(4) = \frac{\pi^4}{90}$;

Assuming these six, Riemann then proved the Prime Number Theorem, namely, that the number of primes less then a given number x asymptotically approaches $\{x/\log_e x\}$. Since then, five of his six conjectures have been proven true.

The famous *Riemann Hypothesis*: All complex solutions of the equation: $1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots = 0$ lie on the line $s = \frac{1}{2} + it$, for some $t \neq 0$, has not yet been proved.

Riemann initiated the study of many more topics, but he died too young to have completed all the projects he started.

Riemann was brought up in a warm family atmosphere, and was of poor health due to poverty at home. Thanks to his father's understanding he did not practice theology, for which he was trained.

$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!}|B_{2n}|, \quad n = 1, 2, \ldots$ (B_n = Bernoulli numbers);

$\zeta(-2m) = 0$;

$\zeta(1-2m) = -\frac{B_{2m}}{2m}$;

$\zeta(-m) = -\frac{B_{m+1}}{m+1}, \quad m = 1, 2, 3, \ldots$;

$\gamma = \sum_{n=2}^{\infty} \frac{(-)^n}{n}\zeta(n)$ (Euler-Mascheroni constant);

$\frac{1}{2}z \coth z = \sum_{n=0}^{\infty}(-)^{n+1}\zeta(2n)\left(\frac{z}{\pi}\right)^{2n}$.

The infinity of primes is a direct consequence of the relation

$$\frac{6}{\pi^2} = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{p_n^2}\right)\cdots,$$

and the fact that the l.h.s. is irrational.

The Riemann Hypothesis (RH) and the Prime Number Theorem[275] *(PNT)*

The functional equation derived by Riemann for the zeta function exhibits a symmetry about $s = \frac{1}{2}$*. Consequently it is expected that the line* $\mathrm{Re}\, s = \frac{1}{2}$ *will play an important role in the theory of the zeta function. On the basis of this functional equation, and some preliminary calculations and asymptotic analyses which he made, Riemann conjectured that all the zeros of* $\Gamma\left(\frac{s}{2}\right)\zeta(s)$ *were on the line* $s = \frac{1}{2} + it$*.*

Despite its apparent simplicity, this statement has never been confirmed or refuted. As a result of the concerted effort of a number of mathematicians — in the pre-computer era: ***E. Landau*** *(1911–1933),* ***T.H. Gronwall*** *(1913),* ***G.H. Hardy*** *(1914),* ***J.E. Littlewood*** *(1914–1928),* ***S. Ramanujan*** *(1915),* ***C. de la Vallée-Poussin*** *(1916),* ***C.L. Siegel*** *(1922–1948),* ***A.E. Ingham*** *(1926–1933),* ***J. Hadamard*** *(1927),* ***E.C. Titchmarsh*** *(1928–1947),* ***A. Selberg*** *(1942–1946),* ***A.M. Turing*** *(1943) — a great deal has been learned about the distribution of zeros of* $\zeta(s)$*.*

It has been proven that:

(1) $\zeta(s)$ *has an* infinity of zeros *in the strip* $0 \le \sigma \le 1$*, all of which are complex (no zeros on the real axis between* 0 *and 1);*

(2) the zeros either *lie on* $\sigma = \frac{1}{2}$ *or occur in pairs symmetrical about this line;*

(3) there is an infinite number of zeros of $\zeta(s)$ *on* $s = \frac{1}{2} + it$ *[****Hardy****, 1914].*

Nevertheless, the RH *remains one of the outstanding challenges of mathematics, a prize which has tantalized and eluded some of the most brilliant mathematicians of* 20^{th} *century. In fact, the* RH *stand today as* the most *important unsolved problem in mathematics. Not only is it tied up with the* prime number theorem*, but many other theorems are* conditioned *on the* RH *— that is, their proofs assume that the* RH *is true. Other results are known to be* equivalent *to the* RH*.*

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275 For further reading, see:

- du Sautoy, M., *Music of the Primes*, Perennial, 2003, 335 pp.
- Derbyshire, J., *Prime obsession*, Joseph Henry Press: Washington, D.C., 2003, 422 pp.

The search for the zeros is a story in itself: In 1903, ***J.P. Gram*** *published values of* t *of the first* 15 *zeros [14.13, 21.02, 25.01, 30.42, 32.93, 37.58, ... to two decimal places], thus rendering the first solid evidence in support of the RH. There was even some wonderment at how Riemann had arrived at his prediction, since his paper on the zeta-function contained no computations at all. It was generally believed that Riemann had based his hypothesis on aesthetics and intuition — two major driving forces of mathematical research.*

However, in 1932, ***Carl Ludwig Siegel*** *proved that there was more to it than aesthetics and intuition. Searching through Riemann's unpublished papers in the archives of the University Library at Göttingen, Siegel discovered that Riemann had indeed computed several zeros of the zeta-function. Not only that, but he had done so by a method superior to those that Gram and others had used after him!*

Siegel cleaned up Riemann's method, and the so-called Riemann-Siegel formula *became the basis for computing zeros of the zeta-function. The list of zeros has grown with the advent of high-speed computers. In* 1952, ***Alan Turing*** *identified the first* 1054 *zeros ("the first" meaning the zeros closest to the real axis — the purported line of zeros is perpendicular to the real axis at the point* $\frac{1}{2}$*).*

The list grew to $25,000$ *in* 1956, 3.5 *million in* 1968 *and* 81 *million in* 1979. *In* 1985, *it reached a staggering* 1.5 *billion zeros —* every *one of which lies on the predicted line. In* 1990, *the* 10^{20}*-th zero of the Riemann zeta-function was found to be* $\frac{1}{2} + [15,202,440,115,920,747,268.629,029,9\ldots]i$. *Thus, to date, mathematicians have amassed impressive amounts of evidence in favor of the hypothesis.*

But that is not all: statistical studies of the distribution of spacing between the zeros of the zeta-function *have led to another conjecture, namely that the distribution of these spacing is similar to that of eigenvalues of random matrices that are studied in* many-particle systems *in physics. This hypothesis suggests that the zeta-function could be used as a model of* quantum chaos.

The importance of $\zeta(s)$ *in the theory of prime numbers lie in the fact that it connects two expressions, one of which contains the primes explicitly, while the other does not.*

The theory of primes is largely concerned with the function $\pi(x)$, *the number of primes less or equal to* x.

Gauss *(1792, age 14!) was first to notice that* $\pi(x)$ *can be estimated by the function* $\left\{\frac{x}{\log_e x}\right\}$ *or*

$$Li(x) = \int_2^x \frac{dt}{\log_e t} = \frac{x}{\ln x} + \frac{1!x}{(\ln x)^2} + \ldots \frac{(k-1)!x}{(\ln x)^k} + O[\frac{x}{(\ln x)^{k+1}}];$$

He did not publish this result. In 1798, ***Legendre***, *independently, suggested that* $\pi(x) \sim \frac{x}{\log_e x - 1.08366}$. *At that time, these relations seemed completely inexplicable, since* $\log_e x$ *arose in differential calculus in connection with problems of continuous growth and decay and was not known to be related in any way to discrete prime numbers. The approximation, in percentage terms, grows better and better as x increases. Gauss, being both a number theorist and the man who founded mathematical statistics, used his "method of least squares" to show that, as x approaches infinity, the errors are likely to eventually approach zero [for* $x = 10^3$, *the error is 16.0%, for* $x = 10^9$, *it is 5.4%, while for* $x = 10^{14}$, *it is only* 3.2 *percent].*

It took 50 *years before anyone made any progress toward proving the Gauss-Legendre conjecture. The first person to do so was* ***P. Chebyshev*** *in* 1850. *He obtained a partial result and his ideas were then imitated by others. But eventually it turned out that his methods would not go any further, and they were abandoned.*

In 1859, ***Riemann*** *published a small paper entitled: "On the Number of Primes Less Than a Given Magnitude" (in German). Its reasoning contained large gaps, and very little was definitively proven, but nearly everything that has been done in the theory of numbers since then has been influenced by that paper.*

For 40 *years, other mathematicians tried to prove the main result enunciated in Riemann's* 8*-page paper — but to no avail. In* 1896, ***Hadamard*** *and* ***de la Vallée-Poussin***, *working independently, finally proved that* $\lim_{n\to\infty} \left\{ \frac{\pi(n)}{n/\log_e n} \right\} = 1$, *the PNT.*

Further research followed: In 1908, ***E. Landau*** *showed that*

$$\pi(x) = Li(x) + O[xe^{-\gamma\sqrt{\ln x}}].$$

In 1914, ***J.E. Littlewood*** *showed that the difference* $\{Li(n) - \pi(n)\}$ *changes from positive to negative infinitely many times as n runs up through the positive integers, although the first change of sign occurs for a very large n [in* 1986 ***J.J. te Riele*** *showed that this number is smaller than* 6.69×10^{370}. *A computer search made as far as* 10^9 *failed to produce such a number. It may never be possible to discover the actual number!]*

An approximation to $\pi(n)$ *which involves the zeta-function explicitly was derived in* 1903 *by* ***Gram*** *from Euler's product formula:*

$$R(n) = 1 + \sum_{k=1}^{\infty} \frac{1}{k\zeta(k+1)} \frac{(\log n)^k}{k!}.$$

Thus, for $n = 10^9$,

$$\pi(n) = 50{,}847{,}534; \qquad \left\{\frac{x}{\log_e x}\right\} = 48{,}254{,}942;$$
$$\left\{\frac{x}{\log_e x - 1.08366}\right\} = 50{,}917{,}519, \qquad R(n) = 50{,}847{,}455;$$

$R(n)$ *thus yields best estimate with a percentage error of only* 1.5×10^{-4}*!* ***Ramanujan*** *discovered (1913) the alternative form*

$$F(x) = \int_0^\infty \frac{(\log x)^t dt}{t\Gamma(t+1)\zeta(t+1)}$$

for the sum.

The importance of the RH lies in the fact that the errors of the approximations to $\pi(x)$ *depend on the zeroes of the zeta-function. The connection between* $\pi(x)$ *and RH also lies behind a great deal of other known facts about primes. If the RH does turn to be true, then the connection with the function* $\pi(n)$ *will enable even more information about the prime numbers to be deduced than is at present known.*

Moreover, the prime number theorem is important not only because it makes an elegant and simple statement about primes and has many applications, but also because much new mathematics was created in the attempt to find a proof. This is typical in number theory and topology, where problems which are very simple to state are often extremely difficult to solve. Mathematicians working on these problems often create new areas of mathematics of independent interest[276]*. Two additional examples stand out:*

(1) the creation of Algebraic Number Theory as a result of work on the Fermat Conjecture;

(2) the creation of Graph Theory as a result of the search for the solution to the 4-color problem.

[276] This phenomenon, which happened over and over again in mathematics, brings to mind the well-known tale about a farmer who had three lazy sons that were loafing around without doing any substantial work. On his deathbed, the farmer told them that a treasure was buried somewhere on the farm. Following his death the sons began to dig the farm inside out in search of the fortune. In doing so, they unknowingly cultivated the land and became very prosperous. The conjecture of Riemann was such a hidden 'treasure'.

1851 CE The first successful *submarine telegraph cable* was laid between Dover and Calais.

1851 CE The *brown rats* (alias *Norway rats*) reach the Pacific coast[277]of the United states after some 50 years of migration from the East coast (average diffusion rate of ca. 300 meter/day). It reached the ports of the New World from Europe as stowaways on ships. Brown rats migrated to Europe from Asia, apparently from *North China*. They are known to have reached Paris in 1753.

1851–1855 CE Tuberculosis ravaged England; Ca 250,000 died.

1851–1897 CE **William Thomson** (**Lord Kelvin**, 1824–1907, England). A distinguished physicist of the 19^{th} century. Kelvin published more than 600 papers on a wide range of scientific subjects, and he patented 70 inventions. Queen Victoria knighted Kelvin for his work as an electrical engineer in charge of laying the first successful transatlantic cable[278] in 1866.

In 1851 he proposed the gas thermometer as the basis of an *absolute temperature scale* (Kelvin scale; 1848), with degree intervals equivalent to those on the centigrade scale but with the fiducial zero point at $-273.7\,°\mathrm{C}$ [today at $-273.15\,°\mathrm{C}{=}{-459.67}\,°\mathrm{F}$], called the *Absolute zero*. [According to classical physics, ideal gases at this temperature contract to solids and all molecular motion ceases.] He coined the word *Thermodynamics* (1849).

In 1852 he discovered with **James Prescott Joule** the '*Joule-Thomson Effect*', according to which gases undergo a change of temperature when made to expand freely [the effect was utilized in 1877 by **Carl von Linde** (1842–1934) to design an Ammonia gas refrigerator]. Greatly interested in the improvement of physical instrumentation, he designed and improved many new devices.

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[277] Some 300 years after **Balboa**, the first European to see the Pacific Ocean in 1513. Since the second half of the 19th century brown rats arrived everywhere with the speed of trains, ships and cargo planes, spreading diseases such as plague, typhus, anthrax and trichinosis.

[278] Experimental testing of physiologists in the 1930's provided important evidence confirming the relevance of telegraphic cable theory to *nerve axons*. **Alan Lloyd Hodgkin** (1946–1947) and his co-workers presented derivations of the Kelvin cable equation for nerve cylinders and included transient solutions as well as methods for estimating the values of key parameters. The application of cable theory to *dendritic neurons* began in the late 1950's, when it became necessary to interpret experimental data obtained from individual neurons by means of intercellular microelectrodes located in the neuron soma.

Among Kelvin's inventions are the mirror galvanometer (1867) and the marine compass free of magnetic influence (1873). In 1876 he proposed the principle of the *differential analyzer*[279] (a misnomer for a mechanical computer, destined to solve mainly ordinary differential equations). In 1897 he finalized his estimate of the age of the earth, using the theory of conductive cooling of a semi-infinite half-space model. He hypothesized that the earth was formed at a uniform high temperature and that its surface was subsequently maintained at low temperature. He then assumed that a thin near-surface boundary layer developed as the earth cooled. Since the boundary layer would be thin compared with the radius of the earth, he reasoned that a one-dimensional heat-equation model could be applied. His calculations then yielded the value of ca 20 million years for the age of the earth. The discovery of radioactivity (1896) showed that his basic assumptions were wrong[280].

Kelvin was also wrong on three other counts: he was convinced that the Eulerian period of 10 months for the free precessional motion of the earth's axis of rotation was real, and ignored the effect of the period lengthening by 4 months due to the non-rigidity of the earth.

He was totally blind to the impact of vectors on physical theory. In 1886 he wrote to Hayward: "*Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been useless to those who have touched them in any way, including Clerk Maxwell*". In a letter to G.F. Fitzgerald in 1890 he wrote: "*Vector is a useless survival, or off-shoot, from quaternions, and has never been of the slightest use to any creature*". Finally, Kelvin did not accept the atomic theory of matter.

Applied mathematicians and physicists will always be grateful to him for discovering the *method of stationary phase* (1887), for asymptotic evaluation of special integrals[281].

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279 The proposal was unfortunately neglected. Not until 50 years later was a workable machine constructed by **Vannevar Bush** and his colleagues at the Massachusetts Institute of Technology.

280 Kelvin also viewed the sun as some sort of a dwindling self-gravitating coal pile that illuminated the earth for only a few tens of millions of years.

Kelvin would not accept geological arguments against his estimate and once, in the heat of a dispute over the earth's age, said that geology was as intellectually respectable as collecting postage stamps.

281 The method is applicable to integrals of the form $K(\lambda) = \int_{-\infty}^{\infty} g(\omega)e^{i\lambda f(\omega)}d\omega$, where $g(\omega)$, $f(\omega)$ are real functions and λ is a large, positive constant. When λ is large, the exponential function $e^{i\lambda f(\omega)}$ will, in general, oscillate very

Kelvin resolved *Olber's paradox* quantitatively and correctly in the framework of a transparent, uniform, and *static* universe. In a paper entitled "On ether and gravitational matter through infinite space", published in the *Philosophical Magazine* (1901), he was the first to show, on the strength of the Kant-Laplace nebular hypothesis, that if one assumes that we live in a universe of *finite age* (or in a universe of unlimited age in which the stars have been shining for only a limited time), then the observed phenomenon of a dark starlit sky would categorically necessitate a cosmological regime in which the size of the visible universe is less than the background limit[282]. This Kelvin

rapidly. If $g(\omega)$ changes slowly, the rapid phase-change will tend to cause cancellations. In total, the integral will approximately vanish except around points $\omega = \omega_0$ where $f(\omega)$ is stationary, i.e., $f'(\omega_0) = 0$. For one stationary point, the quantitative analysis of this statement yields *Kelvin's formula*:

$$\int_{-\infty}^{\infty} g(\omega) e^{i\lambda f(\omega)} d\omega = \left[\frac{2\pi}{\lambda |f''(\omega_0)|}\right]^{1/2} g(\omega_0) e^{i\lambda f(\omega_0) + i\delta} \left[1 + O\left(\frac{1}{\lambda}\right)\right],$$

where $\delta = \frac{\pi i}{4} \operatorname{sgn} f''(\omega_0)$.

282 The treatment of Kelvin elucidates what was previously shown by **Halley** (1721), **Cheseaux** (1744), and **Olbers** (1826). Let all stars be sun-like, of radius a, and uniformly distributed (n per unit volume). Let α denote the fraction of the sky covered by the discs of stars out to radius r. Then $\alpha = \frac{r}{\lambda} \geq 1$, where $\lambda = \frac{1}{\pi n a^2}$ is the mean free path of a light ray. We note that $\alpha = 1$ when the radius r of a surrounding sphere of stars equals λ, and hence λ is also the *background limit* of a star-filled universe. (In this case the entire sky is covered in a distribution of stars of infinite extent and the average distance observable from any position is the background limit λ.)

Let each star have *luminosity* L. The contribution to the *radiation density* u at its center of a shell of radius q and thickness dq is $du = \frac{nL}{c} dq$. By integrating from $q = 0$ to $q = r$ we find $u = u^* \frac{r}{\lambda} \equiv u^* \alpha$, where $u^* = \frac{L}{\pi a^2 c}$ is the radiation density at the surface of a star. Therefore, $\alpha = \frac{u}{u^*}$ demonstrates the truth of Kelvin's statement that α is the ratio of the apparent brightness of our starlit sky to the brightness of our sun's disc. Kelvin thus showed that,

$$\frac{\text{brightness of starlit sky}}{\text{brightness of sun's disc}} = \frac{\text{size of visible universe}}{\text{background limit}} = \frac{\text{fraction of sky}}{\text{covered by stars}}.$$

Since the left hand fraction is much less the unity, any viable theory must explain why the size of the visible universe (the part we see) is much smaller than the background limit. In his estimate for the size of the background limit $\lambda = \frac{1}{\pi n a^2}$, he followed the reasoning of Cheseaux and Olbers, and the astronomical data available to him, which gave him a value of ca 3×10^{15} light-years. To ensure that the *visible* universe remains always smaller than λ, his static uniform model forced him to limit the *age* of the universe (or equivalently, the age of stars in a universe otherwise of unlimited age). Kelvin chose (erroneously), a visible

was easily able to show on the basis of the then available astronomical information.

Although he solved the riddle of cosmic darkness according to the conditions prescribed in its original conception for a uniformly populated and static universe, it can be shown that all the variants of this primitive standard model that resort to absorption, hierarchy, and redshift (owing to expansion) merely accomplish a state of greater darkness in a universe already dark.

Kelvin disbelieved in paradoxes. In his *Baltimore Lectures* (1884) he more than once declared: "*There are no paradoxes in science*". He took the rationalist attitude that paradoxes are the result of misunderstandings; they lie in ourselves and not the external world.

Kelvin was born in Belfast, Ireland. His father James was a professor of mathematics at Glasgow University. He was educated at the universities of Glasgow, Cambridge, and Paris. Kelvin became a professor of natural history at the University of Glasgow in 1846 and remained there until his retirement in 1899. He was married twice (1852, 1874). However, there was no heir to his title, which became extinct.

Electrostatics and Number Theory

In 1853, ***Lord Kelvin*** *used his method of images to calculate, in a very elegant way, the mutual capacitance* (C_{12}) *of a configuration consisting of* 2 *spheres of radii* a *and* b, *with their centers a distance* $c \geq (a+b)$ *apart. He was able to show that the result can be represented in the form of a converging modified Lambert series (**Lambert**, 1771)*

$$C_{12} = \frac{EI}{c} \sum_{n=1}^{\infty} \frac{\alpha^n}{1-\alpha^{2n}}; \quad \alpha = \frac{E-I}{E+I} \leq 1.$$

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universe of 14 million light-years, but even with modern values of 14 billion light-years, the darkness at night is fully guaranteed.

Cheseaux and **Olbers** assumed that the stars shine long enough for light to travel from the background limit. Had they questioned this assumption, they might have realized that there was no need to postulate the absorption of starlight.

Here,

$$E = \sqrt{c^2 - (a-b)^2}, \qquad I = \sqrt{c^2 - (a+b)^2}$$

are the lengths of the external and internal tangents, respectively, to the circles obtained by cutting the spheres with a plane through their centers.

*This problem has drawn the attention of mathematicians and physicists, who tried to improve the convergence of the above series [**E.W. Barnes** (1903), **A. Russell** (1911), **J.H. Jeans** (1915), **W. Smythe** (1939)]. One hundred years after its inception (1953), the problem was revisited by **Balthazar van der Pol**, who noticed that*

$$\sum_{n=1}^{\infty} \frac{\alpha^n}{1-\alpha^{2n}} \equiv \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \alpha^{n(2m+1)}.$$

Therefore,

$$C_{12} = \frac{EI}{c}\sum_{n=1}^{\infty}\left\{d(n) - d\left(\frac{n}{2}\right)\right\}\alpha^n.$$

The Dirichlet divisor function, $d(n)$, characteristic of number theory, represents the number of divisors of n. [$d\left(\frac{n}{2}\right)$ is to be replaced by zero for any odd n.] This function is related to the Riemann zeta-function $\zeta(n)$, via the Voronoi relation[283]

$$\{\zeta(s)\}^2 = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}.$$

*Also, the above series is closely related to the famous divisor problem of **Dirichlet**[284], which is that of determining the asymptotic behavior as $x \to \infty$ of the sum*

$$D(x) = \sum_{n \le x} d(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta^2(z)\frac{x^z}{z}dz$$

(with $c > 1$ and x not integer).

1851–1852 CE **Jean Bernard Léon Foucault** (1819–1868, France). A distinguished experimental physicist. Demonstrated the absolute rotation of

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283 **G. Voronoi** (1904).

284 Dirichlet proved that $D(x) = x\log x + (2\gamma - 1)x + O(x^{1/2})$, where γ is the Euler-Mascheroni constant.

the earth at the Paris exhibition by means of a pendulum (1851). An iron ball of about 30 kg was suspended from the ceiling of the Pantheon from a wire 60 m long, so as to be free to swing in any direction. To an observer in an inertial frame outside the earth, the plane of motion of a pendulum at either pole remains fixed in space while the earth rotates underneath. To an observer on earth the plane of motion rotates relative to earth, and the pendulum's motion is attributed to the action on it of a *Coriolis force*. (In any non-polar location on earth, however, the pendulum's plane of motion is fixed in *neither* frame.)

In 1852 Foucault constructed a refined gyroscope to demonstrate the rotation of the earth. Because the device exhibited the rotation of the earth before his eyes, Foucault named it the 'gyroscope', which means etymologically 'to see rotation' [from the Greek '*gyros*' = ring, '*skopien*' = to view].

During 1850–1865, he produced many important experiments, discoveries and inventions: he used a revolving mirror to measure the speed of light, and showed that light travels slower in water than in air and that its speed varies inversely with the index of refraction. He also made various improvements in the mirrors of reflecting telescopes and invented the *parabolic mirror telescope* which did away with spherical aberration. He discovered the existence of eddy currents, which are produced in a conductor moving in a magnetic field.

Foucault was the son of a publisher at Paris. After an education received chiefly at home he studied medicine, which he abandoned for physical science.

History of Gyroscopic Phenomena and Technologies, I

A. General theory

The earliest appreciation of gyroscopic phenomena appears to date to the time of **Newton** *(1642–1727). It arose from a study of the motion of our planet, which is itself a massive gyroscope. A description of its motion will, in fact, help in understanding some of the essential characteristics of the general case. The earth approximates closely to a free gyroscope, for its axis remains almost fixed in the direction of the North Star, Polaris, irrespective of its transit around the sun. The direction of the axis, however, has been changing*

slowly throughout the centuries, a phenomenon known as "precession of the Equinoxes". Its polar axis is sweeping out a cone, with an apex angle of $46^\circ 54' 16''$ *and a period of* $25,800$ *years.*

Though extremely slow, this motion is similar to the precession of a spinning top. It arises from the gravitational torque to which the earth is subjected by the sun and the moon, as a combined result of its lack of sphericity and the inclination of its axis to the ecliptic plane. A further periodic movement is also present, in which the earth's axis describes a much smaller cone whose diameter at the North Pole is approximately 800 *cm. This movement, known as Eulerian motion, has an observed period of* 428 *days and corresponds to the free oscillation or nutation of a gyroscope. The above phenomena are superposed on the orbital motion of the earth around the sun.*

From the above we may identify three gyroscopic attributes, namely directional stability, precession and nutation. The technological applications of the gyroscope are based upon these properties. For example, directional stability, which may be regarded as the reluctance of a body to change its orientation, provides the basis of modern inertial navigation.[285]

Also, it is found that rate of precession is proportional to applied torque, and as the latter may be produced by the acceleration of a platform upon which the gyro rides, linear acceleration may be measured by angular velocity, and consequently linear velocity by angular displacement. This integrating ability of the gyroscope is made use of in instruments carried by rockets for recording their position in space.

A further characteristic depends on the opposite nature of action and reaction. If forced to precess, a gyroscope exerts a reactive moment proportional to the product of the velocities of spin and precession. Moments of immense magnitude may thus be produced by the precession of high-speed rotors. This feature is utilized in gyroscopic vibration absorbers and in some ship stabilizers.

The mathematical foundation of gyroscopic behavior must undoubtedly be ascribed to ***Euler*** *(1707–1783). His initial work in this field concerned the general motion of a rigid body for which he derived a set of dynamical equations involving relations between applied moment, inertia, angular acceleration and angular velocity. These, known as* Euler's equations*, were stated with reference to axes fixed in the body. Later he established the independence of rotation and translation of a rigid body, and devised the so-called* Eulerian angles *to define its orientation with respect to a system of fixed axes.*

[285] Although it is being replaced with such technologies as *laser gyros* and MEMs (= Micro Electro-Mechanical devices), which detect rotations and accelerations by other, optical and mechanical, means.

From this background came his first direct contribution to gyrodynamics: the general motion of a rigid body, fixed at a point and otherwise free from external force. This problem includes that of the free gyroscope, and was to occupy the attention of mathematicians for many years. The general displacement of the body is, in fact, only expressible in terms of elliptic functions, mainly associated with the name of ***Jacobi*** *(1804–1851). Euler's later contributions included the inertial properties of bodies, which led to the concepts of principal axes and momental ellipsoid.*

There is an important concept which first appears to have been recognized by ***Clairaut*** *(1713–1765) in* 1742*, though credited much later to* ***Coriolis****. This concerns the force to which a particle is subjected when moving on a surface which is itself subjected to rotation. Although this had not been neglected in earlier work, Clairaut indicated its application to a moving frame of reference.*

From the death of Euler in 1783 *until the early part of the* 19^{th} *century, little was added to the theory of the gyroscope. A revival of interest, however, is evident in the work of* ***Poinsot***[286] *(1777–1859) who approached the subject by way of analogy. He demonstrated theoretically that if a free body, fixed at a point, were replaced by its momental ellipsoid, the path of motion of the ellipsoid when rolled on a fixed plane was identical to that of the body. In this representation, the distance of the plane from the fixed point was a function of the energy and momentum of the body. At a later date,* ***Sylvester*** *(1814–1897) showed that if a solid homogeneous ellipsoid were used, not only the path but also the* transit times *at each position would be identical to that of the actual body.*

Many contributions of ***Poisson*** *(1781–1840) are associated with gyrodynamics. In particular, he appears to have been the first to investigate the motion of the spinning top, a much more complex problem than that of the free gyroscope. Because of the torque due to the gravitational force, a top may perform a large variety of complicated motions; and during the latter half of the* 19^{th} *century, much thought was devoted to this subject. Poisson also made a comprehensive study, related to the work of Poinsot, which dealt with the rolling of bodies of various shapes on a plane.*

The following years provided new approaches to gyroscopic problems. ***Peter Guthrie Tait*** *(1831–1901) investigated the motion of the free gyroscope by vector methods, while* ***Edward John Routh*** *(1831–1907) studied the stability of gyroscopic motion and gave a geometrical construction for determining the rise and fall of a spinning top. The contributions of* ***Lord Kelvin*** *(1824–1907) were both practical and theoretical. He made a suggestion for*

[286] **Poinsot** introduced the concept of 'torque' (1804).

using a gyro-compass as early as 1883, and later developed analogies between gyroscopic motion and the motion of electrons in magnetic fields. The work of ***Felix Klein*** *(1849–1925) also deserve mention. He approached the motion of a top by using parameters which later became known as the Cayley-Klein parameters.*

By the turn of the 20^{th} *century gyroscopic theory was virtually complete, and since then emphasis has shifted to gyroscopic applications. This has stimulated much theoretical work involving the solution of specific problems, rather than the discovery of new phenomena. The gyroscope has become the province of the inventor rather than of the mathematician.*

B. Gyro-technology (1744–1930)

The early history of the gyroscope is obscure. Its modern history begins with the Englishman ***Serson***[287]*, who in 1744 constructed a spinning rotor for indicating the position of the horizon at sea, when the real horizon was obscured. It was supported at its centroid (to avoid precession) so as to be free from disturbance by the motion of the ship, and was the forerunner of the 'gyroscopic horizon', used in modern aircraft.*

The early part of the 19^{th} *century saw gyroscopes being used in the teaching of dynamics.*

In 1810 ***Bohnenberger*** *(Germany) constructed the earliest type of gyroscope now in use. In 1819 the English instrument maker* ***Edward Troughton*** *(1753–1835) produced an improved 'gyroscopic horizon' in which the center of gravity of the rotor could be adjusted accurately by means of screwed plugs. In 1832* ***Walter Rogers Johnson*** *(Philadelphia, U.S.A.) constructed an improved type and used it to illustrate the dynamic of rotating bodies. He called it a 'rotascope'. In 1852,* ***Foucault*** *constructed a gyroscope with which he successfully demonstrated the earth's rotation.*

In the last decade of the 19th century, the stage was set for the application of the gyroscope to real world problems. These were quick in coming;

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[287] Serson was sent to sea by the Admiralty to test his instrument, but he was lost in the wreck of the "victory" in 1744. His invention was reported, however, in *Phil. Trans. Royal Soc.* **47**, 1752.

three things drove the transformation of the gyroscope from a child's toy, or inventor's curiosity to that of a usable technology. These were:

- *The increasing use of steel in ships made the vessels unstable.*
- *Unreliability of the magnetic compass within a steel ship.*
- *Preparation of the great powers to conduct underwater warfare in steel hull ships.*

In 1883 ***Lord Kelvin*** *made suggestions for using a gyro-compass, which was indeed designed during* 1908–1911 *by* ***Hermann Anschütz-Kaempfe*** *(Germany) and* ***Elmer Ambrose Sperry*** *(1860–1930, U.S.A.), mainly for the use of polar expeditions. Anschutz and Sperry both built on the properties of the gyroscope: stability and precession. If force is exerted on it, it will react at right angles to the applied force. The characteristic of a gyro combined with other elements of precession, pendulousity and damping will allow the gyro to settle toward the true north.*

In 1908 Anschutz patented the first north seeking gyrocompass with the United Kingdom's Patent Office. The same year Elmer Sperry invented and introduced the first ballistic gyrocompass, which included vertical damping. Both of these first devices were of the single pendulum type. Earlier, ***Schlick*** *(Germany) made first attempts to stabilize a ship against rolls by means of a gyroscope. However, the solution to this problem was finally perfected by Sperry in* 1907*. In 1923,* ***Max Schuler*** *(Germany) introduced his finely-tuned gyro-pendulum*[288].

Since that date, the gyroscope has been used in a variety of ways to steer torpedoes, navigate ships, rockets and missiles, to stabilize the rolling of ships, to counter vibration and to operate innumerable control mechanisms. The small directional unit of the gyro-compass operates by the same principles as the massive rotor of the ship stabilizer, weighting up to 100 *tons.*

The conventional gyroscope, however, always consists of a symmetrical rotor spinning rapidly about its axis and free to rotate about one or more perpendicular axes. Freedom of movement about an axis is normally achieved by supporting the rotor in a gimbal, and complete freedom can be approached by using two gimbals. None other than ***Albert Einstein*** *spent much of his valuable time on the improvement of the gyrocompass during 1915–1925, as a*

[288] With its universal *Schuler-period* of $T = 2\pi\sqrt{\frac{R}{g}} = 84.4^m$, where R is the earth's mean radius.

consultant to the Kiel-based firm Hermann-Anschütz-Kaempfe. After WWI, Einstein and Anschutz collaborated intensively on the development of a fundamentally improved gyrocompass. In the 1930's virtually every navy in the world, except the British and the American, was equipped with gyrocompasses by the Anschütz firm. The construction of these gyrocompasses also involved a patent of Albert Einstein.

As of 1925, Einstein's share in the profits of the gyrocompass project was contractual, receiving 3% of the sales price of each instrument, and 3% of any revenue from licenses. The contract was not with the Kiel firm, but with the Dutch firm Giro, a distribution company founded by Anschütz primarily to evade the ban imposed by the treaty of Versailles on exports of military articles. This firm was liquidated in 1938 since the parent firm in Kiel no longer needed a Dutch branch to circumvent armaments controls. By then Einstein had no longer received any payments from the German Reich. He was thus at least spared any disquieting thoughts on the propriety of earning royalties from a device which guided German U-boats and Japanese aircraft carriers.

1852 CE **Francis Guthrie** (1831–1899, England). Mathematician. Formulated the four-color conjecture. This states that any map on a plane or a sphere can be colored with the use of only four colors, in such a way that no two adjacent domains are of the same color.

Chromatic Numbers (1852–1952)

"I know by the color. We're right over Illinois yet... Indiana ain't in sight... Illinois is green, Indiana is Pink. You show me any pink down here, if you can. No, sir; it's green... I've seen it on the map".

"Indiana pink*?... Well, if I was such a numskull as you, Huck Finn, I would jump over. Seen it on a map! Huck Finn, did you reckon the states was the same color out-of-doors as they are on the map?"*

"Tom Sawyer, what's a map for? Ain't it to learn facts? ...there ain't no two states the same color. You get around that*, if you can, Tom Sawyer".*

Mark Twain, '*Tom Sawyer Abroad*' (1835–1910)

There are many topological questions, some of them quite simple in form, to which intuition gives no satisfactory answer. A problem of this kind, known as the 4*-color problem, arose out of the practical needs of map-makers already in the* 16^{th} *century. These men were familiar with the notion that not more than* 4 *colors are necessary in order to color a map of a country (divided into districts) in such a way that no two contiguous districts shall be of the same color*[289].

The problem was mentioned by ***A.F. Möbius*** *in his lectures in* 1840*, but was first stated as a mathematical conjecture in* 1852 *by* ***Francis Guthrie****. Shortly after he had completed his studies at University College, London, he was coloring a map showing counties of England. As he did so, it occurred to him that, in order to color* any *map [subject to the requirement that no two regions sharing a length of a common boundary should be given the same color], the maximum number of colors required seemed likely to be* 4*. Being unable to prove this, he communicated the problem to* ***Augustus de Morgan*** *(1806–1871), one of the major mathematicians of his time, and through him the proposition then became generally known.*

[289] *Contiguous = districts having a common* line *as part of their boundaries. The map is drawn on a simply-connected surface, such as a plane or a sphere. The number of districts is finite and no district consists of two or more disconnected pieces. The map may or may not fill up the whole surface. Some maps can be colored with fewer than* 4 *colors, such as a chess-board, which requires only* 2*, or a hexagonal tessellation, which requires* 3*.*

Like Guthrie, de Morgan had no difficulty proving that at least *4 colors are necessary (i.e. that there are maps for which 3 colors are not sufficient). He also proved that it is* not *possible for 5 regions to be in a position such that* each *of them is adjacent to the other 4 [this may, at first glance, appear to imply that 4 colors are always sufficient, but it does* not *in fact imply this at all. Numerous false 'proofs' of the 4-color conjecture that appeared during* 1852–1976 *were based upon this invalid implication].*

Unable to solve the problem, de Morgan passed it on to his students and to other mathematicians, among them ***W. Hamilton****, giving credit to Guthrie for raising the question. However, the problem did not seem to attract much interest until* 1878*, when* ***Arthur Cayley****, unable to determine the truth or falsity of the conjecture, called attention to it by asking the members of the London mathematical society if they knew a proof of the conjecture. His question was published in the society's proceedings, and this was the first mention of the problem in print.*

Within a year after Cayley's challenge, ***Arthur Bray Kempe*** *(1849–1922), a London barrister and a member of the London Mathematical Society, published a paper that claimed to prove that the conjecture was true. However, in 1890* ***Percy John Heawood*** *(1861–1955, England) pointed out that Kempe's argument was in error. Heawood was, however, able to salvage enough to prove that* 5 *colors are always adequate.*

Heawood, in trying to attack the problem, investigated a generalization of the original conjecture: The maps studied by Guthrie and Kempe were maps in a plane or on a sphere. Heawood also considered maps on more complicated surfaces containing "handles" and "twists". He was able to derive upper bounds *for the numbers of colors required to color maps on these surfaces (the numbers themselves are known today as the* chromatic numbers*). His method, however, was* not *applicable to the plane*[290]*.*

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290 **Heawood** proved that for a surface of *Euler characteristic* n

$$(= V\ (vertices) - E\ (edges) +\ F\ (faces)),$$

such that $n \leq 1$, the number of colors that suffice to color all maps drawn on the surface is $\frac{1}{2}[7 + \sqrt{49 - 24n}]$, where $[x]$ denotes the largest integer contained in x.

For a sphere $n = 2$; for the torus and the *Klein bottle*, $n = 0$, and for a double-torus, $n = -2$. Though topologically equivalent surfaces *do* have the same n value, topologically different surfaces may or may not.

Thus for the torus, 7 colors are sufficient. For the *Klein bottle*, the formula gives the answer 7, while only 6 colors are needed.

Heawood continued to work on the 4*-color problem for the next* 60 *years (1890–1950). During that time numerous mathematicians and even a greater number of amateurs, investigated the* 4*-color problem, developing in the process a great many mathematical techniques that ultimately proved to have applications elsewhere in mathematics. In fact, much of what is now known as Graph Theory (the geometry of wiring diagrams and airline routes) grew out of the work done in attempting to prove it.*

In 1913*,* ***George D. Birkhoff*** *used the idea of* ***Kempe*** *to develop much of the basis for later progress. Using these results,* ***Philip Franklin*** *(1898–1965, U.S.A.) proved in* 1922 *that every map with* 25 *or fewer zones could be colored with* 4 *colors. In* 1975 *this figure reached* 96*. In* 1950*,* ***H. Heesch****, who had been working on the* 4*-color problem since* 1936*, indicated for the first time that the problem would be solved* only *with the aid of a computer, capable of handling vast amounts of data, and he indeed advocated, and attempted, a computer-aided assault on the problem.*

1853–1869 CE **Johann Wilhelm Hittorf** (1824–1914, Germany). Physicist. Pioneer in electrochemical research. Investigated the migration of ions during electrolysis (1853) and suggested that different ions in a solution impelled with an electric current travel at different rates. He then developed expressions to account for the measured transport numbers. Studied electrical phenomena in rarefied gases, the *Hittorf tube* being named for him. Determined a number of properties of *cathode-rays* (before Crookes) including the deflection of the rays by a magnet (1869).

Hittorf was born in Bonn, Rhenish Prussia. He was a professor of physics at Münster (1852–1890).

1853–1871 CE **William John Macquorn Rankine** (1820–1872, Scotland). Ingenious engineer and physicist. Was among the founders of the science of thermodynamics on the basis laid by **Sadi Carnot** and **J.P. Joule**. His work was extended by **Maxwell**.

Although the word *energy* occurs already in the writing of Aristotle, it was introduced into the language of science by Rankine in 1853. His *manual of the steam engine* (1859) coined most of the modern terms used in the field. Introduced the *Two-Phase Rankine Cycle* in the ideal steam engine and the *Rankine Temperature Scale*. He demonstrated (1865) that the functioning of the propeller is based on the principle of reaction and was first to recognize that the essential point in its action is the acceleration of the air mass passing

through the circular area swept by its blades. He also contributed to the theory of *shock waves* (1870). Rankine also wrote on *fatigue* in the metal of railway axles and on *soil mechanics.*

Rankine was born in Edinburgh and completed his education in its university. In 1855 he was appointed to the chair of civil engineering in Glasgow.

1853–1876 **Heinrich (Zvi Hirsch) Graetz** (1817–1891, Germany). The first Jewish historian of modern times. His eleven-volume *History of the Jews* is one of the great monuments of the 19th century historical writings.

Graetz was born in Posen[291]and received his doctorate from the University of Jena (1846). He momentary thought of entering the rabbinate, but he was unsuited to that career. For some years he supported himself as a tutor, but in 1856 the publication of the 3rd volume of his history made him famous. No Jewish book of the 19th century produced such a sensation as this. The work has been translated into many languages; it appeared in English in five volumes in 1891–1895. In 1854 he was appointed on the staff of the new Breslau Seminary and passed the remainder of his life in this office. In 1869 he was created professor by the government of the Breslau University. He kept this post until his death at Munich.

Graetz made use of a vast number of sources in many languages, but his vision of the Jew was rooted in Deutero-Isaiah. The Jews, he argued, had

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[291] *Poznan* (Posen); a city at the confluence of the Cybina and the Warthe rivers. One of the oldest towns in Poland (ca 800 CE), and the residence of some of the early Polish princes. It became the seat of Christian bishopric about the middle of the 10th century. The original settlement was on the right bank of the Warthe, but the new town established on the opposite bank by German settlers (1250), soon became the more important part of the double city. Posen became a great depot for the trade between Germany and Western Europe on the one hand and Poland and Russia on the other. The city attained the climax of its prosperity in the 16th century (p. 80,000). The intolerance shown to Protestants, the troubles of the Thirty Years' war, the plague and other causes, soon conspired to dwindle its population to 12,000 in the 18th century. Since its annexation by Prussia and the 2^{nd} partition of Poland (1793) its growth has been rapid. The German rule (1793–1806, 1815–) ended in 1918 when most of German inhabitants left the city. During WWII the Germans exiled its citizens and filled it with Germans from the Baltic states. It reverted to Poland after the war.

Jews lived in Poznan from the 12th century to the Holocaust [3000 (1519); 76,00 (1840); 26,599 (1910); 5000 (1920); 0 (1940)]. Its orthodox community constantly supplied the German Jewery with religious spiritual leaders as well as distinguished intellectuals that contributed to Judaism and science.

always been powerful and productive in religious and moral truths for the salvation of mankind. Judaism was, by divine providence, self-created. In that respect it was unlike any other great religion. Its 'sparks' had ignited Christianity. Its 'seeds' had brought forth the fruits of Islam. From its insights could be traced the origins both of scholastic philosophy and Protestantism.

Graetz was not interested in the social and economic motivations of human society, and laid the blame for the persecution of the Jews on the narrowness and bigotry which characterized Christianity during the Middle Ages. This, obviously, aroused resentment among many Christians and German nationalists.

Graetz's history was accepted by the Jews with mixed feelings: although the Jews of the world over hailed it with enthusiasm, it carried no real message to the great masses of East European Jews. He disparaged their study of the Talmud, as vain and useless scholasticism. Knowing little about Jewish mysticism, he had nothing but contempt for *Hasidism*, which was so widespread among them. To him it seemed pure superstition.

He failed to see the beautiful piety which prevailed in Eastern Europe despite hostility and poverty. Had he taken the trouble to learn more about the living Jews of Russia, Poland and Romania, Graetz would have found among them that very loyalty to Jewish life which he claimed so much in the Jews of the Middle Ages and which he was trying to revive among the Jews of Germany.

Nevertheless, Graetz had written a *justification of Jewish life* and, in a sense, gave a *pledge of Jewish continuity*. The *History of Jews* appeared at an opportune moment: Forces were already in motion in Germany which were opposed to that spirit of freedom and democracy which had brought emancipation to the Jews.

1854 CE A wire telegraph was established between London and Paris.

1854 CE **George Boole** (1815–1864, England). Irish logician and mathematician. One of the principal founders of symbolic logic.

Boole was born in Lincoln. His extraordinary mathematical talents did not manifest themselves in early life. During 1832–1849 he was a school teacher. In 1849 however, he was appointed a professor of mathematics in the Queen's College at Cork. He published some 50 papers and a few books and pamphlets. His most important work is "*An Investigation of the Laws of Thought, on which are founded the Mathematical Theories of Logic and Probabilities*" (1854) in which he put logic on a mathematical basis.

Boole was first to put laws of human reason in symbolic form. Today, Boolean algebra is considered as a mathematical system used to solve problem

in logic, probability and engineering. In the context of this algebra, logical statements are formulated symbolically so that they can be written and proved in a manner similar to that used in ordinary algebra.

Boolean algebra deals with relationships between *sets* (collections of entities). Such sets and operations on them (unary or binary) are represented by letters and symbols of operations [e.g. $A \cap B$ represents the set of those elements that are in both sets A and B].

In 1881 Boole's work was extended by **John Venn** (1834–1923), and later in 1901 by **Giuseppe Peano** (1858–1932).

The Algebra of Switching Circuits (1854–1901)

By switching circuit is meant a connected set of circuit elements which may be opened (thereby interrupting a portion of the circuit) or closed.

Let x represent the condition of an element by taking the value

$$x = \begin{cases} 1 & \text{closed} \\ 0 & \text{open} \end{cases}$$

Let the operation + denote elements in parallel *and "multiplication" represent elements in* series.

Since the switching circuit design is an arrangement of wires and switches where an open *switch prevents the flow of current while a* closed *switch permits the flow, the tables below exhaust all possible configurations of a subcircuit consisting of two distinct switches x and y, through each of which current may flow or not.*

(*A*)

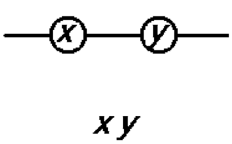
x	y	*parallel circuit*	x	y	$x+y$
on	*on*	*on*	*1*	*1*	*1*
on	*off*	*on*	*1*	*0*	*1*
off	*on*	*on*	*0*	*1*	*1*
off	*off*	*off*	*0*	*0*	*0*

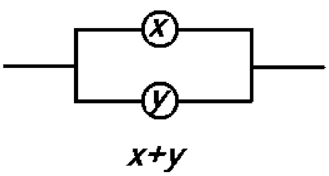
(*B*)

x	y	*series circuit*	x	y	xy
on	*on*	*on*	*1*	*1*	*1*
on	*off*	*off*	*1*	*0*	*0*
off	*on*	*off*	*0*	*1*	*0*
off	*off*	*off*	*0*	*0*	*0*

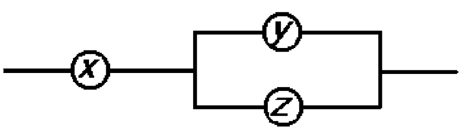
Note that $x + x = x$. Also $1 + x = 1$, because having one element of a parallel pair closed ensures the pair will 'act' closed. Also, to each element x there is an element $\bar{x}$ (with $x + \bar{x} = 1$ and $x \cdot \bar{x} = 0$) which is open when x is closed and closed when x is open.

The two binary operations introduced above can be represented by the diagrams:





Each switching circuit has an associated switching function *which describes whether it is on or off as a whole, as a function of the states of its individual switches. Thus for the circuit*



the switching function would be $f = x(y + z)$ which is unity if and only if $x = 1$ and $(y + z) = 1$ (i.e. y or z or both are closed) and which is zero if and

only if $x = 0$, or if y and z are zero, or if x, y and z are all zero. The overall state of the switching circuit can be specified by the vector (x, y, z).

If x, y and z are circuit conditions (switches), the following algebraic properties hold [$\bar{x}$ is the complement of x defined by $x + \bar{x} = 1$ and $x \cdot \bar{x} = 0$]:

$$
\begin{array}{rll}
\text{(i)} & x + 0 = x & \textit{additive identity} \\
\text{(ii)} & x \cdot 1 = x & \textit{multiplicative identity} \\
\\
\text{(iii)} & x + y = y + x & \multirow{2}{*}{\left.\right\} \textit{commutative laws}} \\
\text{(iv)} & xy = yx & \\
\\
\text{(v)} & (x + y) + z = x + (y + z) & \\
\text{(vi)} & (xy)z = x(yz) & \\
\\
\text{(vii)} & x(y + z) = xy + xz & \\
\text{(viii)} & x + yz = (x + y)(x + z) & \\
\\
\text{(ix)} & x + \bar{x} = 1 & \\
\text{(x)} & x\bar{x} = 0 & \\
\\
\text{(xi)} & x + 1 = x & \\
\text{(xii)} & x \cdot 0 = 0 & \\
\text{(xiii)} & x^2 = x & \\
\\
\text{(xiv)} & \overline{x + y} = \bar{x}\bar{y} & \\
\text{(xv)} & \overline{xy} = \bar{x} + \bar{y} &
\end{array}
$$

(i)	$x + 0 = x$	*additive identity*
(ii)	$x \cdot 1 = x$	*multiplicative identity*
(iii)	$x + y = y + x$	*commutative laws*
(iv)	$xy = yx$	
(v)	$(x + y) + z = x + (y + z)$	*associative laws*
(vi)	$(xy)z = x(yz)$	
(vii)	$x(y + z) = xy + xz$	*distributive laws*
(viii)	$x + yz = (x + y)(x + z)$	
(ix)	$x + \bar{x} = 1$	
(x)	$x\bar{x} = 0$	
(xi)	$x + 1 = x$	
(xii)	$x \cdot 0 = 0$	
(xiii)	$x^2 = x$	
(xiv)	$\overline{x + y} = \bar{x}\bar{y}$	*De Morgan's laws*
(xv)	$\overline{xy} = \bar{x} + \bar{y}$	

Each of these identities can be proved using the above tables. Equations (viii) through (xiii) have no analogues in ordinary algebra. In particular, (viii) is a 'weird' fact since ordinary algebra has instead

$$x^2 + xy + xz + yz = (x + y)(x + z)$$

The switching circuit algebra is an example of a Boolean algebra. It is easy to see that there exists a 1-1 correspondence between a disjunction (join, union) and a parallel circuit (+) on one hand, and between a conjunction (meet, intersect) and a series circuit (·) on the other. Thus, in general Boolean algebra we replace + by the symbol $\vee$ and (·) by the symbol $\wedge$, to remind us of the corresponding set-theoretic operations.

Nonassociative Algebraic systems

By developing algebras satisfying structural laws different from those obeyed by common algebra, **Hamilton**, **Grassmann**, *and* **Cayley** *opened the floodgates of modern abstract algebra. Indeed, by weakening or deleting various postulates of common algebra, or by replacing one or more of the postulates by others, which are consistent with the remaining postulates, an enormous variety of abstract systems can be studied.*

As examples of these systems we mention groupoids, quasi-groups, loops, semigroups, monoids, groups, rings, integral domains, lattices, division rings; Boolean rings, Boolean algebras, fields, vector spaces, Jordan algebras, and Lie algebras, the last two being examples of nonassociative algebras.

It is probably correct to say that mathematicians have, to date, studied well over 200 such algebraic structures. Most of this work belongs to the twentieth century and reflects the spirit of generalization and abstraction so prevalent in mathematics today. Abstract algebra has become the vocabulary of much of present-day mathematics, and has increasingly penetrated even engineering textbooks.

Octonions were discovered by **John T. Graves** *(1843) and independently by* **Arthur Cayley** *(1845). They are sometimes called Cayley numbers or Cayley algebra.*

Every octonion is a real linear combination of unit octonions

$$1, e_1, e_2, e_3, e_4, e_5, e_6, e_7$$

which thus form a basis of a vector space of octonions over the field of real numbers $\mathbb{R}$. The multiplication table for this 8-dimensional algebra, shown below, describes the result of multiplying the element in the i-th row by the element in the j-th column. Multiplication of two general non-basis octonions is defined by means of the distributive laws and the general properties of vector spaces:

$$\begin{aligned}(a + \sum_{j=1}^{7} b_j e_j) \cdot (a' + \sum_{j=1}^{7} b'_j e_j) &= aa' - \sum_{j=1}^{7} b_j b'_j \\ &+ \sum_{j=1}^{7} (a' b_j + a b'_j) e_j + \sum_{j,k=1}^{7} b_j b'_k e_j e_k\end{aligned}$$

	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

The interesting things that one learns from this table are:

- $e_1, e_2, \ldots, e_7$ *are square roots of* -1*:* $e_i^2 = -1, \quad i = 1, 2, \ldots, 7$
- e_i *and* e_j *anticommute when* $i \neq j$*:* $e_i e_j = -e_j e_i$
- *whenever* $e_i e_j = s e_k$ *with* $s = \pm 1$*,* $e_{i+1} e_{j+1} = s e_{k+1}$ *with index addition understood to be defined so that* 8 *equals* 1 *(index cycling property)*
- *whenever* $e_i e_j = s e_k$ $e_{2i} e_{2j} = s e_{2k}$ *where index doubling is again understood to obey the cycling property "*$8 = 1$*" (index doubling property)*
- $e_i(e_j e_k) \neq (e_i e_j)e_k$ *unless* $i = j$ *of* $j = k$ *(non-associativity)*

$$\text{e.g. } e_1(e_2e_3) = e_1e_5 = e_6 \qquad e_6(e_7e_3) = -e_6e_1 = -e_6$$
$$(e_1e_2)e_3 = e_4e_3 = -e_6 \qquad (e_6e_7)e_3 = e_2e_3 = e_5$$

The definitions of the norm, conjugate and inverse are similar to those of quaternions. The norm $N(A)$ *is defined as* $N(A) = A^*A = AA^*$*, where* A^* *is the conjugate octonion, i.e.* e_j *replaced with* $-e_j$*. For a nonzero octonion* A*, the multiplicative inverse* A^{-1} *is also an octonion (division algebra), since* $A^{-1} = \frac{1}{N(A)}A^*$*. For two octonions* A *and* B

- $(AB)^* = B^* A^*$
- $N(AB) = N(A)N(B)$ *(composition algebra* property*)*
- $A(AB) = A^2 B$ *(alternativity* property*)*
- $A^*(AB) = (A^* A)B = N(A)B$

Since octonions do not form an associative algebra, they cannot be represented directly by matrices. The following describe a method of representing octonions:

An octonion A is written as an ordered pair of two 4-dimensional quaternions q_1 and q_2 as $A = (q_1; q_2)$. Then the rule of multiplication of two octonions A and B is

$$AB = (q_1;\ q_2)(q_3;\ q_4) = (q_1 q_3 + \beta q_4^* q_2; \quad q_2 q_3^* + q_4 q_1)$$

where β is a field parameter.

Octonions are the largest of the 4 normed division algebras, but were somewhat neglected due to their nonassociativity. Their relevance to geometry was quite obscure until 1925, when ***Elie Cartan*** *used them to establish symmetry between vectors and spinors in 8-dimensional Euclidean space. Their potential relevance to physics was noticed in 1934 by* ***Jordan***, ***von Neumann*** *and* ***Wigner***, *but attempts to apply octonions quantum mechanics to nuclear and particle physics were met with little success.*

In 1985 it was realized that octonions explain some curious features of string theory. Beside their possible role in physics, octonions seem to tie together some mathematical structures. Today they stand at the crossroads of many interesting fields of mathematics: Clifford algebras and spinors, projective and Lorentzian geometry, Jordan algebras, and the exceptional Lie groups. They seem to have applications in quantum logic, special relativity and supersymmetry.

Sedenions

The word sedenion is derived from sexdecim, *meaning 16. A sedenion is a hypercomplex number constituted from 16 basal elements obeying the multiplication table:*

	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6	e_9	$-e_8$	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$	e_{10}	e_{11}	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$	e_{11}	$-e_{10}$	e_9	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
e_8	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	-1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_9	e_9	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	-1	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
e_{10}	e_{10}	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	e_3	-1	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
e_{11}	e_{11}	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	$-e_2$	e_1	-1	$-e_7$	e_6	$-e_5$	e_4
e_{12}	e_{12}	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6	e_7	-1	$-e_1$	$-e_2$	$-e_3$
e_{13}	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7	$-e_6$	e_1	-1	e_3	$-e_2$
e_{14}	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	-1	e_1
e_{15}	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	-1

The sedenions form a 16-dimensional algebra over the reals. *Like octonions their multiplication is neither commutative nor associative. But in contrast to octonions the sedenions do not even have the property of being* alternative. *They do, however, have the property of being* power-associative: $A^nA^m = A^{n+m}$.

The sedenions have multiplicative inverses, but they are not a division algebra because they have zero divisors.

Every sedenion is a real linear combination of the unit sedenions $1, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}$ and e_{15}, *which form a basis of the* vector space *of sedenions.*

A sedenion may be represented as an ordered pair of two octonions. The product of the sedenions S and T is then given as

$$ST = (A;\ B)(C;\ D) = (AC + \gamma D^*B; \quad BC^* + DA)$$

where B^* is the conjugate of B and γ is a field parameter.

Classical Group Theory[292] *(1770–1903)*

A group G is a set of objects, symbols or operations (elements), $a, b, c, \ldots$ for which there exists a certain binary composition law (usually called 'multiplication') that associates with each ordered pair of elements a unique element (called their 'product'), such that the following conditions are satisfied:

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[292] To dig deeper, see:

- Hall, G.G., *Applied Group Theory*, American Elsevier Publishing Company: New York, 1967, 128 pp.
- Smirnov, V.I., *Linear Algebra and Group Theory*, McGraw-Hill, 1961, 464 pp.
- Barnard, T. and H. Neill, *Mathematical Groups*, Teach Yourself Books, 1996, 218 pp.
- Lyubarskii, G.Ya., *The Applications of Group Theory in Physics*, Pergamon Press, 1960, 381 pp.
- Bishop, D.M., *Group Theory and Chemistry*, Dover, 1993, 300 pp.
- Kramer, E.E., *The Nature and Growth of Modern Mathematics*, Princeton University Press, 1982, 758 pp.

- *Closure:* If a and b are two elements of G, then the product ab is also an element of G.
- *Associativity:* For any 3 elements a, b, c multiplication is associative, i.e. $(ab)c = a(bc)$.
- *Existence of identity:* There exists an element I called the identity such that $aI = Ia = a$ for every element a of G.
- *Existence of inverse:* Corresponding to each element a of G there exists an element denoted by a^{-1} and called the inverse of a, such that $aa^{-1} = a^{-1}a = I$ for every a of G.

Of course $ab \neq ba$ in general. A group is said to be finite if the number of elements in it is finite. The number of distinct element is called the *order* of the group. Otherwise, it is said to be an *infinite group*. A group G is said to be *Abelian* or *commutative* if in addition to the group postulates we also have $ab = ba$ for any pair of elements a, b of G.

To illustrate the group properties, consider the group of rigid geometrical operations that transform the figure of a square into itself. There are 8 independent operations *in the plane*, generated by rotations and reflections, which achieve this goal. Labeling the corners of the squares with numbers 1 through 4, beginning at the upper right and going around clockwise, it is clear that the numbers will be permuted by each of the 8 operators as indicated in the following list:

Identity	(I)	${}^{4}_{3}\square^{1}_{2}$
90° Counterclockwise rotation	(R_1)	${}^{1}_{4}\square^{2}_{3}$
180° Counterclockwise rotation	(R_2)	${}^{2}_{1}\square^{3}_{4}$
270° Counterclockwise rotation	(R_3)	${}^{3}_{2}\square^{4}_{1}$
Reflection through horizontal axis	(H)	${}^{3}_{4}\square^{2}_{1}$
Reflection through vertical axis	(V)	${}^{1}_{2}\square^{4}_{3}$
Reflection through diagonal joining corners 1 and 3	(D_1)	${}^{2}_{3}\square^{1}_{4}$
Reflection through diagonal joining corners 2 and 4	(D_2)	${}^{4}_{1}\square^{3}_{2}$

These are the 8 basic geometrical operations for transforming a square into itself. If, for example, we effect R_2 after R_1, the combined effect of both operations is exactly the same as would have been obtained from the original square with R_3 alone, so we can write symbolically $R_2R_1 = R_3$, which illustrates closure. Proceeding in this manner, the entire 'multiplication table' for the square operators can be established, with each particular entry being the product of the same-row element in the left column (left factor) with the same-column element of the topmost row (right factor), as is shown in the following table:

	I	R_1	R_2	R_3	H	V	D_1	D_2
I	I	R_1	R_2	R_3	H	V	D_1	D_2
R_1	R_1	R_2	R_3	I	D_1	D_2	V	H
R_2	R_2	R_3	I	R_1	V	H	D_2	D_1
R_3	R_3	I	R_1	R_2	D_2	D_1	H	V
H	H	D_2	V	D_1	I	R_2	R_3	R_1
V	V	D_1	H	D_2	R_2	I	R_1	R_3
D_1	D_1	H	D_2	V	R_1	R_3	I	R_2
D_2	D_2	V	D_1	H	R_3	R_1	R_2	I

The table exhibits that the 8 operators do indeed form a group, but since $D_2R_3 = H$, $R_3D_2 = V$ etc., the group is not Abelian. Any subset of elements in a group (usually a smaller subset) which in themselves satisfy the group postulates is called a subgroup of the initial group. Thus $\{I, R_2\}$ is a subgroup of order 2, while $\{I, R_1, R_2, R_3\}$ is a subgroup of order 4.

A group such as that of the square has a certain formal or abstract structure which does not depend upon geometrical associations for its meaning. Thus, a set of eight matrices could be determined which would satisfy the group postulates just as the geometrical operators did. These are, for example, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $R_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; $R_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$; $R_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$;

$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; $V = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$; $D_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $D_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, *under*

ordinary matrix multiplication. The elements of this group of matrices and those of the group of operators can be put into one-to-one correspondence with each other which preserves the multiplication table, and the groups are then said to be isomorphic *to one another. The matrices are said to form a* representation *of the group.*

A second example is the group S_n of rearrangements (permutations) of n objects, known as the symmetric group. *A typical element of S_5 might be written as [24153], which means: put the second object first, the fourth object second, etc. Two elements are "multiplied" by performing first the rearrangement on the right, then the rearrangement on the left. For example, $[24153][51234]abcde = [24153]eabcd = acedb = [13542]abcde$, where letters represent the 5 objects; therefore $[24153][51234] = [13542]$. The order of S_n is obviously $n!$.*

An example of a group with a different kind of binary operation (composition law) is the group of 6 functions: $I(x) = x$; $A(x) = \frac{1}{1-x}$; $B(x) = 1 - \frac{1}{x}$; $C(x) = \frac{1}{x}$; $D(x) = 1 - x$; $E(x) = \frac{x}{x-1}$. The law of composition is the substitution of one function into the other as a function of a function, e.g., $AE = A\left(E(x)\right) = A\left(\frac{x}{x-1}\right) = \frac{1}{1-\frac{x}{x-1}} = 1 - x = D(x)$.

The general representation *of a group G is a group of square non-singular matrices, one matrix M for each group element g, with matrix multiplication as the composition law. If the matrices corresponding to different elements of G are themselves different, the two groups are isomorphic. If, however, one matrix M represents more than one group element of G, the group is said to be* homomorphic *to the matrix group. An isomorphism of a group onto itself is an* automorphism.

Consider a particular representation D. The matrix associated with the group element g, will be written as $D(g)$. We can form another representation D' in quite a trivial way by defining $D'(g) = S^{-1}D(g)S$, where S is any *non-singular matrix. Such representations, connected by the* equivalence transformation, *are said to be* equivalent, *and will, in practice, be considered to be the* same *representation. From two representations of the same element, $D^{(1)}(g)$ and $D^{(2)}(g)$, we can form a new representation*

$$D(g) = D^{(1)}(g) \oplus D^{(2)}(g) = \begin{pmatrix} D^{(1)}(g) & 0 \\ 0 & D^{(2)}(g) \end{pmatrix}.$$

If $D^{(1)}$ and $D^{(2)}$ have dimensionalities n_1 and n_2, the dimensionality of $D(g)$ is clearly $n_1 + n_2$. The representation D is said to be reducible *and splits into two smaller representations $D^{(1)}$ and $D^{(2)}$. A representation which is*

not of the above form and cannot be brought into this form by an equivalence transformation, is called an irreducible *representation. The irreducible representations of a group are the "building blocks" for the study of group representations, since an arbitrary representation can be decomposed into a linear combination of irreducible representations*[293]*.*

Since the same group can be represented by infinite number of equivalent matrix groups which are the same for most physical purposes, it would seem preferable to identify the particular group by something that is invariant *under similarity transformations. One such invariant is the* trace *of the matrix. We therefore define the* character *of the* i^{th} *representation* $\chi^{(i)}(g)$ *as the trace of* $D^{(i)}(g)$*. In the case of* S_3*, for example, we have the following irreducible representation with its associated characters:*

293 A *tensor* in n-dimensional space is reducible if there exists a less general class of the same rank which transforms onto itself by any $\mathbb{R}^n$ rotation (the *group* of these rotations is called $SO(n)$, a subgroup of the orthogonal group $O(n)$).

Vectors are always *irreducible* since for any two given vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ it is always possible to find a rotation $\mathfrak{R}$ such that $\mathfrak{R}\cdot\boldsymbol{a}=\boldsymbol{b}$. With *dyadics* (second rank tensors) the situation is different since in general, for any given two dyadics $\mathfrak{A}$ and $\mathfrak{B}$ one *cannot* always find a rotation $\mathfrak{R}$ such that $\mathfrak{R}\cdot\mathfrak{A}\cdot\mathfrak{R}^{-1}=\mathfrak{B}$, because not all matrices are similar. In fact, the group of all dyadics is reducible into three groups: For an arbitrary dyadic,

$$\mathfrak{A}=\frac{1}{3}\lambda\mathfrak{I}+\frac{1}{2}(\mathfrak{A}-\mathfrak{A}^T)+\left[\frac{1}{2}(\mathfrak{A}+\mathfrak{A}^T)-\frac{1}{3}\lambda\mathfrak{I}\right],$$

where $\lambda=\text{trace }\mathfrak{A}$, $\mathfrak{I}$ is the unit dyadic and $\mathfrak{A}^T$ is the transpose of $\mathfrak{A}$. Then, with $\mathfrak{B}=\mathfrak{R}\cdot\mathfrak{A}\cdot\mathfrak{R}^{-1}$, $\lambda=\text{trace }\mathfrak{B}=\text{trace }\mathfrak{A}$:

- $\mathfrak{R}\cdot(\lambda\mathfrak{I})\cdot\mathfrak{R}^{-1}=\lambda\mathfrak{I}$.
- $\mathfrak{R}\cdot(\mathfrak{A}-\mathfrak{A}^T)\cdot\mathfrak{R}^{-1}=\mathfrak{B}-\mathfrak{B}^T$.
- $\mathfrak{R}\cdot\left(\frac{1}{2}\mathfrak{A}+\frac{1}{2}\mathfrak{A}^T-\frac{1}{3}\lambda\mathfrak{I}\right)\cdot\mathfrak{R}^{-1}=\frac{1}{2}\mathfrak{B}+\frac{1}{2}\mathfrak{B}^T-\frac{1}{3}\lambda\mathfrak{I}$.

These three sub-representations (identity, skew-symmetric and symmetric traceless) are irreducible.

g	$D(g)$	rotations	$\chi(g)$
[123]	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$	$R_z(0)$	2
[231]	$\frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} = A$	$R_z(120\,°)$	-1
[312]	$\frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} = B$	$R_z(240\,°)$	-1
[321]	$\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} = E$	$R_{E}(180\,°)$	0
[213]	$\frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} = D$	$R_{D}(180\,°)$	0
[132]	$\frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = C$	$R_{C}(180\,°)$	0

Note that $D(g)$ *serves also to represent the 6-element crystallographic dihedral group with three axes of symmetry, when 3 identical atoms are fixed at the vertices of an equilateral triangle in the x–y plane:* $(0, 1)$, $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ *and* $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. *There are 3 in-plane rotations and 3 reflections which leave the triangle invariant. These 6 symmetry operations are: 3 rotations* R_z *about a z-axis perpendicular to the x–y plane at its origin with the respective angles* 0, $120\,°$, $240\,°$, *a reflection* R_{C} *about the y axis, a reflection* R_{D} *about an altitude drawn from* $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, *and finally a reflection* R_{E} *about an altitude drawn from* $\left(\frac{-\sqrt{3}}{2}, -\frac{1}{2}\right)$.

It is an indication of the versatility of the group concept, that in addition to the groups whose order is finite or denumerably infinite, there are groups the elements of which form a continuum. For example, the real numbers form an Abelian group L under addition. Also, the 2×2 *active rotation matrices* $R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ *form an Abelian group* $O(2)$ *under matrix multiplication, since* $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$ *[clearly L and* $O(2)$ *are not isomorphic, but rather homomorphic].*

The group O_2 *is an example of a group whose elements depend on one or more continuous parameters. The functional dependence is not only continuous but also differentiable to any order provided a suitable set of parameters*

is used in each region of the group manifold. Such groups are known as Lie groups. Obviously, we cannot write down a "multiplication table" in the ordinary sense for such a group. The neighborhood of the identity (i.e. of $\theta = 0$ *in the* $O(2)$ *example) is of special significance, since the identity element is* $I = R(0)$, *and the inverse of* $R(\theta)$ *is* $R(-\theta)$. *In the neighborhood of the unit element the group structure of* L *and* O_2 *is similar. Such a neighborhood is known as a group germ, and both* L *and* O_2 *have isomorphic group germs. Indeed, for small angles* $\delta\theta$, *the infinitesimal rotation operator takes the linear form*

$$R(\delta\theta) = \begin{pmatrix} 1 & \delta\theta \\ -\delta\theta & 1 \end{pmatrix}.$$

Some examples of Lie groups will indicate the range of applications of the ideas. Generally, a Lie group of r *parameters has elements which depend on* r *real parameters in an* r *dimensional space and are so related that if* $A(\boldsymbol{\gamma}) = A(\boldsymbol{\beta})A(\boldsymbol{\alpha})$, *where* $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ *each have* r *components, then the parameters satisfy* $\boldsymbol{\gamma} = f(\boldsymbol{\beta}, \boldsymbol{\alpha})$ *and each component of* f *is analytic in all components of* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$[294]. *Some of the more important Lie groups are:*

- $\boldsymbol{GL}(n)$: *The general linear group in* n *dimensions consist of all real non-singular* $n \times n$ *matrices. It has* n^2 *parameters ranging over a non-compact domain.*
- $\boldsymbol{SL}(n)$: *The special linear (or unimodular) group is a subgroup of* $\boldsymbol{GL}(n)$ *which consists of matrices whose determinant is unity. This condition reduces the number of parameters by one.*
- $\boldsymbol{O}(n)$: *The orthogonal group in* n *dimensions consists of all real* $n \times n$ *matrices* A *which satisfy* $AA^T = I$. *There are* $\frac{1}{2}n(n-1)$ *angle parameters that have a compact domain. Only proper rotations can be reached by starting from the identity and applying successive infinitesimal rotations. The subgroup of* $O(n)$ *consisting of proper rotations only (no reflection allowed) is* $SO(n)$, *which can also be defined as a set of matrices* A *of* $O(n)$ *such that* $\det A = 1$.
- $\boldsymbol{U}(n)$: *The unitary group in* n *dimensions consist of all* $n \times n$ *matrices* U *with complex elements satisfying* $UU^+ = I$, *where* $U^+ = (U^*)^T$. *The number of real parameters is* n^2. *[In 2 dimensions* $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *where* $aa^* + bb^* = 1, \quad cc^* + dd^* = 1, \quad ac^* + bd^* = 0$.*]*

[294] In general, more than one chart of coordinates $\boldsymbol{\alpha}$ may be needed to cover the *group manifold*, with analytic maps $\boldsymbol{\alpha}'(\boldsymbol{\alpha})$ relating the coordinates of any two intersecting chords inside their overlap region

- $\boldsymbol{SU}(n)$: *The special unitary group is a subgroup of* $\boldsymbol{U}(n)$ *whose matrices have a determinant of unity. This reduces the number of real parameters to* $n^2 - 1$.

- *The affine group in three dimensions consists of all transformations of the form* $\boldsymbol{r}' = A\boldsymbol{r} + \boldsymbol{a}$, *where A is a non-singular matrix and $\boldsymbol{a}$ is a vector. This group has 12 parameters. One important subgroup is the group of rigid rotations and translations, in which A is restricted to orthogonal matrices.*

- *The fractional linear (or projective) group, under substitution in two variables, consists of all transformations of the type* $\overline{x} = \frac{a_{11}x+a_{12}y+a_{13}}{a_{31}x+a_{32}y+a_{33}}$, $\overline{y} = \frac{a_{21}x+a_{22}y+a_{23}}{a_{31}x+a_{32}y+a_{33}}$ *and has 8 parameters.*

  In one variable, all expressions of the form $w = \frac{az+b}{cz+d}$ *form a group under substitution, where a, b, c, d are parameters which satisfy* $ad - bc = 1$. *If w and z are complex variables and (a, b, c, d) are complex parameters then this is a group of conformal representations. One particular finite subgroup of this continuous group has already been mentioned earlier. It consists of the 6 functions* $\left\{z, \frac{1}{1-z}, 1-\frac{1}{z}, \frac{1}{z}, 1-z, \frac{z}{z-1}\right\}$.

- *A one-parameter group of general transformations: Consider the transformation of coordinates (not axes) given by* $x' = f(x, y; a)$; $y' = \phi(x, y; a)$, *where a is an arbitrary parameter. The transformation carries the point* $M(x, y)$ *in the x–y plane to another point* $M'(x', y')$ *in the same plane. To each value of the parameter a corresponds a definite transformation. Varying this parameter, we obtain an infinite number of different transformations.*

Suppose that we carry out in succession two different transformations of this set, corresponding to two values of a and b of this parameter. The first transformation will carry the pair (x, y) *into* (x', y') *according to the above equations. The second transformation will carry* (x', y') *into a third pair* (x'', y'') *such that* $x'' = f(x', y'; b)$; $y'' = \phi(x', y'; b)$. *Substituting* (x', y') *from the first transformation, we find* $x'' = F(x, y; a, b)$, $y'' = \Phi(x, y; a, b)$, *which defines a point-transformation depending on two parameters a and b.*

We shall say that the set of transformations f and ϕ form a continuous one-parameter group if the new transformation belongs to this set. It is necessary and sufficient for this that $x'' = f(x, y; c)$; $y'' = \phi(x, y; c)$, *where* $c = \psi(a, b)$ *for some function* ψ.

Examples are: $\{x' = x + a;\quad y' = y + 2a\}$; $\{x' = x\cos\alpha - y\sin\alpha;\ y' = x\sin\alpha + y\cos\alpha\}$; $\{x' = ax;\quad y' = a^2 y\}$. *But* $x' = x + a$; $y' = y + a^2$ *do not form a group.*

Such a group contains the identity transformation

$$f(x, y; a_0) = x, \qquad \phi(x, y; a_0) = y$$

for some value $a = a_0$ *of the parameter. If* ϵ *is small, the transformation* $x_1 = f(x, y; a_0 + \epsilon)$; $y_1 = \phi(x, y; a_0 + \epsilon)$ *will be such that* x_1 *differs only infinitesimally from* x, *and* y_1 *from* y. *This transformation therefore differs only infinitesimally from the identity transformation, and is said to be an infinitesimal transformation.*

The idea of a group is one which pervades the whole of mathematics, both pure and applied. In the particular form of the study of symmetry, *group theory can claim to have its origin in prehistoric times. Nowadays, group theory is developed in an abstract way so that it can be applied in many different circumstances — but many of these still concern symmetry.*

The notion of group *(though not the term) originated (1770) in the works of* **Lagrange** *and* **Vandermonde** *on* permutation groups. *Lagrange moved toward the definition of the group concept in his attempts to solve the general* quintic equation. *He proved that the number of elements in a group is divisible by the number of elements in any of its subgroups (1770, Lagrange's Theorem).* ***A.L. Cauchy*** *(1815) investigated permutation groups and discovered several basic theorems.* ***Galois*** *(1829 to 1832) laid the foundations to the theory of groups in his group-theoretical approach to the problem of solvability of algebraic equations by radicals.* ***Auguste Bravais***[295] *(1811–1863, France) studied the symmetry of crystalline lattices by means of rotations and translations of their patterns into themselves. He thus advanced both crystallography and group theory (1848 to 1851).*

A. Cayley *(1854) presented the first technical definition of a group, listing postulates and representing a finite group by its multiplication table. He later (1878) asserted the equivalence of isomorphic groups*[296]. ***Ludvig Sylow*** *(1832–1918, Norway), discovered (1872) the fundamental theorem of permutation groups*[297].

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[295] French physicist. Served in the navy (1831–1857), and as a professor at the École Polytechnique (1845 to 1856). Demonstrated (1850) the 14 possible lattice configurations, known as *Bravais lattices*.

[296] *Cayley theorem*: Every group of finite order is isomorphic to a subgroup of the permutation group $S_n = \begin{pmatrix} 1\ ,2\ ,3\ ,\ldots,n \\ s_1, s_2, s_3, \cdots, s_n \end{pmatrix}$, also known as the symmetry group for some n.

[297] If p is a prime number and α is a positive integer such that p^α divides the order of the group but $p^{\alpha+1}$ does not, then the group has a subgroup of order p^α.

The progress that was made by means of theory of groups in the solution of algebraic equations of higher degree, induced mathematicians of the mid 19^{th} century to attempt to use the theory of groups in the solution of equations of other forms, in the first instance the solution of differential equations, which play such an important role in the applications of mathematics. This attempt was rewarded with success.

Although the place occupied by groups in the theory of differential equations is entirely different from their role in the theory of algebraic equations, the applications of the theory of groups to the solution of differential equations led to substantial extension of the very concept of a group and to the creation of a new theory of the so-called continuous group (Lie groups)*, which have proved to be extremely important for the development of many branches of mathematics and physics.*

M.S. Lie[298] *established (1874 to 1891) the theory of* continuous groups *and applied it to the classification and integration of ODE.* **Georg Frobenius** *(1896 to 1903) expanded the study of group representations to all finite abstract groups. He introduced and developed the concepts of* reducible *and* completely reducible *representations.* ***Eliakim Moore*** *(1862–1932, U.S.A.) developed (1893) the theory of the group of* automorphism *of any finite group.*

1854–1863 CE **Francesco Brioschi** (1824–1897, Italy). Mathematician, engineer and architect. Contributed to the theory of *determinants* (1854) and the application of *elliptic modular functions* to the solution of algebraic equations of the 5^{th} and 6^{th} degree.

Brioschi was born in Milan, and graduated from the University of Pavia (1845). He taught mechanics, architecture and astronomy at Pavia and the Istituto Tecnico Superiore in Milan (1863–1897).

1854–1883 CE **Gustav Robert Kirchhoff** (1824–1887, Germany). Distinguished physicist. Made important contributions to the theory of circuits, using *topology*, and to elasticity.

[298] **Lie** and **F. Klein** were students together in Berlin in 1869–1870 when they conceived the notion of studying mathematical systems from the perspective of transformation groups which left these systems invariant. Thus, Klein in his famous *Erlangen program*, pursued the role of *finite* groups in the studies of regular bodies and the theory of algebraic equations, while Lie developed his notion of continuous transformation groups and their role in the theory of ODE.

Born and educated at Königsberg, Prussia. A student of **Gauss**. Served as a professor of physics at Breslau (1850), Heidelberg (1854) and Berlin (1875–1887). Made numerous important contributions to mathematical physics. In 1857 he showed that the mechanical forces manifested by static and current electricity were related by a constant which has the dimensions of velocity. By comparing the attractive force of two static charges with the magnetic force produced when they are discharged, he demonstrated that the constant has the same magnitude as the velocity of light. Established the fundamental laws which govern the distribution of currents in a network of conductors (*Kirchhoff circuit laws*).

During 1859–1861, Kirchhoff and his collaborator **Robert Wilhelm Bunsen** (1811–1899) invented the Kirchhoff-Bunsen spectroscope for chemical analysis of metals placed in a flame, whereby the bright lines in the spectra of elements could be accurately recorded. They conjectured that each element produces a characteristic spectrum. By discovering the metal *cesium* they demonstrated how new elements could be discovered via spectroscopic analysis.

Furthermore, Kirchhoff found that for any emitting body in thermal equilibrium, the coefficient of emission and the coefficient of absorption are in a ratio that is a function only of wavelength and temperature. He introduced the concepts of *blackbody* and *emissivity*. He mapped the solar spectrum, and showed that elements such as sodium can be detected in the atmosphere of the sun by means of the dark (absorption) lines they cause in the spectrum[299].

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[299] The heated vapors produced an *emission spectrum*: bright lines against a dark background. The nature of these bright lines depended on the elements present in the vapor. Each element produced *its own pattern of bright lines*, and the same line in precisely the same position was never produced by two different elements. The emission spectrum served as a sort of fingerprint of the elements present in the glowing vapor. In the course of their studies, Kirchhoff and Bunsen detected lines that were not produced by any known element. They suspected the presence of new and hitherto undiscovered elements and were able to verify the fact by chemical analysis. The new elements were named *cesium* and *rubidium* from Latin words meaning *sky blue* and *red* respectively, signifying the colors of the lines that led to the discovery (the first elements to be discovered spectroscopically).

Kirchhoff and Bunsen then worked with light from a glowing solid (which produced white light that consisted of a continuous spectrum) and passed that light through a cool vapor. They found that the vapor absorbed certain wavelengths of light, and that the spectrum that was formed after the light had passed through the vapor was no longer completely continuous, but was crossed by dark lines which marked the position of the absorbed wavelengths. This was

In 1883 he generalized the Helmholtz harmonic time solution of the scalar wave equation (1860) to the case of transient waves.

1855 CE **Heinrich Geissler** (1814–1879, Germany). Physicist. Developed a mercury pump which he uses to produce the first good vacuum tubes. Such tubes were used in 1869 to produce "cathode rays", leading eventually to the discovery of the electron.

1855 CE **Henry Bessemer** (1813–1898, England). Inventor and manufacturer. Developed the *Bessemer process* of converting pig iron to steel. In this process a blast of air burns most impurities out of the molten pig iron. In 1830 he established his own steelworks at Sheffield and financed the experiments that promoted his invention. It was soon adopted throughout the world.

1855 CE **William Parsons** (1800–1867, England and Ireland). Astronomer. First to record observations of the spiral structure of galaxies. The nature of these spiral nebulae remained a source of speculations until 1924.

Using a large telescope of his own design (built 1827–1845) he was able to distinguish spiral structure in what we know today as the *whirlpool galaxy* (M51, in the constellation of Canes Venatici, 15 million light years away from earth).

William Parsons, the 3^{rd} earl of Rosse, was born in York, England. He was rich, liked machines, and was fascinated with astronomy. Accordingly he set about the business of building gigantic telescopes. In February 1845 he completed a 183 cm reflecting telescope at his estate in Birr Castle, Ireland. The contraption was mounted at one end of a 18.3 m tube that was controlled by cables, straps, pulleys, and cranes. For a brief time it was the largest telescope in the world.

In the course of 20 years, Lord Rosse examined many of the nebulae that had been discovered and catalogued by **William Herschel**, and observed that some of these have a distinct spiral structure. Because he did not have any photographic equipment, he had to make *drawings* of what he saw. His drawing of the M51 captured the salient features of modern photographs of the galaxy. Views such as this inspired Lord Rosse to echo **Kant**'s (1755) theory that these objects might be "*island universes*" — vast collections of stars far beyond the confines of the Milky Way.

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an *absorption spectrum*, and it seemed clear that the solar spectrum was an example of this.

The poor weather over Birr Casle usually limited the capabilities of the big telescope. Nevertheless, the whirlpools of light stood clearly against the black background of space.

1855–1859 CE **John Henry Pratt** (1809–1871, England and India). Geophysicist and clergyman. Introduced (with **Airy**) the concept of *isostasy compensation* and calculated the average depth of density compensation to be 100 km. Gave 43 km as the difference between equatorial and polar radii of the earth.

Pratt postulated (1855) density difference in the crust of the earth: lower density under mountains, high density in the lowlands — to explain the too nearly constant values obtained for gravity of a given latitude. In the same year, Airy offered a different explanation (though based on the same principle) of the gravity data. Both proposals have their merits but are oversimplification of the actual situation.

Pratt was archdeacon of Calcutta (1850–1871).

The Hindu Puzzle

Before the middle of the 18th century, geologists and geodesists regarded mountains as composed of matter of much the same density as the rest of the crust, and it was not recognized that their weight would be expected to produce any deformation of the matter below them, nor that the density of the matter below a mountain range might differ systematically from that of the matter at an equal depth below a plain or even an ocean. Now, if a mountain is considered merely as an extra mass superposed on a previously uniform crust, and its deforming effect in the interior is ignored, it is possible to compute all components of its contribution to gravity on bodies in its neighborhood. The attraction can also be found experimentally and the result compared with that calculated.

The experiment was carried out on several mountains during the 18th century with unexpected results. Indeed, the measurements of ***P. Bouguer*** *(1749) of the deflection of gravity due to the mountain Chimborazo (Andes), of* ***Maskelyne*** *(1774) at Schiehallion (England) and* ***Petit*** *(1849) in the Pyrenees, showed that the attraction of mountains was generally nearer to zero than to the values calculated on the supposition that the underlying matter was of normal density.*

The modern development began with the discussions of ***Pratt*** *(1855) and* ***Airy*** *(1855) on the deflections of gravity observed by* Everest *during the great land survey in India (1830–1843): the distance between Kaliana (some 100 km south of the Himalayas range, and Kalianpur (600 km further south), was determined in two precise ways – by measurement over the surface and by reference to astronomical observations – and the results disagreed by some 150 meters over 600 kilometers. This may seem to be a small amount, but it was an intolerable surveying error even by 19th century standards.*

The astronomical method of measuring distances uses the angles of stars w.r.t. the vertical, which are defined by a plumb line (a weight suspended on a string). To account for the difference, it was proposed that the plumb line was tilted toward the Himalayas because of the gravitational attraction of the mountains on the plumb bob, causing an error in the distance measurement. When, however, the calculation was actually made, it was found that the mountains should have introduced an even larger error – one of about 450 meters – thus compounding the puzzle!

Following an earlier suggestion by ***Cavendish*** *(1772),* ***Pratt*** *and* ***Airy*** *(1855), independently, came forward with an explanation for the discrepancy that contained the basis of the* principle of isostasy *(the actual word isostasy*

was coined in 1889 by ***C.E. Dutton****). Accordingly, the enormously heavy mountains are not supported by a strong rigid crust below, but that they float in a "sea" of rock. Thus, the excess mass of the mountain above sea-level is* compensated *by a deficiency of mass in an underlying root (since the lighter mountain-base material locally displaces the denser "rock-sea" material). This root provides the buoyant support, in the manner of all floating bodies such as a ship with a deep hull or an iceberg. The plumb bob "feels"* both *the excess mass on top and the deficiency of mass below; hence the reduced deflection.*

The principle of isostasy *is thus the Archimedes principle of buoyancy applied to the flotation of continents and mountains, which holds that the relativity light continents float on a more dense mantle. The supportive root must develop as part of a process that provides buoyancy and keeps the load from sinking*[300]*. A simplified quantitative treatment of the above idea develops as follows:*

If the mass of the earth is M, its mean density $\bar{\rho}$, and its radius a, the acceleration of gravity at the surface in the absence if any disturbance is

$$g_0 = G\frac{M}{a^2} = \frac{4}{3}\pi G\bar{\rho}a.$$

At height h above the surface, in the free air, the intensity is

$$g_0\frac{a^2}{(a+h)^2} \simeq g_0(1 - \frac{2h}{a}).$$

If instead of the space between sea-level and height h being empty, it is filled by matter of density ρ_m and a shape of width $\sim h$, the theory of Newtonian attraction predicts an additional contribution of $(2\pi G\rho_m h)$ to the intensity of gravity above it. In order to remove *the gravitational attraction of* local topography *the* ***P. Bouguer*** *gravity anomaly* correction, *$\Delta g = -2\pi G\rho_m h$, must be applied to the data.The overall excess of gravity at height h over its value at sea-level should be*

$$-[\frac{2g_0h}{a}(1 - \frac{3}{4}\frac{\rho_m}{\bar{\rho}})].$$

Although the Bouguer gravity formula is effective in removing the gravitational influence of local topography, it is not effective in removing the influence

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[300] One variant should be mentioned: if for some reason (e.g. regional *heating*) a part of the upper mantle becomes less dense than the adjacent mantle, it will also exert a buoyant force that can support elevated topography above it without the need for a crustal root. Here the lower density mantle serves as a root.

of regional topography: a mountain or valley with a small horizontal scale, say 10 km, can be supported by the elastic lithosphere without deflection and consequently does not influence the density distribution at depth. However, the load due to a mountain range with a larger horizontal scale, say 100 km, deflects the lithosphere downwards as well as the so-called Moho discontinuity in which it is embedded. Because crustal rocks are lighter than mantle rocks, this results in a low density root for the mountain range with a large horizontal scale. The mass associated with the topography of the mountains is compensated *at depth by a low-density root. According to* ***Pratt****, the density of the root varies horizontally as a function of the elevation h according to $\rho_p = \rho_0 \frac{w}{w+h}$, where ρ_0 is the surface density corresponding to zero elevation and w is referred to as the depth of compensation. According to* ***Airy*** *(1855), the density of the crust ρ_c and the mantle ρ_M are assumed to be constant. A crust feature with an elevation h has a crustal root of thickness*

$$b = \frac{\rho_c}{\rho_M - \rho_c} h,$$

derived from the principle of hydrostatic equilibrium.

Clearly, compensation in the lithosphere may be a complex combinations of both the Pratt and the Airy models.

The resolution of the 'Indian Puzzle' not only led to the notion of isostasy but also introduces gravity surveying *as a method for detecting mass variations in the interior by their corresponding gravity variations.*

1855–1858 CE **Rudolf Ludwig Carl Virchow** (1821–1902, Germany). Pathologist and political leader. Founded *cellular pathology.* Extended and applied cell theory to problems of pathology[301] and disease and set forth the principle that the outward symptoms of disease are merely the reflections of impairment at the level of cellular organization. He also advanced the notion that all cells arise from pre-existing cells: "*Omnis cellula e cellula*". His book

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301 *Pathology* — the study of disease. Pathology took definite form with **Giovanni Morgagni**'s (1682–1771, Italy) *Seats and Causes of Disease* (1761), correlating disease symptoms with the underlying pathology of the organs.

Xavier Bichat (1771–1802, France) introduced the concept of *tissue* to underlie pathological anatomy. **Karl von Freiherr Rokitansky** (1804–1878, Austria) systematized modern post-mortem protocol and described many specific conditions such as gastric ulcer and acute yellow atrophy of the liver.

Cellular Pathology (1858) established the cell as the fundamental pathological unit and permitted such processes, as *inflammation*, *tumor growth* (cancer) and degeneration, to be understood in cellular terms.

Virchow was born at Schivelbein, in Pomerania. He took a M.D. degree in Berlin (1843). Professor of pathological anatomy, Würzburg (1849) and Berlin (1856). Carried on research on blood, phlebitis, tuberculosis, rickets, tumors, trichinosis, etc. Made sanitary reforms in Berlin; established farms utilizing sewage for fertilizing the land. Founder and leader of the German Liberal party. Member of the German Reichstag (1880–1893); opposed policies of Bismarck.

1855–1868 CE **Jules-Antoine Lissajous** (1822–1880, France). Physicist. Studied the vibrations of bodies under combined excitations of different frequencies phases and amplitudes[302]. Found a simple way visualizing and studying these vibrations by reflecting a light beam from a mirror attached to a vibrating object onto a screen (1855–1857); e.g. by successively reflecting light from mirrors on two tuning forks vibrating at right angles. The resulting *Lissajous-figures* could be seen only because of persistence of vision in the human eye (no oscilloscopes available at that time!).

Lissajous entered the Ecole Normale Superieure (1841) and was awarded a doctorate thesis on vibrating bars using Chladni's sand pattern method to determine nodal positions (1850).

1855–1868 CE **Nathanael Pringsheim** (1823–1894, Germany). Botanist. One of the first to demonstrate sexual reproduction in algae (1855). Observed sperm penetration of the egg of *Oedogonium.*

1855–1876 CE **Alexander Bain** (1818–1903, Scotland). Philosopher and psychologist. Referred to by many as the first real psychologist. Developed psychology as a discipline apart from philosophy and physiology and

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[302] These had been observed earlier (1815) by **Nathaniel Bowdich** (1773–1838, USA), mathematician and astronomer. Consider a particle forced to vibrate harmonically, with two simultaneous motions at right angles

$$x = x_m \cos(\omega_x t + \phi_x); \quad y = y_m \cos(\omega_y t + \phi_y).$$

The path of the particle in the $x - y$ plane is a *Lissajous curve.* If $\frac{\omega_x}{\omega_y}$ is a rational number, the angular frequencies are commensurable and the curve is *closed* i.e. the motion repeats itself at regular intervals. Even for $\omega_x = \omega_y$, $x_m = y_m$ the curves will depend on the phase-difference $\phi_x - \phi_y$. If $\frac{\omega_x}{\omega_y}$ is irrational, then the curve is *open.* Lissajous observed *beats* when his tuning forks had slightly different frequencies; it showed as a rotating ellipse for the case $\omega_x = \omega_y$.

attempted to relate known physiological facts to psychological facts. Extended the associationist approach to all areas of psychological functioning, including habit and learning. Coined the term "trial and error"; wrote the first textbook on psychology in English (1855, 1859); and founded the first psychological journal, *Mind*, in 1876.

Bain was born in Aberdeen. Studied mathematics, physics and philosophy at Marischol College and later came under the influence of **John Stuart Mill** (1842). Appointed professor of mathematics and natural philosophy in the University of Glasgow but in 1846 resigned his position and devoted himself to literary work. Lived in London since 1848 and acquired wide influence as a logician and grammarian. Guided the awakened psychological interest in of British thinkers of the second half of the 19th century.

1855–1878 CE **David Edward Hughes** (1831–1900, England and USA). Inventor. Invented (1855) a keyboard telegraph with rotating type-wheel printer that grew into modern *telex* industry. Invented the *carbon microscope*[303] (1878).

Efforts to improve the telephone transmitter invented by Alexander Graham Bell (1876) led to the development of the microphone. Other microphone inventors included: **Philip Reis** (1861), **Emile Berliner** (1877) and **Thomas Edison** (1877).

Hughes was born and died in London. He emigrated with his parents to the United States (1838). In 1850 he became a professor of music at the College of Bardstown, Kentucky. He abandoned his academic career (1854) and moved to Louisville to manufacture his type-printing telegraph machines. In the succeeding ten years it came into extensive use all over Europe. It used a keyboard in which each key caused the corresponding letter to be printed at a distant receiver. It worked a bit like a 'golfball' typewriter and was produced before the typewriter was invented. The modern teleprinter, telex system and the computer keyboard are all direct descendants of his invention. His invention of the loose-contact carbon microphone became vital to *telephony* and later to *broadcasting* and *sound recording*. Hughes refused to patent his inventions and revealed it to the general public.

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[303] **Charles Wheatstone** was first to use the word *microscope* (1827). **Hughes** (1878) revived the term in connection with his discovery that a loose contact in a circuit containing a battery and a telephone receiver would give rise to sounds in the receiver corresponding to vibrations impinged upon the diaphragm of the mouthpiece or transmitter.

1856 CE **Norman Robert Pogson** (d. 1891, England). Amateur astronomer. Proposed a quantitative scale of stellar magnitude that is now generally adopted.

1856 CE **Philipp Ludwig von Seidel** (1821–1896, Germany). Mathematician. Presented the earliest systematic treatment of third-order geometrical aberrations[304] which was extremely important in the design and construction of lens system in cameras and other optical instruments. [In earlier optical system such as telescopes, only those points and rays were considered which lie in the immediate neighborhood of the axis. The resulting theory is known as *Gaussian optics*].

Seidel entered the University of Berlin in 1840 and studied under **Dirichlet**. He moved to Königsberg where he studied under **Bessel**, **Jacobi** and **Franz Neumann**. He obtained his doctorate from Munich University (1846) and he went on to become a professor there.

1856 CE **Adolf Eugen Fick** (1829–1901, Germany). Physiologist. Developed fundamental laws of diffusion in living organisms. Professor at Zürich (1852–1868) and Würzburg (1868–1899).

Fick's law consists of the observation that for small concentration gradients, the diffusive flux $\boldsymbol{J}$ is proportional to the gradient of the concentration c [$\boldsymbol{J} = -D\,\mathrm{grad}\,c$, where D is the diffusion coefficient and the minus sign shows that the flow is in the direction from higher to lower concentration]. This law treats diffusion from the *macroscopic* point of view.

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[304] First order geometrical ray theory (Gaussian optics) is based on the assumption that the optical system is restricted to operate in an extremely narrow region about the optical axis — this is known as the *paraxial approximation*. Mathematically, one takes here $\sin\varphi \approx \varphi$. Obviously, if rays from the periphery of a lens are to be included in the formation of the image, the statement that $\sin\varphi \approx \varphi$ is unsatisfactory. Seidel retained the first *two* terms in the expansion

$$\sin\varphi = \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \frac{\varphi^7}{7!} + \dots$$

which resulted in a *third* order theory. Departures from first-order theory are then embodied in the five *primary aberrations*: *spherical aberration*, *coma*, *astigmatism*, *field curvature* and *distortion* ('Seidel aberrations'). The difference between the results of exact ray tracing and the computed primary aberrations are the sum of all contributing *higher aberrations*. More refined treatment of aberrations is based on *physical optics* via *diffraction theory*. **K. Schwarzschild** (1905–1906) extended Seidel's analysis to include 5th order terms.

Fick was the first to put diffusion on a quantitative basis by adopting the mathematical equation of heat conduction derived some years earlier by **Fourier** (1822). Moreover, he recognized that the transfer of heat by conduction is due to *random molecular motion*. He was one of the first to actually experiment with *contact lenses* on animals and then, finally, fit contact lenses to human eyes.

1856 CE Discovery of the first Neanderthal remains in Germany.

1856–1901 CE **Pierre Eugène Marcellin Berthelot** (1827–1907, France). Chemist and politician. One of the most distinguished chemists of the 19^{th} century. His interests were extraordinarily wide and his work was highly original and of a fundamental character. First to synthesize organic compounds from inorganic ones[305].

By the synthetic production of numerous hydrocarbons, natural fats and sugars, he showed that organic compounds can be formed by ordinary methods of chemical manipulations, and obey the same laws as inorganic substances. He held that chemical phenomena are not governed by special laws peculiar to themselves, but rather are explicable in terms of the general laws of physics, and are in operation throughout the universe (1860). Discovered the *partition law*[306] for solubility of a substance in a mixture of liquids (1872).

Berthelot was born in Paris, the son of a doctor. After distinguishing himself at school in history and philosophy, he turned to the study of science. In 1851 he became a member of the staff of the College de France, and in 1865 became a professor of organic chemistry there. He was appointed inspector general of higher education in 1876, minister of public instruction (1886–1887), and held the portfolio of foreign affairs in the cabinet of 1895–1896.

1856 CE **Christoph Hendrik Diederik Buys-Ballot** (1817–1850, Holland). Meteorologist. Professor at Utrecht University (1847). Formulated the law determining the swirl direction of large storms and hurricanes[307] [*counterclockwise* in the Northern Hemisphere as viewed from

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[305] When he began his active career it was generally believed that on the whole, organic chemistry must remain an analytical science and could not become a synthetic one, because the formation of organic compounds required the intervention of *vital activity* of some form.

[306] The weights of dissolved substance per unit volume of each liquid are in constant ratio.

[307] Big storms in the atmosphere are usually centered on low-pressure areas. Independently, **William Ferrel**, in the United States (1859), gave a mathematical formulation of atmospheric motions on a rotating earth and applied his theory to the general circulation of both the atmosphere and the oceans.

above the North Pole (direction of earth's spin) and *clockwise* in the Southern Hemisphere – in accord with the *Coriolis effect* (1835)].

1857–1866 CE A *submarine telegraphic cable* was laid across the Atlantic Ocean from Ireland to Newfoundland. **Otway H. Berryman** made a line of soundings from the U.S.S. *Arctic* to verify the existence of a submarine ridge on which it was proposed to lay the telegraph cable (1856) [see Table 4.4]

1857–1880 CE **Joseph Wilson Swan** (1828–1914, England). Inventor. Notable for his achievements in photography, synthetic textiles and *electric lighting*. A rival of Edison.

Swan was born in Sunderland and after leaving school at 13 was apprenticed to a druggist. In 1846 he joined a pharmaceutical business in Newcastle which also manufactured photographic plates, and thus Swan was led to one of the advances in photography with which his name is associated – the production of extremely rapid dry plates based on his observation that heat increases the sensitiveness of a gelatino-bromide of silver emulsion (1857). The patent was bought by **George Eastman**, founder of Kodak, and helped make photography cheaper and thus widely popular.

He was one of the first (since 1848) to undertake the production of electric lamp in which light should be produced by the passage of an electric current through a *carbon filament*, but it was not until **Herman Sprengel** (1834–1906, Germany and England)developed his mechanized *air-pump* (1865) that it became possible to achieve the necessary vacuum in the bulb. A *scientific American* article of July 1879 mentioned how Swan had designed an incandescent lamp using carbon shaped into a form of a cylinder.[308]

In 1880 he established a small factory near Newcastle and within three years he was manufacturing 10,000 lamp bulbs a week. In 1883 he amalgamated his business with Edison to form the 'Ediswan' Electric Light Company.

[308] Edison admitted to have read this article. Swan defenders claim Edison stole this idea from Swan. Edison backers claim Edison read this article after he designed his own carbon filament. Apparently, neither of them was the sole inventor. More than 30 experimenters have been known to work on the perfection of incandescent electric lighting during 1802–1879 and carbon had been an ingredient of experimental light bulbs 50 years before Edison.

Table 4.6: DEVELOPMENT OF WORLD-WIDE ELECTROMAGNETIC TELEGRAPHY COMMUNICATIONS (1831–1866)

1831	**Joseph Henry** (USA) sent an electric charge over 1.5 km of a single wire, where an electromagnet produces a force on a suspended permanent magnet that rang a bell.
1831	**C.F. Gauss** and **W. Weber** (Germany) built an electromagnetic telegraph that operated over a distance of 2 km. It used a mirror-galvanometer as a receiving device.
1836–1837	**John Daniell** and **Charles Wheatstone** improved the voltaic cell, creating a stable source of current.
1838	**Carl August von Steinheil** (1801–1870, Germany) discovered the possibility of using the *earth* for a return conductor in telegraphy (*grounding*). He also invented a telegraph system in which characters are printed on a paper ribbon.
1839	**William Oshaughnessy** (British) laid a telegraph cable which crossed the Ganges in India.
	Samuel Morse (USA) laid down a telegraph cable in the port of New York (Detected the use of an electromagnet for transmitting signals in 1837).
1844	**Samuel Morse** (USA) set up a 60 km telegraph line between Washington DC and Baltimore.
1846	New York City was linked with Washington DC.
1851, Dec 31	England and France combined efforts to lay the first submarine cable across the English Channell from Dover to Calais (ca 33 km).
1852	England was linked telegraphically to Ireland, Belgium, Holland, and Denmark.
1853	The Rhine was crossed at Worms.
1854	Turkey was linked with Crimea, across the Black Sea.

1855 **David Edward Hughes** (England and USA) invented a keyboard-telegraph with rotating type-wheel printer that grew into modern *telex* industry.

1855 The Mediterranean was crossed from Italy to Corsica, from Corsica to Sardinia and from Sardinia to Bône.

1856 Twelve companies in the USA combined to form the *Western Union Telegraph Company*.

1858, Aug 18 The first cablegram from America to Europe was sent across the Atlantic. It took 35 minutes to arrive. The text consisted of a congratulation of US President Buchanan to Queen Victoria. It read:

"Europe and America are united by telegraphy. Glory to God in the highest, on earth peace, goodwill toward man. It is a triumph more glorious because for more useful to mankind than was ever won by a conqueror on the field of battle."

The line remained in service for 23 days only when the cable broke. Silence fell over the Atlantic for the next 6 years.

1861, Oct 24 Transcontinental coast to coast telegraph line was established in the USA. That day, Stephen J. Field, chief justice of California, sent the first message to President Abraham Lincoln. It declared California's loyalty to the Union. The transcontinental telegraph ended the *pony express*, which had operated only about 19 months.

1865 A telegraph system between India and England. It took, on the average, 6 days to telegraph a message overland between the two countries.

1866, Sept 08 **Cyrus West Field** (1819–1892, USA) and **Lord Kelvin** (1824–1907, England) succeeded in laying a telegraph cable across the Atlantic Ocean. The Old and the New Worlds were joined together again, for "better or for worse".

Searching for a better filament for his bulbs, Swan dissolved cellulose in acetic acid and extruded it through narrow jets into a coagulating fluid. Soon afterwards **Chardonnet** (1839–1924, France) adopted this process to make *rayon*, and it was further developed by **Charles Cros** (1855–1935, England) and **Edward Beran** (1856–1921, England) who, in conjunction with **Samuel Courtaulds** (1893–1881, England) laid the foundation of the *synthetic textile industry*. Swan also made significant improvement to lead-plate batteries by designing cellular lead plates which held the lead oxide more securely.

Swan was a self-taught experimentalist and entrepreneur, much like Edison in the USA. He also elected FRS (1874), knighted (1904), and received many other honors.

1858 CE **Archibald Scott Couper** (1831–1892, Scotland). Chemist. Introduced the idea of the *valence bond* and drew the first structural formulas independently of **Kekulé** in Germany. Proposed the tetravalency of carbon and the ability of carbon atoms to bond with each other. Couper recognized two valencies of carbon, one in CO and one in CO_2. He also assumed that carbon atoms form the basic bone of organic compounds. Couper, who was only 27, had sent his contribution to his former teacher, **Charles Adolphe Würtz** (1817–1884) in Paris, to be presented to the French Academy of Sciences. But Würtz unaccountably neglected to do this. His work first appeared through the intervention of **Jean Baptist André Dumas** (1800–1884) a little less than a month after Kekulé's publication, and consequently Kekulé received most of the credit.

Couper's health, poor since childhood, thereafter failed, and after a nervous breakdown he made no further contributions to chemistry[309]. Kekulé, whose conceptions were not quite as close to modern ideas as Couper's, went on to develop the structure of benzene.

Couper was born in Kirkintillach, Scotland.

1858–1862 CE *Discovery of the sources of the Nile.* For thousands of years the people of Egypt revered the Nile as a sacred river. They did not know where it originated, nor what caused its annual flooding. But they did know that without it their civilization might never have come into being.

With a length of 6650 km from its farthest headstream to the Mediterranean, the Nile is the world's longest river. Its drainage basin, estimated at 3,349,000 km^2, includes parts of 9 countries and encompasses about $\frac{1}{10}$ of Africa's land area.

[309] H.C. Brown, *J. Chem. Educ.* **11** (1934) 331; **36** (1959) 104–110; O.T. Bently, *ibid.*, 319–320, and E.N. Hiebert, *ibid.*, 320–327.

Down through the ages the source of the great river was shrouded in mystery despite many efforts to discover it. The task was complicated, as it turned out, by the fact that the Nile has not one but three major sources, since its northward flow unites the waters of its longest branch, the so-called *White Nile*, with those of the *Blue Nile* and the smaller *Atbara*.

If the ancient Egyptians knew of the Blue Nile and its source, the knowledge was lost. It was not until 1615 that the Jesuit missionary **Pedro Paez** (1564–1622, Spain), working in the service of the Portuguese, visited the source of the Blue Nile in the Ethiopian highlands (h. 1830 m; 11 °N, 37 °E), southeast of Lake Tana. It was rediscovered by **James Bruce** (1730–1794, Scotland) in 1770.

The White Nile proved a more difficult problem. In 150 *CE*, the Greek astronomer Ptolemy placed its headwaters in a range called the *Mountains of the Moon* — a range that has since been identified as the Ruwenzori Mountains on the border between Uganda and Zaire. Although Ptolemy was not far from the truth, attempts to confirm his theory were unsuccessful.

In the early 1800's, European knew little more about Africa than the Phoenicians had known in 500 BCE. However, the impetus given to research and exploration in the prosperous Victorian era, made possible the organization of a series of British expeditions in an effort to unveil the last mysteries of the '*Dark Continent*'. During the second half of the 19^{th} century these expeditions finally discovered the river's ultimate headstream, the *Kagera River*, which rises in the present-day Burundi (h. 2130 m; 2 °20′S, 29 °20′E) and flows northeast 400 km into *Lake Victoria*. The overflow from the northern end of this lake, in turn, is the beginning of the White Nile proper.

Flowing northward through *Lake Kyoga*, the White Nile plunges 37 m over *Murchison Falls* and begins a rapid descent from the lake plateau to the low flat plains of southern Sudan. There the river slows drastically as it spreads out across a broad marshy area, and eventually after a northward journey of 800 km is joined by the Blue Nile at Khartoum. Although much shorter than the White Nile, with a length of 1370 km, the Blue Nile carries 63% of the Nile's total waters (annual average 10^9 m^3), being fed by summer rains on the Ethiopian highlands. It is this sudden influx of water that accounts for the annual flooding of the arid lower Nile Valley.

The discovery of the sources of the White Nile is a most exciting human drama: In January 1858 *Lake Tanganyika* was discovered by **Richard Burton** (1821–1890, England) and **John Hanning Speke** (1827–1864, England). On Aug. 03, 1858, Speke discovered *Lake Victoria Nyanza* (he did not know at that time that he had reached the head reservoir of the White Nile). On the 28^{th} of July 1862 Speke, at the head of a new expedition, stood by the *Ripon*

Falls, where the Nile issues from Lake Victoria. In his journey he discovered the *Kagera River*, now known as the most remote headstream of the Nile.

On March 14, 1864, **Samuel White Baker** (1821–1893, England) reached *Lake Albert Nyanza*. These discoveries virtually solved the Nile problem so far as the source of the main stream was concerned, but there remained much to be done before the hydrography of the whole Nile basin was made known.

The project was achieved in two steps: During 1874–1889 **Henry Morton Stanley** (1841–1904, Wales) filled in the gaps left by Speke and Baker, exploring the *Kagera*, *Lake Kyoga*, *Lake Albert Nyanza*, and the '*Mountains of the Moon*'. Between 1891–1908, British and German teams made accurate surveys of the entire source region.

Stellar Brightness, Magnitudes, Luminosities, and Spectra

Hipparchos *(ca 150 BCE) and* ***Ptolemy*** *(ca 150 CE) used a scale of magnitude to indicate the* apparent brightness *of stars on their charts. The notion of brightness is based on the amounts of light energy, or* luminous flux *(erg per sec per cm^2) received from stars, which are among the most important and fundamental observational data of astronomy. It is used in estimating the distances and the actual output of stellar energy.*

By 1856*, stellar photometry had developed to such a degree that accurate magnitudes could be determined by visual methods. Photography was adapted to astronomy at about the same time.* ***John Frederick William Herschel***[310] *(1792–1871, England) and* ***N.R. Pogson*** *noticed that an average first-magnitude star was about* 100 *times brighter than a star of* 6^{th} *magnitude, i.e., it delivers to earth somewhat more than* 100 *times as much*

[310] His father, **William Herschel**, devised a simple method to measure the relative intensity of starlight (photometry): let two stars appear with different brightness in the field of view of a telescope. They can be made to appear equally bright, when viewed one at a time, by adjusting the aperture size for either of them. The ratio of the relative light *aperture areas* then serves as an approximate measure of the relative light intensities arriving from the two stars.

light as a star that is just barely visible on a dark night. Therefore, a difference of five magnitudes corresponds to a luminous flux ratio of $100:1$.

Since the physiology of sense perception is such that equal differences of brightness correspond to equal ratios of light flux energies, Pogson suggested that the ratio of light flux corresponding to a step of one magnitude be $\sqrt[5]{100} = 2.512$. *By assigning a magnitude* 1.0 *to the bright stars Aldebaran and Altair, Pogson's new scale gave magnitudes that agreed roughly with those in current use at the time. Thus, for each difference of ca* 5 *magnitudes, the ratio of brightness increases (or decreases) by a factor of* 100. *In general, if* m_1 *and* m_2 *are the magnitudes corresponding to stars from which we receive visible light flux in the amounts* ℓ_1 *and* ℓ_2, *the difference between* m_1 *and* m_2 *is* $m_1 - m_2 = 2.5\log_{10}\frac{\ell_2}{\ell_1}$.

The star possessing the highest apparent brightness, Sirius, sends us about ten times as much light as the average star of the first magnitude and so it has the magnitude $1.0 - 2.5 = -1.5$. *Venus, the brightest planet, is of magnitude* -4. *The sun, with a magnitude of* -26.5, *sends us* 10^{10} *as much light energy as Sirius; and we also receive* 10^{10} *times as much light from Sirius as from the faintest star that can be photographed with a* 5 *meter telescope. Magnitudes are determined by eye estimates (visual), by blue-sensitive photographic plates (photographic* magnitude) *and with photo cells (photoelectric* magnitude).

Since the stars are not all at the same distance from the sun, it is desirable to calculate all magnitudes as if *all were at the same distance. The term* absolute magnitude *means that the magnitude of a star is calculated for a standard distance of* 10 *parsecs (32.6 light-years), assuming the light intensity to vary inversely with the square of the distance [the absolute magnitude of the sun is about* $+5$*]. The extreme range for absolute magnitudes observed for normal stars in* -10 *to* $+19$, *a range of a factor of more than* 10^{11} *in intrinsic light output* $[2.512^{29} = 10^{11.6}]$.

The difference between the star's apparent magnitude m *and its absolute magnitude* M *is a measure of its distance*[311]. *The light energy-flux arriving at the earth is called a star's* brightness *and is usually expressed in erg per* cm^2 *per sec. The eye receives* 2.512 *times more energy per* cm^2 *per sec from a* 3^{rd}*-magnitude star than from a* 4^{th} *magnitude star.*

In determining a star's absolute magnitude, astronomers must make allowance for non-visible radiation and for the absorption and scattering of light in the atmosphere. The apparent magnitude of a star that we see in the sky

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311 From the definition of magnitudes $m - M = 2.5\log_{10}\frac{\ell(10)}{\ell(r)}$, where r is the *actual* distance in parsecs. Combining this with the inverse-square law $\frac{\ell(10)}{\ell(r)} = \left(\frac{r}{10}\right)^2$, we obtain $m - M = 5\log_{10}\frac{r}{10}$.

could be misleading if the star happens to emit a significant fraction of its radiation at non-visible wavelengths [e.g., a very luminous and hot star with surface temperature of $35,000$ K appears deceptively dim to our eyes simply because most of the star's radiation is emitted at ultraviolet wavelengths. Furthermore, the earth's atmosphere is opaque to many non-visible wavelengths, and thus a sizable fraction of the radiation from the hottest stars and the coolest stars simply does not penetrate the air to get at our eyes or telescopes].

To cope with this difficulty, astronomers have defined the bolometric magnitude *of a star as the star's apparent magnitude measured* above *the earth's atmosphere and over* all *wavelengths. In recent years, satellites have allowed us to determine the bolometric magnitude of many stars. The absolute magnitude deduced from the bolometric magnitude is called the* absolute bolometric magnitude *(M_{bol}) of a star and is always smaller than the star's absolute visual magnitude (M) deduced from ground-based observations at visible wavelengths alone, per fixed* distance *estimate.*

By comparing satellite and ground-based data, astronomers have figured out how much they must subtract from a star's absolute visual magnitude to get its absolute bolometric magnitude. This correction is called the bolometric correction *(BC). The* luminosity *(L) of a star[312] is its total energy output in units of erg·sec^{-1}. The star's absolute bolometric magnitude is directly related to its luminosity[313], assuming the correct distance was used in computing the former.*

A discovery by ***P.A. Secchi*** *in 1863, opened the field of* stellar spectroscopy*: in addition to its magnitudes and luminosity, a star could be characterized by its* spectrum*, or* spectral type*. The physical interpretation of this phenomenon became available only in the early 1900's, when Niels Bohr explained the structure of the hydrogen atom[314].*

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312 For the *sun*: absolute bolometric magnitude = +4.75; absolute visual magnitude = +4.85; $L_\odot = 3.90 \times 10^{33}$ erg·sec^{-1}; energy flux = $E_\odot = \frac{L_\odot}{4\pi R_\odot^2} = 6.41 \times 10^{10}$ erg·cm^{-2}·sec^{-1}. Using the Stefan-Boltzmann law, we find for the sun's surface temperature $T_\odot = \left\{\frac{E_\odot}{\sigma}\right\}^{1/4} = 5800$ K.

313 $M_{\text{bol}} = 4.75 - 2.5\log_{10}\left(\frac{L}{L_\odot}\right)$. Knowing the star's M_{bol} and using this equation, astronomers can calculate how much energy is released from the stars surface each second.

314 Each dark line in a stellar spectrum is due to the presence of a particular chemical element in the atmosphere of the star observed. The differences in both continuous stellar spectra and their elemental absorption lines are due to the widely differing *temperatures* in the outer layers of the various stars. Hydrogen, for example, is by far the most abundant element in all stars, but hydrogen lines

1858–1866 CE **Max Johann Sigismund Schultze** (1825–1874, Germany). Anatomist, zoologist and cytologist. Altered the conception of the cell (1861); advanced the correct theory of retinal function (1866); demonstrated minute nerve endings in the ear (1858) and nose (1863); introduced new techniques in histology.

Schultze was born at Freiburg (Baden). He studied medicine in Berlin; Professor at Halle (1854–1859) and Bonn (1859–1874).

Schultze recognized the *protoplasm* with its nucleus as the fundamental common substance for all forms of life (emphasizing the living protoplasm and *not* the membrane). Advanced the *duplexity theory* of retinal function by identifying the *retinal cones* as color receptors and the *rods* for night vision.

1858–1872 CE **Siegfried Heinrich Aronhold** (1819–1884, Germany) and **Rudolph Friedrich Alfred Clebsch** (1833–1872, Germany). Mathematicians. Developed independently a consistent symbolism in invariant theory: a method for systematic investigation of algebraic invariants. We now recognize in this symbolism as well as in Hamilton's vectors, Grassmann's 'gap' products and Gibbs' dyadics — special aspects of *tensor algebra*.

Clebsch was born at Königsberg in Prussia, and studied at the university of that town. During 1858–1863 he held the chair of theoretical mechanics at the Polytechnicum in Carlsruhe. In 1863 he accepted a position at the University of Giessen. In 1868 he went to Göttingen, and remained there until his death. He worked successively in the fields of mathematical physics, the calculus of variations, partial differential equations of the first order, the general theory of curves and surfaces, the theory of invariants and Abelian functions. He introduced the topological concept of a *genus* of a curve.

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do not necessarily show up in a star's spectrum: if the star is much hotter than 10,000 K, high energy photons pouring out of the star's interior easily knock electrons out of the hydrogen atoms in the star's outer layers, ionizing the gas. The hydrogen ions have no electrons in their lower energy levels to absorb photons and produce Balmer lines. Conversely, if the star is much cooler than 10,000 K, the majority of photons escaping from the star do not posses enough energy to boost many electrons up from the ground state of the hydrogen atoms. These unexcited atoms also fail to produce Balmer lines. A prominent set of Balmer lines is a clear indication that the star's surface temperature is about 10,000 K. Only then is an appreciable number of atoms excited to the second energy level, from which they can absorb additional photons and rise to still higher levels of excitation. These photons correspond to the wavelengths of the Balmer series, which is the part of the spectrum that is readily observable.

Aronhold was born in Angerburg to Jewish parents, and died in Berlin. He was educated at the University of Königsberg (1841–1845) under **Bessel**, **Jacobi**, **Hesse** and **F. Neumann** and in Berlin under **Dirichlet** and **Steiner**. From 1852 to 1854 Aronhold taught at the Artillery and Engineer's School at Berlin. He also taught at the Royal Academy of Architecture at Berlin from 1851. Aronhold was appointed professor at the Royal Academy of Arts and Crafts. In 1864 he became a professor at the Berlin Royal Academy of Architecture.

1858–1875 CE **Joseph William Bazalgette** (1819–1891, England). Civil Engineer. Planned and constructed London's main drainage system and Thames embankment (1860–1874). It consisted of 130 km of large intersecting sewers, draining more than 250 km^2 of buildings, and calculated to deal with some 1.7 million m^3 a day. The cost was 4.6 million pounds.

As late as 1850, towns and cities were plagued by three problems: dispersing of the *rain-water* which might cause floods, of the miscellaneous *rubbish* which in the course of times would make the streets impassable, and of the decomposing of *organic matter* which was not merely an offensive nuisance but a grave danger to health.

Since the Great Fire of London (1666), dumping places for rubbish has been officially provided in the streets of the City, from which refuse was removed by a paid staff of 'rakers'. The content of the privies were removed by 'night-soil men' at times when the streets were deserted. Much was sold for agricultural uses and the rest was tipped into Thames. By the end of the 18th century a town like London, with more than a million inhabitants, was driven to attempt a number of solutions, all of which proved increasingly inadequate. One of those was the *water-closet*: when flushed, discharged the content directly to a cesspool (in the basement or under the garden) which was emptied at something like annual intervals. But the cesspool constituted a double danger to health, from the effluvia which commonly entered the house and from the leakage which tainted wells, rivers and water-pipes.

By 1840 the situation in London became horrible: evil-smelling mudbanks proclaimed the fact that the river fleet and the other rainwater sewers were discharging vast quantities of household effluent into the Thames, to be carried to and for on the tide even in the heart of the capital. In 1855, after 20,000 Londoners had died in two *cholera epidemics*, a Metropolitan Board of Works was set up, with **Bazalgette** as its chief engineer.

Bazalgette built five main sewers running parallel to the course of the Thames, three on the north bank and two on the south, which would be capable of dealing with all household sewage, together with the normal flow of rainwater. At first all sewage was discharged into the Thames. Later, however, chemical clarification of the river waters was established.

1858–1882 CE **Friedrich August Kekulé** (von Stradonitz, 1829–1896, Germany). Distinguished organic chemist. Made far-reaching contributions to chemical theory, especially in regard to the constitution of *carbon* compounds. Established, simultaneously with **Couper**, the 4-valence[315] of carbon (1858) and the fact that carbon atoms can chemically combine with one another to form *chains*. First to perceive the correct structure of *Benzene*.

Kekulé drew chemical structural formulae in which he represented each atom in the molecule as possessing a number of hooks, or *bonds* equal to its valence, and then wrote those bonds into a formula so that the atoms seemed held together in tinker-toy fashion. For over a century now, chemists have been able to use the Kekulé system as a guide to the possible structure of new compounds and to the number of *isomers* possible in a given case. The system has been greatly refined and made at once more complicated and more flexible, but its main outline still stands.

Kekulé was born in Darmstadt. While studying architecture at Giessen he came under the influence of **Liebig** and was induced to take up chemistry. From Giessen he went to Paris, and then visited England. On his return to Germany he started a small chemical laboratory at Heidelberg and in 1858 was appointed professor of chemistry at Ghent. In 1865 he was called to Bonn to fill a similar position.

1858–1901 CE **Peter Guthrie Tait** (1831–1901, Scotland). Physicist. Creator of new methods in quaternion analysis, many of which were later transferred to vector analysis. Changed the emphasis in quaternion analysis towards its usefulness as a tool for physical science. Extended the ∇ (Nabla) operator to vector fields, and developed it as a fundamental tool in modern vector analysis.

Tait did important work on the 'Four-Color' problem[316], and wrote an analysis of the physics of golf balls in flight!

He refused fellowship in the Royal Society of London, declaring that a fellowship in the Royal Society of Edinburgh was quite good enough for him.

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[315] The name *valence* (from a Latin word meaning *power*) was suggested in 1852 by the chemist **Edward Frankland** (1825–1899, England).

[316] For further reading, see:

- Wilson, R., *Four Colors Suffice*, Princeton University Press, 2005, 262 pp.

History of Algebraic Equations[317]

The solution of polynomial equations continued to occupy the center stage in the algebra of the early 19th century. In the previous century, the efforts of ***Euler***, ***Vandermonde***, ***Lagrange*** *and* ***Ruffini*** *to solve algebraically general equations of degree greater than 4 came to nought. Indeed, the quintic equation boggled the minds of the finest mathematicians of Europe for about 300 years!*

So ended also the efforts of mathematicians to furnish a general algebraic solution to the binomial equation $x^n - 1 = 0$. ***Gauss***, *however, opened (1801) the 18th century with the algebraic solvability of the cyclometric equation* $x^p - 1 = 0$ *(where* p *is a prime*[318]*), which is the equation of the division of the circle into* p *equal parts. The latter refers to the fact that the roots of this equation are (by de Moivre's theorem)*

$$x_k = \cos(k\frac{2\pi}{p}) + i\sin(k\frac{2\pi}{p}), \quad k = 1, 2, \ldots, p$$

and the complex numbers x_k, *when plotted geometrically, are the vertices of a* p*-sided regular polygon that lie on the unit circle.*

Gauss then showed that the cyclometric equation is solvable in radicals if and only if $p = 1 + 2^{(2^n)}$ *[Fermat's Number], namely*

$$p = 3,\ 5,\ 17,\ 257,\ 65537, \ldots.$$

For an arbitrary n, *Gauss proved that* $x^n - 1 = 0$ *is solvable in radicals if and only if* $n = 2^\alpha p_1 p_2 p_3 \ldots p_n$ *where all prime factors are distinct Fermat's*

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317 For further reading, see:

- Dehn, E., *Algebraic equations*, Dover Publications Inc: New York, 1960, 208 pp.

318 The case of p prime takes care of $x^n - 1 = 0$. For if $n = pq$, let $y = x^q$ and the problem reduces to $y^p - 1 = 0$ which is solvable. Moreover $x^q =$ const. can be solved if q is a prime, and if not, q can be decomposed in the same manner that n was.

Numbers and α is a positive integer or zero. Clearly this implies that a regular polygon with n sides can be constructed with compass and ruler only when n is of the above form.

***Gauss** (1801) proved the Fundamental theorem of Algebra stating that every polynomial equation of degree n,*

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

in which the a_i ($i = 1, 2, \ldots, n$) are real or complex numbers, has at least one solution in the domain of complex numbers. If this solution is denoted by x_1, one finds that $f(x) - f(x_1) = (x - x_1)[x^{n-1} + \cdots + a_{n-1}] = 0$. By the fundamental theorem, the expression in the square brackets also has a solution, say x_2. If this process is iterated, one obtains a product representation of $f(x)$, from which it follows that an equation of degree n has exactly n roots, which need not necessarily be distinct from each other.

*The question of the solution of general equations of degree higher than four was settled by **Abel** (1826). He first proved the theorem:*

"The roots of the equation solvable by radicals *can be given such a form that each of the radicals occurring in the expression for the roots is expressible as a* rational function *of the roots of the equation and certain roots of unity".*

Abel then used this theorem to prove the impossibility of solving by radicals the general equation of degree greater than four.

In 1828, Abel discussed the general solution of the cyclometric equation. *One of the reasons why it was so difficult to arrive at an understanding of the solvability of equations of higher degree, was the fact that* special *equations of higher degree can be solvable by radicals.*

In particular, there are two classes of equations which can be solved; The first are those in which the polynomial can be written as a product of polynomials of lower degrees. The second class are those whose polynomial can be decomposed into powers of a polynomial of a lesser degree.

Even for cubic and quadratic equations, the results can be extremely complicated. If the parameters in equations like these are symbolic; there can also be some subtlety in what the solutions mean: the results you get by substituting specific values for the symbolic parameters into the solution may not be the same as what you get by doing the substitutions in the original equation.

An example of an equation solved by radicals is

$$x^6 - 9x^4 - 4x^3 + 27x^2 - 36x - 23 = 0,$$

where one solution is $x = \sqrt[3]{2} + \sqrt{3}$.

In another example, the equation

$$x^5 + 20x + 32 = 0$$

has a root

$$\begin{aligned} x_1 = \frac{1}{5}\Bigg(& -\sqrt[5]{2500\sqrt{5} + 250\sqrt{50 - 10\sqrt{5}} - 750\sqrt{50 + 10\sqrt{5}}} \\ & - \sqrt[5]{-2500\sqrt{5} + 750\sqrt{50 - 10\sqrt{5}} + 250\sqrt{50 + 10\sqrt{5}}} \\ & + \sqrt[5]{2500\sqrt{5} + 750\sqrt{50 - 10\sqrt{5}} + 250\sqrt{50 + 10\sqrt{5}}} \\ & - \sqrt[5]{2500\sqrt{5} - 250\sqrt{50 - 10\sqrt{5}} + 750\sqrt{50 + 10\sqrt{5}}} \Bigg). \end{aligned}$$

Indeed, it was shown (1885) by ***J.S.C. Glashan*** *(1844–1932),* ***G.P. Young*** *(1819–1889), and* ***C. Runge*** *(1856–1927) that all irreducible solvable quintics, with the quadratic, cubic and quartic terms missing, have the following form, with* ν *and* μ *rational*

$$x^5 + \frac{5\mu^4(4\nu + 3)}{\nu^2 + 1}x + \frac{4\mu^5(2\nu + 1)(4\nu + 3)}{\nu^2 + 1} = 0.$$

The previous quintic example is a special case for $\mu = 1$, $\nu = \frac{1}{2}$.

Thus, the next step after Abel's work was to discover general criteria *for the solvability of algebraic equations with* $n > 4$. *This task fell to* ***Galois*** *(1830–1), using a novel approach that revolutionalized mathematics. An important point to be emphasized is that "algebraic" solution requires expression in a* finite *number of arithmetic steps. Solution of general equations of degree higher than 4 is possible if an* infinite *number of steps is permitted. But such solutions are nonalgebraic, and are sometimes expressed in terms of special non-algebraic (transcendental) functions. One may think of such functions as formulated by the* infinite series *so important in analysis, and in this way realize that an infinite number of arithmetic steps is involved.*

Now, ***Girard*** *(1629) had already shown that trigonometric functions (which are nonalgebraic or transcendental functions) are effective in obtaining solutions when the Cardano's formula yields irreducible results. Therefore, mathematicians after Galois' day conceived the idea that the* elliptic functions, *which generalize ordinary trigonometric functions, might offer a means of expressing solutions of some higher-degree equations that are not*

solvable algebraically. Hence, ***Charles Hermite*** *(1858) succeeded in solving the general quintic equation ($n = 5$) in terms of* elliptic modular functions.

Another part of Galois' ideas is his theory of fields, which is needed to clarify the notion of rational functions. *The idea of a field was introduced by* ***Abel*** *(1826). By a field of numbers he meant (like Galois) a collection of numbers such that the sum, difference, product, and quotient of any two numbers in the collection (except division by zero) are also in the collection. Hence, rational numbers, real numbers and complex numbers form fields. A polynomial is said to be* reducible *in a field (usually the field to which its coefficients belong) if it can be expressed as the product of two lower-degree polynomials over the field.*

After Galois, ***Bravais*** *(1849),* ***Cayley*** *(1849),* ***Jordan*** *(1869),* ***Sylow*** *(1872),* ***Sophus Lie*** *(1874),* ***Frobenius*** *(1887) and* ***Hölder*** *(1889) continued researches in the theory of groups. With them the study of groups assumed its abstract form (independent of the solution of algebraic equations), and developed at a rapid pace. The notion of group came to play a major role in geometry, and in algebra it became an important factor in the 20th century rise of* abstract algebra.

The technique of solving the quintic equation (and higher order equations) by means of transcendental functions (especially elliptic modular functions*) was perfected over a period of some 200 years. We shall next give a brief survey of these efforts since* ***E.S. Bring's*** *reduction of the general quintic (1786) into a canonical form.*

George Birch Jerrard *graduated at Trinity College (1827) and heard about* ***E.S. Bring's*** *Lund publication (1786) only in 1861. Both achieved reduction of the quintic be means of the* ***Tschirnhausen*** *substitution (1683) method, through which the reduction is effected by the extraction of only square and cubic roots. However, Jerrard found a* single *Tschirnhausen transformation that converts an nth degree polynomial equation ($n \geq 5$) into an nth degree polynomial equation in y in which the coefficients of* $y^{n-1}, y^{n-2}, y^{n-3}$ are all zero.

Thus, the general quintic

$$x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 = 0$$

can be transformed, via a single Tschirnhausen transformation, into the ***Bring-Jerrard*** *form*[319] $\quad y^5 - A_4y + A_5 = 0.$

[319] This simplification has, however, a 'price': The evaluation of (A_4, A_5) from $\{a_1, a_2, a_3, a_4, a_5\}$ requires the solution of three quadratic equations and one cubic equation. The final result of the general case is therefore quite messy.

Starting from the Bring-Jerrard canonical form, mathematician of the second half of the 19th century concentrated their efforts on infinite series solution of the quintic[320]. ***Chebyshev*** *(1838),* ***Eisenstein*** *(1844) and others considered the function* $y(x) = x^5 - x - \rho$ *together with its inverse* $x = x(y, \rho)$.

The formal series expansion of $x(y;\rho)$ *is then calculated for small* y *values, yielding at* $y = 0$,

$$x(0;\rho) = -\rho - \rho^5 - 5\rho^9 - 35\rho^{13} - 285\rho^{17} - 2530\rho^{21} - 23,751\rho^{25} - 231,880\rho^{29} + O(\rho^{33}).$$

These are the first 8 terms of a series having the general term $\{\frac{-\rho^{4k+1}}{4k+1}\binom{5k}{k}\}$. *This series can be summed using hypergeometric functions; one of the roots of the quintic is*

$$x_1(\rho) = -\rho\ {}_4F_3\left(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; \frac{1}{2}, \frac{3}{4}, \frac{5}{4}; \frac{3125}{256}\rho^4\right). \tag{1}$$

In principle this method will work for any polynomial.

During 1860–1862, ***James Cockle*** *(1819–1895) and* ***Robert Harley*** *(1828–1910) developed a method for solving algebraic equations of the type* $x^p + bx^q + \rho = 0$ *based on differential equations. They showed that these equations have roots that can be represented in terms of hypergeometric functions of one variable. In particular they solved the quintic equation* $x^5 - x - \rho = 0$.

Their idea was to consider the function $x(\rho)$ *and derive for it a linear differential equation based on the algebraic equation. Assuming the form:*

$$a_1\frac{d^4x}{d\rho^4} + a_2\frac{d^3x}{d\rho^3} + a_3\frac{dx^2}{d\rho^2} + a_4\frac{dx}{d\rho} + a_5x + a_6 = 0,$$

the repeated differentiation of the original algebraic equation w.r.t. ρ *leads to a system of 5 linear equations in the* a_j, *yielding*

$$\begin{array}{lll} a_1 = (256 - 3125\rho^4)/1155, & a_2 = -6250\rho^3/231, & a_3 = -4875\rho^2/77, \\ a_4 = -2125\rho/77, & a_5 = 1, & a_6 = 0. \end{array}$$

The general solution of the differential equation is then a linear combination of four independent solutions with as yet unknown four parameters. These

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For this reason, mathematician in the second half of the 19th century devised various ingenious 'bypasses' of the Tschirnhausen transformation.

[320] **Lambert** (1757) was first to suggest (based on the ideas of Girard in 1629) a solution based on infinite series.

in turn are found by substituting the known general solution back into the quintic equation, and expand it about $\rho = 0$.

Collecting terms in ρ and setting the coefficients equal to zero, there results a system of 4 linear equations for the four parameters. This completes the procedure of deriving the five roots of the given quintic. The explicit form of one of them is the same as (1).

***Hermite** (1858) produced an elliptic function solution of the quintic equation by combining previous ideas of **Galois**[321] with the **Bring-Jerrard** solution. Here indeed, in the case $n = 5$, the modular equation of order 6 depends on an equation of order 5. Conversely, the general quintic equation may be made to depend upon this modular equation of order 6. Thus, assuming the solution of this modular equation, Hermite was able to solve (not by radicals) the general quintic equation, analogously to the trigonometric solution of the cubic equation.*

Hermite's explicit solution of the Bring-Jerrard reduced equation $x^5 - x - a = 0$ is given in the form: $x_j = \frac{1}{\lambda} z_j$, $j = 1, 2, 3, 4, 5$, where

$$\lambda = 2 \cdot 5^{3/4} \cdot p^{1/4} \cdot (1 - p^2)^{1/2}, \quad p = \tan \frac{\alpha}{4}, \quad \sin \alpha = \frac{16}{\sqrt{5^5 - a}}$$

$$\begin{aligned}
z_1 &= (v_1 - v_2)(v_2 - v_6)(v_4 - v_5) \\
z_2 &= (v_2 - v_4)(v_3 - v_1)(v_5 - v_6) \\
z_3 &= (v_3 - v_2)(v_4 - v_3)(v_6 - v_1) \\
z_4 &= (v_4 - v_2)(v_5 - v_4)(v_1 - v_3) \\
z_5 &= (v_5 - v_2)(v_1 - v_4)(v_3 - v_4)
\end{aligned}$$

$$v_m = p^{5/4} \mathrm{sn}(K - 4\omega_m)\, \mathrm{sn}(K - 8\omega_m), \qquad m = 1, 2, 3, 4, 5, 6$$

$\mathrm{sn}\, u =$ Jacobi's elliptic function

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[321] Among other results demonstrated and announced by Galois may be mentioned those relating to the modular equations of the theory of elliptic functions (derived by Jacobi in 1829): for the transformations of order 5, 7, 11, the modular equations of orders 6, 8, 12 are reducible to the orders 5, 7, 11 respectively, but for n prime and > 11, the reduction is not possible.

$$K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - p^2 \sin^2 \theta}}, \qquad K' = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - p_1^2 \sin^2 \theta}}, \qquad p^2 + p_1^2 = 1$$

$$\omega_1 = \frac{K}{5}, \qquad \omega_2 = i\frac{K}{5}, \qquad \omega_3 = \frac{1}{5}(K + iK'),$$

$$\omega_4 = \frac{1}{5}(K + 2iK'), \quad \omega_5 = \frac{1}{5}(K + 3iK'), \quad \omega_6 = \frac{1}{5}(K + 4iK')$$

In 1877, ***Klein*** *published Lectures on the Icosahedron and the Solution of Equation of the Fifth Degree. In this book and a later article he gave a complete solution of the quintic in terms of ratios of hypergeometric functions.*

Klein used a Tschirnhausen transformation to reduce the general quintic to the form

$$z^5 + 5az^2 + 5bz + c = 0.$$

He found the solution of this reduced quintic by first solving the related icosahedral equation

$$z^5(z^{10} + 11z^5 - 1)^5 - \lambda[z^{30} - 10005(z^{20} + z^{10}) + 522(z^{25} - z^5) + 1]^2 = 0,$$

where λ can be expressed in radicals in terms of $\{a, b, c\}$. A solution of the icosahedral equation using hypergeometric function is

$$z = \frac{\lambda^{-1/60}\ {}_2F_1(-\frac{1}{60}, \frac{29}{60}; \frac{4}{5}; 1728\lambda)}{\lambda^{11/60}\ {}_2F_1(\frac{11}{60}, \frac{41}{60}; \frac{6}{5}; 1728\lambda)}.$$

Gordan *(1878) described an alternative method, avoiding the difficult Tschirnhaus transformation to the quintic form.* ***Brioschi*** *(1858) found a simpler Tschirnhausen transformation that takes the general quintic into the same form*

$$z^5 + 5az^2 + 5bz + c = 0$$

and requires only a single *square root. Another Tschirnhaus transformation of the same kind yields the* ***Brioschi quintic***

$$u^5 - 10\lambda u^3 + 45\lambda^2 u - \lambda^2 = 0$$

which depends on a simple parameter λ. ***Kiepert*** *(1878) transformed the Brioschi equation into the Jacobi Sextic equation*

$$s^6 + \frac{10}{\Delta}s^3 - \frac{12g_2}{\Delta^2}s + \frac{5}{\Delta^2} = 0,$$

$$\Delta = -\frac{1}{5}, \qquad g_2 = \frac{1}{12}\sqrt[3]{\frac{1-1728\lambda}{\lambda^2}}.$$

This sextic is then solvable with Weierstrass Elliptic functions.

In 1884 ***F. von Lindemann*** *expressed the roots of an* arbitrary *polynomial in terms of theta functions.* ***Birkeland*** *(1905) showed that the roots of an algebraic equation can be expressed using hypergeometric functions of several variables.* ***Mellin*** *(1915) solved an arbitrary polynomial equation with the aid of Mellin integrals.*

One could assume that the quest for new solutions of the quintic would slow down after the beautiful results given by Hermite and Klein, but this was not to be: a new generation of mathematicians asked for more, and the 300-year old race continued.

Carl Woldemar Heymann *(1855–1910) solved trinomial equations using integrals. In the same vein,* ***Robert Hjalmar Mellin*** *(1854–1933) solved an arbitrary polynomial equation, using his Mellin transform. Finally,* ***Paul Emile Appell*** *(1855–1930) and* ***Joseph Marie Kampe de Fériet*** *(1893–1982) recognized (1926) the hypergeometric functions in the series solution of the quintic.*

Table 4.7: MILESTONES IN THE HISTORY OF ALGEBRAIC EQUATIONS

ca 2000 BCE

• Babylonians solve quadratic in radicals.

ca 300 BCE

• **Euclid** demonstrates a geometrical construction for solving a quadratic.

1515

• **Scipione del Ferro** (University of Bologna, Italy) gave an algebraic closed-form solution of the cubic equation $x^3 + px = q$, probably basing his work on earlier Arabic sources.

1579

• **Francois Viete** (France) gave a trigonometric solution for the 'irreducible' case of the cubic equation.

1669

• **Newton** introduced his iterative method for the numerical approximation of roots.

1757

• **Johann Heinrich Lambert** gave infinite series solutions of the trinomial equation $x^m + x + r = 0$.

1767

• **Lagrange** expressed the real roots of a polynomial equation in terms of a continued fraction. He showed (1770) that algebraic equation of degree five or more cannot be solved by the methods used for quadratics, cubics and quartics.

1799

• **Paolo Ruffini** gave an incomplete proof of the unsolvability of the quintic equation by means of algebraic functions of the coefficients. **Abel** (1826) gave a complete proof.

• **Gauss** proved the fundamental theory of algebra. In 1801 he solved the cyclotomic equation $x^{17} = 1$ in square roots.

1832

• **Galois** discovered the connection between solutions of algebraic equations and group theory, and showed that the general equation of degree $n > 4$ is not solvable in radicals.

1858

• **Hermite**, **Kronecker** and **Brioschi** independently solved a general

quintic in terms of elliptic modular functions. Earlier studies of **Jacobi** (1829) of modular equation (for elliptic functions)

$$u^6 + v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0$$

is fundamental for the Hermite solution.

1870

• **Camille Jordan** showed that algebraic equations of *any* degree can be solved in terms of the modular functions.

1877

• **Felix Klein** solved the icosahedral equation in terms of hypergeometric functions, thus rendering a closed-form solution of a principal quintic.

1884–1892

• **Ferdinand von Lindemann** expressed the roots of an arbitrary polynomial in terms of theta functions.

1895

• **Emory McClintock** gave series solutions for all the roots of a polynomial.

1905

• **R. Birkeland** showed that the roots of algebraic equation can be expressed using hypergeometric functions in several variables.

1915

• **Mellin** solved an arbitrary equation with the aid of Mellin integrals.

"*From so simple a beginning, endless forms most beautiful and most wonderful have been, and are being evolved*".

Charles Darwin, in the final words of '*The Origin of the Species*' (1859)

"*God created men because he was disappointed in the monkey*".

Mark Twain

1858–1871 CE **Charles Robert Darwin** (1809–1882, England). A British naturalist whose theory of evolution through natural selection caused a revolution in the biological sciences, and had strong impact on natural philosophy and all of the sciences — especially geoscience, astronomy, chemistry, linguistics and anthropology. His book "*On the Origin of the Species*[322] *by Means of Natural Selection, or the Preservation of Favoured Races in the Struggle for Life*" (1859), gave facts on which he based his concept of gradual changes of plants and animals.

According to Darwin, *biological evolution* is the process whereby new species arise from earlier by accumulated changes. As this process of *speciation* proceeds with time, increasing number of species appear, becoming increasingly different.

After many years of careful study, Darwin attempted to show that higher species had come into existence as a result of gradual transformation of lower species and that the process of transformation could be explained through the selective effect of the natural environment upon organisms. He thus concluded that the principles of *natural selection* and *survival of the fittest*[323] govern all life. According to Darwin, nature does not optimize the good of the species, only the good of the individual, upon which natural selection acts. In 1871 Darwin published *The Descent of Man*, outlining his theory that man came from the same group of animals as the chimpanzee and other apes.

Darwin was born in Shrewsbury and was educated at the Universities of Edinburgh and Cambridge. Soon after his graduation at the age of 22 he sailed

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[322] *Species* can be defined as a population which reduces all the individuals that could mate or are likely to mate with one another.

[323] This prong of Darwin's theory is a tautology: who are the fittest? The ones that survive! So this is a circular statement: Through the process of natural selection, the "fittest" survive.

aboard *H.M.S. Beagle*[324] on a 5-year cruise (1831–1836) around the world's oceans. During an exploratory voyage along the coast of South America, the Galapagos[325] and other Islands of the Pacific, Darwin searched for fossils and studies plants, animals and geology.

Destined for the church, Darwin was happily prepared to champion the Book of Genesis. But everything he encountered on the voyage — from the primitive people of Tierra del Fuego to the famous finches of the Galapagos Islands, from earthquakes and eruptions to fossil seashells gathered at 4000 m elevation in the Andes — conspired to wean the young scientist from the simple faith of the *Beagle's* commander, Captain **Robert FitzRoy** (1805–1865)[326], and force upon him the subversive conclusions of '*The Origin of the Species*'. Darwin was influenced in the formulation of his theory by the writings of the economist **Robert Malthus** (1766–1834) and the geologist **Charles Lyell** (1797–1875).

Lyell, in his "*Principles of Geology*" (1830–1833), had restated the thesis already advanced some 50 years earlier by the Scottish physician, Landowner and agriculturist **James Hutton** (1726–1797), that the earth's physical appearance was the result of the same geological processes that are still active at the present time. This idea of vast changes brought about by natural causes, which Lyell had applied to the inorganic world, Darwin applied to the world of organisms. In searching for an explanation of organic evolution, Darwin was impressed by Malthus' account [*Essay on Population*, 1798] of the intense competition among mankind for the means of subsistence.

Darwin never stopped working. "*When I am obliged to give up observation and experiment*", he said, "*I shall die*". He was working on 17 April, 1882; he died two days later. He was interred at Westminster Abbey, with Huxley, Hooker and Wallace among the pall-bearers.

The popular success of *The Origin of the Species* distinguishes it from most other novel ideas in the history of science. Isaac Newton's *Principia* was, and still is, inaccessible to the general reader: its mathematical argument is so

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[324] For further reading, see:

- Moorehead, A., *Darwin and the Beagle*, Penguin Books: England, 1971, 280 pp.

[325] The bishop of Panama reached the Galapagos Ills in 1535 and named them after their giant *tortoises.* These creatures allegedly may reach the age of 200 years. Thus, some of those living there today may have *seen* Darwin!

[326] Later Vice-Admiral, hydrographer and a meteorologist pioneering in weather prediction. Sick in body and mind, he took his own life in a spasm of righteous despair.

obtruse that it took many years of patient analysis before the *scientific* community fully understood its implications. Darwin's book, on the other hand, is amazingly simple for a major scientific book; it is written in such straightforward English that anyone who is capable of following a logical argument can recognize what it means.

Although Darwin made many important observations of his own, the facts which would have supported his theory were already known and had been widely discussed before. Moreover, by 1859 the scientific atmosphere was saturated with the possibility of evolution. It was only a matter of time before someone stumbled on this idea. Why then, had no one thought of it before?

What happened to Darwin's predecessors (and to some of his contemporaries as well) was that their vision was obscured by a strong *preconception* (i.e. not because they were short of facts, but because they had reasons for 'seeing' these facts in a different way). These were: the Biblical notion of special creation (*Creationism*) and the Greek philosophical notion of Ideal Forms (*Essentialism*). Darwin's theory overturned the *catastrophic* history of the world (as promulgated by the world's leading religions), changing it from a series of separate tableaux into a slow-motion picture.

Although Darwin had already formulated the essential outlines of his theory as early as 1839, he delayed its publication for 20 years, waiting for the scientific world to become thoroughly familiar with the issue of evolution. The sources of his scruples were fear of controversy and persecution, his own religious beliefs, and finally his scientific caution; the mechanism which he had invoked was contrary to all of the most dearly held beliefs of Victorian Christianity. In his notebook Darwin had grimly reminded himself of the persecution meted out to other scientists who had flouted traditional belief.

The most important factor was, however, Darwin's doubt about the *scientific credibility* of his own theory. He recognized that evolution could not be observed *directly*. The only way of overcoming this difficulty was to collect such an overwhelming mass of *indirect evidence* that the deduction could be inescapable.

The objections to his theory gave Darwin serious trouble. The first was raised by the zoologist **Jackson St. George Mivrat** (1827–1900, England), who argued that although natural selection might account for the success of well-established adaptations, it could not possibly explain the *initial* stages of their development. The biological usefulness of the eye is self-evident, but how did such an organ get started in the first place? In other words, there must have been a stage at which the incipient organ had no recognizable function, and would therefore have conferred no selective advantage. Therefore, useful

organs must have developed *with a view* to a function they would eventually serve!

Darwin's answer was that a random novelty which gains a foothold by conferring one kind of biological advantage might end up conferring a different sort of advantage altogether (e.g., a primitive feather probably served as a heat insulator, and only subsequently developed its aerodynamical advantage. It is a mystical nonsense to suppose the feather emerged in order to realize the remote possibilities of flight!).

The second objection was the *absence of intermediate types*. Darwin was confidence that subsequent research would restore the episodes of gaps in the fossil record, but this has not happened. There is now overwhelming evidence pointing to the conclusion that certain forms remained stable for long periods of time, only to be suddenly succeeded by new forms altogether. Thus, whilst the process of imperceptible change has an all-important part to play in the origin of the species, it is often superseded by *abrupt transformations* which result in the emergence of comprehensively new designs.

The third objection raised against Darwin's theory arose when **Lord Kelvin** calculated from the temperature of the earth's interior that Darwin has grossly overestimated the age of the earth and hence that an evolutionary mechanism which is based on the slow accumulation of small invisible novelties simply does not have such huge lengths of time at its disposal.

Darwin rightly suspected that Kelvin's calculations would turn out to be wrong. If he'd lived longer, he would have been gratified to discover that the earth was even older than he supposed[327].

Darwin's theories, although they partially accounted for the origin of the species, did not at all account for the origin of life. Only since 1953 have we had the scientific basis to make biochemical guesses about that.

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[327] Darwin did indeed project a vista of slow, gradual, steady, progressive change. In his day, it was a temerity to suggest that the earth was older than a few millions of years. Darwin ventured to ascribe many millions of years to the earth's antiquity. There was good circumstantial evidence for it — but, more than that, he *needed* what then seemed to be a vast amount of time for his notions of how evolution works to hold. The fossil record, as it is known today, suggests that some of the specific ideas that Darwin (and many of his successors) had on *how life evolves*, may well be at least partly wrong; rather than a stately progression, the gross history of life shows a mixture of status quo and revolutions. But the *order* is there: more primitive forms antedate more advanced. Some episodes in evolution proceed faster than others, but the fossil record abundantly affirms the general notion that *life has evolved*.

It is sometimes claimed by historians of science that Darwin 'borrowed' material from other writers (e.g. **Lamarck**, **E. Darwin**, **P.L. Maupertuis**) and lifted his central ideas (without giving due credit) from a number of precursors, including earlier evolutionists and formulators of the principle of *natural selection* (e.g. **Blyth**, 1835). On the other hand, his defenders claim that Darwin, like any scientist, had *influences*, but that he was honest in his theoretical developments and was working as a bona fide scientist of his day.

Darwin's theories consist of seven main hypotheses. He was neither original nor claimed to be on *transmutation*, *the struggle for existence*. He extended or modified earlier theories of *common descent*, *biogeographical speciation* and *natural selection*. *Sexual selection* was his own theory, not influenced either by earlier formulation or by Wallaces' independent discoveries. His theory of *heredity* was not original except in his specific and mistaken hypothesis of pangenesis. Table 4.8 summarizes the "evolution of evolution".

Table 4.8: ORIGIN OF EVOLUTIONAL HYPOTHESES

Hypothesis	**Original to Charles Darwin**	**Influenced**	**First author**
Transmutation of species	No	Possibly, by Lamarck, E Darwin, and Lyell's anti-Lamarckian arguments	Lamarck or Erasmus Darwin in the scientific tradition
Struggle for existence	No	Yes, by numerous scientists, and writers (e.g., Malthus, Tennyson)	Heraclitos
Common descent	No, but first to propose single ancestor of all life	Yes, by numerous scientists, especially von Baer and Owen	Maupertuis
Biogeographical speciation	No	Numerous scientists, esp. Wallace	Gmelin, von Buch
Natural selection	No	Yes, by Blyth (1835)	Patrick Matthew (1831) and William Charles Wells (1813)
Sexual selection	Yes	Possibly by comments by Erasmus Darwin	C. Darwin
Heredity (use and disuse)	No	Yes, possibly by Lamarck	Ancient
Heredity (pangenesis)	Yes	Yes	C. Darwin

Details of the hypotheses are given below:

1. **Transmutationism** (also called by Darwin "Descent with Modification"). This word means in context that species change ("mutate", from the Latin) from one species to another. It is in opposition to the prevailing Aristotelian views that species were natural kinds that were eternal.

2. **Common descent.** This is the view (not held by all evolutionists prior to Darwin or even after) that similar species with similar structures (homologies) were similar because they were descended from a common ancestor. Darwin tended to present the cases for limited common descent – i.e., of mammals or birds – but extended the argument to the view that all life arises from a common ancestor or small set of common ancestors.

3. **Struggle for existence.** This is the view that more organisms are born than can survive. Consequently, most of those zygotes that are fertilized will die, and of those that reach partition (birth) many will either die or not be able to reproduce. The competition here is against the environment, which includes other species (predators and organisms that use the same food and other resources). This is *inter*specific (between species) competition.

4. **Natural selection.** This is a complex view that species naturally have a spread of variations, and that variants that confer an advantage on the bearer organisms, and are hereditable, will reproduce more frequently than competitors, and change the "shape" of the species overall. Notice here that this competition is mostly *intra*specific, i.e., between families of the same species (and indeed of the same population).

5. **Sexual selection.** Many features of organisms are obvious hindrances (such as the tails of birds of paradise), and these often occur in one sex only. Darwin argued that there was competition for mating opportunities and any feature that initially singled a member of one gender out as a good mating opportunity would become exaggerated by the mating choices of the opposite gender. Competition here is between conspecifics of the same gender.[328]

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[328] *The struggle for survival* is above all a struggle for reproduction. Individuals are constantly and automatically tested for their ability to multiply under certain conditions of existence and produce descendants that can live in certain territories. The struggle between males for the possession of females results in the strongest and most wily having the most descendants.
The neo-Darwinists of the beginning of the 20^{th} century argued that the decisive factor in natural selection is not the *struggle for life* (an expression that came from Herbert Spenser and not from Darwin) but the differential rate of reproduction within a given species.

6. **Biogeographic distribution.** Darwin and Wallace were concerned to explain why species were found in the areas they were, and argued that dispersal of similar, but related, species was due to their evolution in one place and migration into other regions.

7. **Heredity.** Darwin knew very little about what we would call the principles of genetics. He accepted the prevailing and old view that the use of features of the organism would change the way those features were inherited.

In conclusion, Darwin and Darwin alone can be seen to be responsible for the theory of *sexual selection*. He was the first person to scientifically posit common descent for *all* life. He and Wallace independently uncovered the causes of biogeographical distribution, though not of the phenomenon itself, and of natural selection in a time of limited resources and change, despite prior sketches. The idea of a transmutation of species was not original to any 19th century scientist, although Darwin and Wallace, along with **Huxley**, **Haeckel**, **Gray**, **Hooker**, **Lyell** and others, were chiefly responsible for its acceptance by the scientific and general community and the success of the view of differentiating and branching evolution.

All biologists until **Weismann** accepted some version of the use and disuse theory of heredity that is known today as "Lamarckism". Even then, the views known as Mendelian genetics were not widely accepted until the turn of the 20th century. Darwin's pangenesis was a heroic but doomed effort.

D'Arcy Thompson said of him (1915):

> "That wise student and pupil of the ant and the bee, who curiously conjoined the wisdom of antiquity with the learning of today; whose Provencal verse seems set to Dorian music; and who, being of the same blood and marrow with Plato and Pythagoras, saw in Number *le comment et le pourquoi des choses*, and found in it *la clef de voûte de l'Univers*".

After Darwin, there were many attempts to extend the idea of evolution into social affairs and explain everything and anything by the same principle of the 'survival of the fittest'. Few of these speculations were well founded but they gave rise to a particular concept of progress and a direction of change.

Evolution did away with the idea that the living world is a finished product. This opened the door to ideas of progress (and regress) and to speculations about what the world might be like in the future. These ideas come more naturally to life scientists. Physical scientists who study the mathematical laws of nature lay much emphasis upon the unchanging character of those laws.

Before the twentieth century, the most successful applications of those laws were to the motions of the moon and the planets. The changes seen in the astronomical realm were slower, simpler, and more predictable than those in the living world. Not until the twentieth century would astronomers have to come to terms with radical new theories about the origin and evolution of stars and galaxies, and the discovery of the expansion of the universe.

There is abundant evidence in ancient history and the geological record for flood and fire catastrophes that can be associated with impacts of objects such as fireballs, comets and asteroids falling from the heavens in significant numbers, and causing widespread damage, extinctions and loss of life. Consequently, a preoccupation with the sky was an integral part of the earliest civilizations; a fear of certain heavenly phenomena was built on an awareness that the sky presented a real threat to one's survival.

In contradistinction, the Newtonian world pictured everything to be under control, ordered on the whole, and unchanging on the average. The ancient view of a sky filled with arbitrary events capable of devastating civilization therefore gave way to one in which the universe acts with unthreatening and clockwork regularity.

The Newtonian approach was seductive because it implied that we live in a more-or-less predictable and hence comfortable world; the random collision with a comet did not fit such a picture.

Darwin's concept of evolution, invoked *gradual* biological change triggered by equally gradual changes in otherwise benign environment.

Worldview XXIII: Darwin

* * *

"The preservation of favorable variations and the rejection of injurious variations, I call Natural Selection, or Survival of the Fittest. Variations neither useful nor injurious would not be affected by natural select and would be left a fluctuating element."

* * *

"I believe there exists, and I feel within me, an instinct for truth, or knowledge or discovery, of something of the same nature as the instinct of virtue, and that our having such an instinct is reason enough for scientific researches without any practical results ever ensuing from them."

* * *

"False facts are highly injurious to the progress of science, for they often endure long; but false views*, if supported by some evidence do little harm, for everyone takes a salutary pleasure in proving their falseness."*

* * *

"As many more individuals of each species are born than can possibly survive; and as consequently there is a frequently recurring struggle for existence, it follows that any being, if it vary ever so slightly in a manner profitable to itself . . . will have a better chance of survival, and thus be naturally selected."

* * *

"If insects had not been developed on the face of the earth, our plants would not have been decked out with beautiful flowers but would have produced such poor flowers as we see in our fir, oak, nut and ash trees, as grasses, docks and nettles which are fertilized through the action of the wind."

Science Progress Report No. 9

The Oxford Meeting[329] *(June 19, 1860)*

Darwin's theory was presented to the Linnaean Society of London in 1858. It had rather little impact. The president (a dentist interested in reptiles) claimed that the year had not "been marked by any of those striking discoveries which at once revolutionize, so to speak, the department of science on which they bear; it is only at remote intervals that we can reasonably expect any sound and brilliant innovation that shall produce a marked and permanent impression on the character of any brand of knowledge."

"On the Origin of Species" was put out by the publisher John Murray in 1859, *and the first edition of* 1250 *copies sold out on the day of publication.*

"The interest aroused was intense; but the subject was too novel and heretical and so, most scientists did not take sides, preferring to reserve their judgment. But it was too revolutionary an issue to lie dormant. What Darwin was saying, or at any rate suggesting, was that the world had not been created in a week, and certainly not in the year 4004 *BCE, as revealed through the obscure calculations of* **James Ussher** *in* 1650. *It was inconceivably older than this, it had changed out of recognition, and was still changing: all living creatures had changed as well, and man, far from being made in God's image, may have begun as something much more primitive. The story of Adam and Eve, in brief, was a myth. This was intolerable: people were furious at the idea that they might share a common lineage with animals. They wrongly thought that he was saying that man had descended from an ape; in fact, what he did believe was that modern man and modern apes have diverged in the remote past from a common line of ancestors."*

"Naturally, the church entered the fray: by 1860, *when Darwin's book had run through* 3 *editions, the clergy were throughly aroused and chose to come out and join battle at the meeting of the British Association for the Advancement of Science, set to take place at Oxford on June* 19, 1860. *The clergy arrived at the meeting, led by the formidable figure of* **Samuel Wilberforce**, *the Bishop of Oxford, a man whose impassioned eloquence was a little too glib for some people (he was known as 'Soapy Sam'), but whose influence was very great indeed."*

[329] Includes quotations from "*Darwin and the Beagle*" by **Alan Moorehead**, Penguin Books, 1971.

*“Wilberforce announced beforehand that he was out to ‘smash Darwin’. He was supported by the anatomist **Richard Owen**[330], who was a rabid anti-Darwinist, and who probably supplied the Bishop with scientific ammunition. Darwin was ill and could not come, but his old teacher, **J.S. Henslow** presided, and he had two ardent champions in **Thomas Henry Huxley** (1825–1895) and the botanist **Joseph Hooker**.”*

“Wilberforce, with his priestly clothes and his air of confident episcopal authority, accused Darwin of merely expressing sensational opinions that flew in the face of the divine revelations of the Bible. The Bishop, rising to the height of his peroration, then turned to Huxley, who was sitting on the platform, and demanded to know if it was through his grandmother of his grandfather that he claimed to be descended from the apes.”

“Huxley was not a man to provoke lightly. When he heard how ignorantly the Bishop presented his case, ending with his ‘insolent question’, he said in an undertone, “The Lord hath delivered him into my hands”. He got up and announced that he would certainly prefer to descend from an ape rather than from a cultivated man who prostituted the gifts of culture and eloquence to the service of prejudice and falsehood.”

“Uproar ensued. The undergraduates clapped and shouted, the clergy angrily demanded an apology, and the ladies from their seats under the windows fluttered their handkerchiefs in consternation. One of them even collapsed from shock and had to be carried out. Amid the hubbub, a slightly grey-haired man got to his feet. His thin aristocratic face was clouded with rage, and he waved a Bible aloft like an avenging prophet. Here was the truth, he cried, here and nowhere else. Long ago he had warned Darwin about his dangerous thoughts. Had he but known then that he was carrying in his ship such a ... He was shouted down and the rest of his words were lost. There were those in the audience who recognized Vice-Admiral **Robert FitzRoy**, *and it must have been disturbing to hear him so passionately denounce his old shipmate.”*

So ended the overture to the future ‘Monkey Trial’ of 1925. *Darwin lived on for another* 22 *years after the Oxford meeting, and his health somewhat improved. His reputation grew steadily and he was given an honorary Doctor’s degree at Cambridge (but not Oxford!), and when he attended a lecture at*

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[330] **Richard Owen** (1804–1892), a comparative anatomist, knew more than enough biology to recognize the truth. But he was driven by wounded pride to write a long spiteful article in which he deliberately twisted the facts in an effort to discredit the new theory. Darwin wisely disregarded these objections, insisting that he could have written much more damaging criticism himself.

the Royal Institution, the whole assembly rose to their feet and applauded him.

On the Galapagos Islands there is a biological research station maintained by the Charles Darwin Foundation. Charles Darwin is now recognized as the man who, as ***Julian Huxley*** *(1887–1975) said: "provided a foundation for the entire structure of modern biology", but during his lifetime he received no official honor from the state. The Church was strong enough to see to that.*

Evolution — Origins and Impact

The idea of evolution was nothing new. Both in the general sense of gradual development of human society from simple to more complex institutions, and the more narrow biological sense that all organisms had evolved out of more elemental forms, the concept had deep roots in Western thought. Darwin's major contribution was to provide an observational basis for what has previously been a mere hypothesis.

The first evolutionist on record was ***Anaximander*** *(ca 560 BCE) who believed in dynamical natural hierarchy and taught that man evolved from aquatic animals.*

Aristotle *(384–322 BCE) noted that species were characterized by their reproductive isolation. He wrote extensively on the classification and structure of over 500 species of animals from the Mediterranean area. He accepted the idea of the origin of life as a spontaneous event, but was also concerned about the problem of heredity. His classification of life embraced a complete gradation from the lowest to the highest organism — man.*

Spontaneous generation of living creatures from nonliving matter became increasingly suspect in the 17^{th} century. The physician ***Francesco Redi*** *(1621–1697, Italy) became convinced that the maggots found in meat were derived not from the meat itself but from eggs laid by flies.*

The voyages of discovery of the 15^{th} *and* 16^{th} *centuries, and the invention of the microscope revealed a diversity of animal and plant form and function unknown to Aristotle. With these new observations, changes in classification took place.* ***John Ray*** *(1627–1705, England), a naturalist, introduced (1686) the present idea of* species *(based on common descent) and higher categories in classification. Ray showed that groups of similar species could be classified into sets, which he called* genera*. (This system is the basis for the international one still in use today.) In 1749* **George Buffon** *(1707–1788), in the first volume of his Histoire naturelle, defined species as a group of inbreeding individuals who cannot breed successfully outside the group.*

Carl Linnaeus[331]*(1707–1778, Sweden), a naturalist, developed the present system and method of biological classification (taxonomy).* ***Erasmus Darwin*** *(1731–1802, England), grandfather of Charles Darwin, a physician, poet and naturalist, was impressed by the extent of changes in form — within the lifetime of individual animals (frogs, for example), by influence of selective breeding in horses and dogs, and by differences due to climate. He also noted the close affinities of the mammals, which (he reasoned) implied their common origin.*

The notion of natural hierarchy was further elaborated on by ***Leibniz****, while* ***Julien Offroy de La Mettrie*** *(1709–1751, France) prefigured the conception of progress through a struggle for existence in his book* Man as Mechanism *(1748). The encyclopedist* **Denis Diderot** *(1713–1784) suggested (1754) that the hierarchy was not static but resulted from continuous development through time. The simpler organisms came first, the more complex ones evolved from them in progressive stages. Thus, the idea of evolution was there already in the* 18^{th} *century, although another hundred years were to elapse before its mechanism was explained plausibly.*

Jean Baptiste de Lamarck *(1744–1829, France), soldier and biologist, published (1809) his work 'Zoological Philosophy'. In it he expounded a consistent and well-reasoned theory according to which species descend from other species by gradual change over many generations. He argued that species retain constant characteristics only in unchanging environments, but plants and animals will change their form to adapt to their new environment (he thought, of course, that* individual organisms *change, and knew nothing of mutations and natural selection).*

Lamarck's work clearly anticipated many fundamental ideas that were later to be popularized by others, but it was his misfortune to be ahead of

[331] Yet, Linnaeus still believed in the biblical story (*Gen, 8*, 19) that animals and all other living forms started to migrate over the earth from Noah's Ark after the Flood had subsided. Today, it seems ridiculous even to children.

his time. His theory was strongly criticized by leading naturalists of his day and, as a result, never received the attention or credit it deserved. Man's everyday experience provided little support for species' development: in spite of circumstantial evidence, no one had yet seen one species turn into another. It remained for Darwin, writing 50 *years later, to convince the scientific world of the truth of evolution. To be in this receptive mood biology had to progress to a point where the existence of evolution should seem reasonable and therefore deserve a scientific explanation. This proper climate for evolution theory was created by advances in five independent scientific fields:*

(1) Embryology: The work of ***Christian Heinrich Pander*** *(1794–1865,* Germany and Russia*) and* ***Karl Ernst von Baer*** *(1792–1876,* Germany*) showed that the early development of the embryo is similar for wide classes of animal species (1828–1837).*

(2) Paleontology: The emerging understanding of fossils as remains of living creatures and the realization that many fossils were of species that no longer existed. The father of paleontology *was the naturalist* ***Georges Cuvier*** *(1769–1832,* France*), who studied the fossil vertebrates of the Paris basin and attributed the succession of fossil forms to a series of simultaneous extinctions caused by* natural catastrophes *(1796).*

(3) Geology: The understanding that the time required for the evolution was available in the earth's history — namely, that the development of geological features required great stretches of time. This step was necessary since evolution at the species level is not *observable during a human lifetime*[332]*. These concepts were introduced by* ***James Hutton*** *(1727–1797),* ***Abraham Gottlob Werner*** *(1749–1817,* Germany*, 1774),* ***Charles Lyell*** *(1797–1875,* England*, 1830) and* ***Robert Chambers*** *(1802–1871, Scotland, 1852).*

*(4) Economy: The "*industrial revolution*"mostly benefited only a minority, the middle class, while it brought utmost misery and destitution to the growing proletariat. A tremendous population growth and large-scale urbanization inflicted great miseries on the working classes.* ***Robert Malthus*** *(1766–1834, England), clergyman and economist, was unconvinced that man is perfect. Disbelieving the universal peace, equality and bounty predicted by the politicians and utilitarian philosophers of the* 18*th* *century, Malthus wrote an anonymous "Essay on Population"*

[332] Except for organisms of very short lifespan, namely insects and microorganisms. Such creatures often evolve to adapt themselves to artificial agents (pesticides and drugs) devised to eradicate them.

(1798). In it he stated that human population cannot expand indefinitely. Populations tend to expand at a geometric rate of increase with which food supplied can never keep pace: Famine, disease and war, Multhus argued, will limit the increasing size of the human populations. Darwin read Malthus in 1838, *and it struck him at once that in a 'struggle for existence' favorable variations would tend to be preserved and unfavorable ones to be destroyed*[333].

(5) *Philology: The orientalist* ***William Jones*** *(1746–1794, England) had drawn attention to the phonetic similarities between certain key words in Latin, Greek and Sanskrit (1790). By 1816, the philologist* ***Franz Bopp*** *(1791–1867, Germany) suggested that all European languages had descended from the same Indo-European root.*

Alfred Russel Wallace *(1823–1913, England), surveyor and naturalist, independently suggested the theory of natural selection and its relation to the geographical distribution*[334] *of the species. Wallace ventured into the Amazon (1848), expressly for the purpose of solving the problem of the origin of the*

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333 This also occurred to **Karl Marx** (1818–1883), who (1848) proclaimed in his *Communist Manifesto* that civilization is an organism evolving irresistibly by circumstantial selection. In *Das Kapital* (1867) he claimed that "*the relation of the bourgeoisie to society was grossly immoral and disastrous and that it concealed and defended the most infamous of all tyrannies and the basest of all robberies*". Marx thus became an inspired prophet in the mind of every generous soul whom his book reached.

334 The *geographical distribution* of species was immensely important for Darwin's ideas on evolution. The English ornithologist **Philip Lustley Sclater** (1829–1913) studied the geographical distribution of birds (1858) and Wallace, elaborating on his ideas, divided the globe into six major biogeographical areas and eventually published the classic text of 19^{th} century *zoogeography*, *The Geographical Distribution of Animals* (1876).

From earliest times, travelers noted that different kinds of plants and animals are to be found in different parts of the world. Only in the 18^{th} century, however, was special attention diverted to questions concerning the geographical distribution of living things. In the 19^{th} century, the geographic distribution of plants was studied by **Alexander von Humboldt** (1769–1859), **Augustin Pyramus de Candolle** (1778–1841), **Charles Lyell** (1797–1875), and **Edward Forbes** (1815–1854). The latter hypothesized that the existence of land bridges in the past explained the similarities among the faunas and floras of Britain and various continental areas. Recent *continental drift* theory has reintroduced the idea that geographical distribution must be understood in part in terms of *geological history*. It is clear today that Wallace reached independently

species. He could not have gone to a better place for clues: the lowland Amazon basin has an astonishingly high number of species. Overwhelmed with the abundance of life, he soon began to discern patterns in the jumble of life around him. He then decided that the rivers, with their forceful water dynamics, were creators of much of the diversity, affecting and shaping both the landscape and the forest animals as well. He argued that the rivers, so broad and uncrossable, were acting much like fences, keeping the species apart.

Already convinced of the fact of evolution, he conceived the idea of natural selection while lying sick with fever in the Moluccas (February 1858). He recalled the Essay of Population *by Robert Malthus, which he had read twelve years before, and saw its application to evolution in a flash of intuition. In June* 1858, *Wallace sent Darwin his essay, entitled "On the Tendencies of Varieties to Depart Indefinitely from the Original Type". Faced with this unnerving anticipation of his own hard-won new synthesis, he expedited the publishing of his theory (1859), and simultaneously arranged for a joint paper with Wallace to be presented to the Linnean Society the following month.*

By applying the idea of evolution to all living organisms, including man, Darwin destroyed many of the most cherished beliefs of his contemporaries. Yet, to an age that worshiped science, the thought that man was just as much subject to the logic of science as was everything else in nature also held a great fascination. Underlying much of Darwin's work was the idea of progress, *an idea dear to the* 19^{th} *century. History, the study of man's past, suddenly appeared in a new light — as a march toward some far-off, lofty goal.*

The theory of evolution and the origin of species began to change our sense of human time. The pace of technological change led people to wonder about the shape of the future. The notions of natural and supernatural, which had seemed so firm when science was merely experimenting and measuring, became shaky when science began constructing and destroying. Things that had seemed fantastic became actuality, from planes and rockets to wonder drugs and superbombs. In response to this sense of technological change and fantastic possibilities in a future that became increasingly more real, new fictional forms began to emerge.

In order to appreciate the nature of the first future shock, one must imagine how people of the pre-modern era visualized the future. In ancient times most people saw the future as being simply a continuation of the present — until the end of the world. For many, the myth of a golden age, from which men had fallen and to which they might be restored at the end of time, provided some

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the same explanation for evolution as Charles Darwin did. Unfortunately he is remembered in the history of science as *'The man who was not Darwin'*.

comfort. But the notion that the world would change regularly was simply not a part of human thought until modern time. Plato, who saw as far as anyone, saw only cycles — as tyranny, oligarchy, democracy, anarchy, and, once again, tyranny succeeded one another in time. Others saw history as having involved a steady decay from gods to heroes to men, which could only be renewed by the gods returning to earth, possibly destroying it, and beginning the cycle again. The idea of steady and irreversible growth in human capabilities was unthinkable until a few hundred years ago, and the idea of humanity as the product of an evolution from less highly organized forms of life would have seemed fantastic beyond blasphemy until the last century.

Darwin's scheme depends on the interaction between individuals and their environment: random mutations introduce diversity among individuals and the environment acts as a filter, selecting through differentiated reproductive rates the ones best tuned to their surroundings.

Although this mechanism of natural selection acts upon individuals*, it is* species *that evolve, and through them all higher populational entities, like the set of species in a given ecological system and in the final account, the entire biosphere. The ensuing freedom in the choice of evolutional unit has often led to confusion in the analysis of these phenomena.*

Darwin assumed that geological times are long enough to provide opportunity for major modifications of species, so that they could be transformed into different species. Yet he never attempted to deal with the origin of life.[335]

One of Darwin's opponents was the Swiss naturalist and geologist ***Jean Louis Rodolphe Agassiz*** *(1807–1873). He studied many kinds of animals in Europe and America. As a geologist he showed that glaciers once covered large areas of the earth. He became noted for his work on fossil forms of fishes. Agassiz established a zoological laboratory on an island in Buzzard's*

[335] Since the early 1960s, developments in the field of *molecular biology* demonstrated that all organic life is programmed by the DNA, which is essentially a specific coded statement (like an alphabetic statement), with digital (discrete) but also analog aspects. Simplistic calculations show that if the hemoglobin protein evolved by chance there would be one chance in 10^{650} of it actually arising. Similarly, the specificity of the T4 bacteriophage is represented by the number $10^{78,000}$, with only one chance in $10^{78,000}$ of it actually occurring by random shuffling. When these figures are set against the age of the universe (10^{18} sec), it seems as if there is no possibility of *life* evolving through Darwin's theory of natural selection, operating on chance mutations. This paradox of the apparent statistical impossibility of Darwinism on the molecular level may just reflect our present ignorance of knowing how to calculate the correct probabilities of early life processes.

Bay, off the coast of Massachusetts, to provide a place to study animals in their natural surroundings. He believed that animal species do not *change, and criticized Darwin's theory of evolution.*

Agassiz, the son of a Protestant pastor, was born in Motier, on the shore of the Lake of Morat. Educated at first at home, then spending four years at the gymnasium of Bienne, he completed his elementary studies at the academy of Lausanne. Having adopted medicine as his profession, he studied successively at the Universities of Zürich, Heidelberg and Munich, where he extended his knowledge of natural history. In 1829 *he took a degree of doctor of philosophy at Erlangen, and in* 1830 *that of doctor of medicine at Munich.*

Agassiz came to the United States in 1846*, and in* 1848 *became a professor of zoology and geology at Harvard.*

Another 'heretic' was **Jean Henri Casimir Fabre** *(1823–1915,* France*), one of the greatest naturalists of the* 19^{th} *century. He spent his life observing insects and spiders, mostly in the gardens and fields near his home in Sérignan. Fabre was called by many the* Poet of Science *who "thinks as a* philosopher*, sees as an* artist*, and feels and expresses himself like a* poet*".* ***Charles Darwin****, in his* Origin of Species*, called him* 'the incomparable observer'*, and* ***Victor Hugo*** *crowned him as* The Homer of Insects.

Fabre was born in the small upland village of Saint-Léons in the Rouergue Mountains of southern France. His parents were so poor that, when Henri was five, they sent him to live with his grandparents on a farm at Malaval; when he was six, he had already an enormous curiosity about nature. Smock-clad and barefoot, the boy keenly studied every new and strange animal and plant — he looked, examined and made mental notes, always driven by his insatiable desire to know. At seven, Fabre returned to his parents' home to begin his schooling. He worked while attending school and in 1842*, at eighteen, he obtained his diploma from the Normal College of Avignon. He then began his teaching career as a primary schoolteacher at Carpentras. Here, his meager salary was often in arrears. While at Ajaccio, Corsica, where he taught science for a few years, he contracted malaria and was forced to return to the mainland. Finally (1852) he became a teacher at the Lycée of Avignon. Here he labored for nearly* 20 *years; when he left, his rank, title and salary were the same as when he began.*

During those years, on his precious Thursday afternoons — the traditional half-holiday of the French school system — and the summer holidays, the schoolmaster became a schoolboy again, devoted to the study of insects. The hours spent along the banks of the Rhone filled his notebooks with entries.

These activities, while doing little to enhance his stature with his school superiors, gave him a local reputation in this strange field of endeavor. ***Louis***

Pasteur[336] *(1822–1895) was sent to see Fabre when he began his study of the silkworm disease. Victor Duruy, energetic minister of public instruction during the reign of Napoleon III, was so impressed by the obscure provincial teacher, that he invited him to Paris, with the hope that he might become a tutor of the imperial family. He conferred upon him the Ribbon of a chevalier of the Legion of Honor, but the simple son of the Rouergue peasant was ill-fitted for life in the royal court. He fled from the great city declaring that he had "never felt such loneliness before".*

Back in Avignon, he outraged his superiors when he admitted girls to his science classes. The clergy denounced him from the pulpit, and in 1870*, when the German armies were overrunning France, Fabre was dismissed from the Lycée and ejected from his house with his wife and five small children. He was saved by his friend* **John Stuart Mill** *(1806–1873, English economist and philosopher), then living at Avignon. Mill loaned Fabre $600 to see him through the crisis. During the next* 9 *years, Fabre found sanctuary in a house at the edge of Orange and supported himself by writing books on popular science. He continued, however, to observe and record the life of the insects. In* 1879 *he was able to buy a small foothold of earth, sun-scorched and thistle-ridden, at the edge of the village of Sérignan. It was inhabited by wasps, and wild bees, and all manner of other creatures — to which he devoted the remaining years of his life. During the next* 3 *decades (1879–1907) he issued his 10-volume saga Souvenirs Entomologiques. Oftentimes, Fabre felt that he had reached the end of his strength and that his grand scheme would not be fulfilled. In the final paragraph of Volume III he wrote: "Dear insects, my study of you has sustained me in my heaviest trials. I must take leave of you for today. The ranks are thinning around me and the long hopes have fled. Shall I be able to speak to you again?".*

Fabre never accepted the theory of evolution. He was an empiricist, and opposed to hypotheses: "I observe, I experiment and I let the facts speak for themselves". This attitude, in addition to his isolation and remoteness from centers of research, his narrow knowledge of entomological literature, and his not being a trained entomologist, caused him to ignore the role of instincts in the action of many insects and look upon them as 'programmed' machine-like creatures that stick to their course like a train on its rails. Consequently he was dominated by the general rule and failed to ascribe much significance to exceptions to the rule as modern research workers have found they must do. Yet his harvest of facts is invaluable to students of experimental biology, since he led the study of living *entomology at a time when that science seemed*

[336] Pasteur questioned the theory of evolution, because Darwin did not base his ideas on experimental proof. Louis said: "Do not put forward anything you cannot prove by experimentation."

preempted by those whose horizon of interest was limited to the dead insect and the pinned specimen. Each of his experiments was an adventure — and he was able to transmit his enthusiasm to others, never losing sight of humanity in his writings, which possess a charm that defies definition.

His work was recognized when he was already in his eighties. In 1910 *the President of France came to Sérignan to visit him.*

***Faraday**, **Maxwell** and **Kelvin** also rejected Darwinian evolution: they were religious men who adopted without question the view that nature laws were imposed by Divine decree.*

*Finally, many other 19th century scientists did not accept the theory of evolution. Among them: **C. Babbage**, **J.F.W. Herschel**, **James Joule**, **G. Mendel**, **W. Ramsay**, **Lord Rayleigh**, **B. Riemann**, **G.G. Stokes** and **R. Virchow**.*

1859–1860 CE **William Ferrel** (1817–1891, USA). Meteorologist. Applied the theory of the *Coriolis effect* to the general circulation of atmospheric and oceanic currents. Accepted the theory of **James Pollard Espy** (1785–1860, USA) that the energy of cyclones is largely due to the latent heat of condensation when air ascends (1840), and went on to show that differential heating is the initial cause of both cyclones and the general circulation. He attempted to analyse quantitatively the effect of horizontal temperature gradients on the horizontal pressure field at different levels in the atmosphere. From this work he derived the concept of the *thermal wind*.

1859 CE, Aug 27 *Birth of the Oil Industry.* **Edwin Laurentine Drake** (1819–1880, USA), a retired railroad conductor, *drilled* an oil well at Titusville, PA, USA. He found oil 21 meters bellow the surface. Drake used a wooden rig and a steam-operated drill similar to the cable-tool drills of today. He drove an iron pipe 12 meters long into the ground to solid rock, and drilled inside the pipe. This pipe served as a casing. Drake put a pump on the well, which produced 10 to 35 barrels a day. The company sold the oil for $20 a barrel. Other men drilled wells nearby, after Drake showed them how to do it. As a result, the price of oil dropped to 10 cents a barrel in less than three years. In the early 1860's, over 600 oil companies were incorporated in Pennsylvania.

In early times man used petroleum that seeped to the surface from underground springs. The ancient Egyptians coated mummies with *pitch* (natural

asphalt). The Chinese found natural gas while drilling for salt, and used natural gas for fuel as far back as 1000 BCE. About 600 BCE, **King Nebuchadnezzar** used asphalt to build the walls and pave the streets of Babylon. the Assyrians and Persians also used asphalt to build their cities. Boatmen on the Euphrates River made vessels of woven reeds smeared with asphalt. American Indians used petroleum for fuel and medicine hundreds of years before the white man came; remains of their ancient oil wells have been found in the oil regions of Pennsylvania, Kentucky and Ohio.

Some historians believe that the first oil industry began in Romania, which produced about 2000 barrels of oil already in 1857. Workmen used bags and buckets to bring up oil from hand-dug wells. Also in 1857, **James Miller Williams** of Canada dug an oil well and established a refinery near present-day Oil Springs, Ontario. He distilled and sold oil for lamps. But most historians trace *the start of the industry on a large scale* to Drake's well (1859).

James Young (1850, England) started the commercial production of *paraffin* from crude oil by slow distillation, thus creating the paraffin oil-shale industry. The first off-shore oil wells were drilled in 1900.

Drake's pioneering endeavor started a process that would eventually fund much large-scale geological research in search for more oil.

1859–1879 CE **James Clerk Maxwell**[337] (1831–1879, Scotland). The greatest mathematical physicist since Newton, and one of the great theoretical physicists of all time. Made revolutionary investigations in electromagnetism and the kinetic theory of gases, along with substantial contributions in several other theoretical and experimental fields: (1) Color vision, (2) the theory of Saturn's rings, (3) geometrical optics, (4) photoelasticity, (5) thermodynamics, (6) the theory of servomechanisms, (7) viscoelasticity, (8) relaxation processes. He wrote 4 books and about 100 papers.

His greatest achievement was the construction of a unified field theory for electricity and magnetism that integrated the accumulated experimental results known since **Coulomb** (1785), **Oersted** (1820), **Ampère** (1827), **Faraday** (1831) and **Gauss** (1833). He then represented all known electromagnetic phenomena by four partial differential equations, known as *Maxwell's equations* (1873). These represent: absence of magnetic monopoles, electrostatic field of charges (Coulomb's law in Gauss' form), law of induction (Faraday's law) and the magnetic effect of current (Ampère's law in Stokes'

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[337] For further reading, see:

- Everitt, C.W.F., *James Clerk Maxwell, Physicist and Natural Philosopher*, Charles Scribner's Sons: New York, 1975, 205 pp.

form). This last law was modified by Maxwell by adding a term responsible for the *displacement current*, such that the whole system of equations could render *electromagnetic waves*.

Thus, Maxwell established the theory of the electromagnetic fields, putting the field notion of **Faraday** on a solid mathematical footing. He showed that these fields can propagate as *electromagnetic waves*, carry with them a definite amount of energy and move with the velocity of light.

At one time it was thought that gravity, magnetism and electricity are the result of bodies acting on each other via '*action at a distance*'. According to this idea (originated by Newton for the case of gravitation), given any two bodies there is an '*action*' between them, that causes them to attract each other by a force that is proportional to each of their masses and inversely proportional to the square of the distance between them. In the '*field*' picture, a body surrounds itself by a field of force, which exists *whether or not* a second body is present to feel it and be attracted. A field of force is related to a *potential*, whose variation in space determines the field (force per unit test mass; test-charge in the case of electricity).

Maxwell unified the regimes of optics, electricity and magnetism. Moreover, his electromagnetic theory predicted the existence of X-rays, gamma rays, radio waves and ultraviolet and infrared radiation.

These predictions were soon to be verified. In 1883, **George Francis FitzGerald** (1851–1901, Ireland), professor of natural philosophy at Dublin, pointed out that if Maxwell's theory were valid, it should be possible to generate electromagnetic waves purely electrically — by varying an electric current periodically in a circuit. [**Kelvin** had demonstrated in 1853 that the discharges of a Leyden jar, and other electrical condensers, are oscillatory phenomena.] Accordingly, FitzGerald suggested that a discharging condenser would be a good source of the electromagnetic radiation predicted by Maxwell's theory, and he showed that the shorter their wavelength, the greater the amount of energy they would carry, and thus the easier they should be to detect.

During 1886–1889 **Heinrich Hertz** (1857–1894, Germany) confirmed Maxwell's theory by producing, transmitting and receiving electromagnetic waves in the laboratory. They were shown to be transverse and propagate with the velocity of light.

In spite of this, Maxwell's electromagnetic theory was slow to gain general acceptance. In the words of Max Born (1933): "*It seems to be characteristic of the human mind that familiar concepts are abandoned only with the greatest reluctance, especially when a concrete picture of phenomena has to be sacrificed*".

Indeed, Maxwell himself and his followers tried for a long time to describe the electromagnetic field with the aid of mechanical models. It was only gradually, as Maxwell's concepts became more familiar, that the search for an "explanation" of his equations in terms of mechanical models was abandoned.

In 1861 Maxwell created the science of quantitative colorimetry. He proved that all colors may be matched by mixtures of 3 spectral stimuli[338], provided that subtraction as well as addition of stimuli is allowed. He revived **Thomas Young**'s 3-receptor theory of color vision and demonstrated that color blindness is due to the ineffectiveness of one or more receptors. He also projected the first *color photograph* and made other noteworthy contributions to physiological optics. [Helmholtz' paper of 1852 contained useful work, but he overlooked the essential step of putting negative quantities in the color equations and explicitly rejected the 3-color hypothesis.]

In 1859 Maxwell finished his study on the rings of Saturn (**Huygens**, 1655). He proved mathematically that a model assuming broad, rigid, thin sheets of matter would break apart and concluded that Saturn's rings are composed of "*an indefinite number of unconnected particles*". [Supporting observational evidence came 4 decades later when **James Edward Keeler** (1857–1900, U.S.A.), working at Lick Observatory, observed (1895) Doppler shifts in sunlight reflected from the Saturnian rings.]

The problem of determining the motion of large numbers of colliding bodies came to Maxwell's attention while he was still investigating Saturn rings. Then, when he read the new papers by **Rudolf Clausius** (1858 and 1859) on the kinetic theory of gases, he set forth to go a step further and remove the simplifying assumption that all molecules of any one kind have the same speed. This led him (1860) to a statistical formula for the distribution of velocities in a gas at uniform temperature. Maxwell's idea of describing actual physical processes by a *statistical function* was an extraordinary novelty.

He next applied the distribution function to evaluate coefficients of viscosity, diffusion and heat conduction, as well as other properties of gases not studied by Clausius. He interpreted viscosity as the transfer of momentum between successive layers of molecules moving, like Saturn rings, with differential transverse velocities. Finally, Maxwell evaluated the distribution of energy among different modes of motion of the molecules — translational, rotational, etc. [*equipartition law*].

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[338] Artists had indeed known centuries before Maxwell and Helmholtz that the 3 so-called primary pigments, red, yellow and blue, yield any desired hue by mixture; but the weight of Newton's claim that the prismatic spectrum contains 7 primary colors clouded interpretation of the phenomenon.

It is important to note that Maxwell was the first to state that the second law of thermodynamics[339] is *statistical in nature* (1868).

Maxwell introduced the concepts of *'curl'*, *'gradient'* and *'convergence'* (negative 'divergence') of vector fields. He also introduced the distinction between axial and polar vectors, and gave a physical treatment of the two classes of tensors later distinguished mathematically as covariant and contravariant. He gave (1871) a simple physical interpretation of the Laplace operator[340].

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[339] **Maxwell** nevertheless did not quite comprehend The Second Law of Thermodynamics (SLT). In his *Theory of Heat* (1871) he posed a way to defy SLT, saying: "The second law is undoubtedly true as long as we can deal with bodies only in mass, and have no power of pressing or handling the separate molecules of which they are made up". But he postulated that a "being" [known as: *"Maxwell's Demon"*] small enough to manipulate molecules, should be able to defy SLT and use *all* the available heat energy without expending any in the process, in effect creating a perpetual motion machine of the 2^d kind.

To prove this, Maxwell described a vessel with two chambers, A and B, which were connected by a tiny hole, which the Demon can quickly open or close so as to allow only the swifter (hotter) molecules to pass from A to B, and only the slower (cooler) molecules to pass from B to A. He will thus, without expenditure of work, raise the temperature of B and lower that of A, in contradiction to SLT [i.e. constructing a virtual air-conditioner that needs no power supply!].

In 1922, within months of submitting his doctoral thesis, **Leo Szilard** (1898–1964; then a student of Max von Laue, Max Planck and Albert Einstein in Berlin) wrote a paper on thermodynamic equilibrium: *"On the Decrease of Entropy in a Thermodynamic System by the Intervention of Intelligent Beings"*. In it he argued convincingly that *thinking generates entropy*, demonstrating that Maxwell's demon could *not* decrease entropy in the system (since his selective opening and closing of the valve must involve what today would be called "data processing", a.k.a. *thinking*)and thus could not violate SLT. It thus turns out that there is a deep connection between SLT and *information theory*.

Modern developments in the fields of *quantum computing*, *reversible computing* and *nanotechnology* continue to provide new twists on the theme of Maxwell's Demon and the relations between information, statistics, and thermodynamics.

[340] The quantity

$$\nabla^2\Phi$$

is a measure of the difference between the value of the scalar function Φ at a given field point and the *average values* of Φ in an infinitesimal neighborhood of that point. Indeed, define the average of $\Phi(\boldsymbol{r})$ at P inside a sphere of radius R centered about $P(\boldsymbol{r})$,

$$\Phi_{\text{av}} = \frac{1}{V}\int_V \Phi(\boldsymbol{r}+\boldsymbol{\eta})d^3\boldsymbol{\eta},$$

In the introduction to his book: *Treatise on Electricity and Magnetism* (1873) Maxwell abandoned Faraday's description of the electric field as a state of elastic stress in the ether, arguing that under this concept, no measurements could be made. He exchanged it for a mere *mathematical law*. This led to a revolutionary change in our attitude toward the physical world.

Up to Maxwell's time, physics had been divided into its various disciplines according to the human senses (the *anthropomorphic* classification), e.g. optics, acoustics, heat etc.; every observed phenomenon would be reduced to the appropriate sensing organ and classified accordingly. But with the discovery of new phenomena which could not be classified by this method it became necessary to divide physical phenomena according to the respective *mathematical laws*, as Maxwell did. This approach bears the advantage that parallels can be drawn between different phenomena that are subjected to the same *mathematical law* (say, the Laplace equation $\nabla^2\phi = 0$ which appears in hydrodynamics, potential theory, and electricity).

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where $V = \frac{4\pi}{3}R^3$, $0 \le |\boldsymbol{\eta}| \le R$. Expand Φ in a Taylor series about $\boldsymbol{r}$; for small R,

$$\phi(\boldsymbol{r}+\boldsymbol{\eta}) = \Phi(\boldsymbol{r}) + \boldsymbol{\eta}\cdot\nabla\phi + \frac{1}{2}\boldsymbol{\eta}\boldsymbol{\eta}\colon \nabla\nabla\phi + O(R^3).$$

Since

$$\frac{1}{V}\int_V \phi(\boldsymbol{r})d^3\boldsymbol{\eta} = \phi(\boldsymbol{r}),$$

$$\frac{1}{V}\int_V \boldsymbol{\eta}\boldsymbol{\eta}d^3\boldsymbol{\eta} = \frac{3}{5}R^2\mathfrak{I},$$

$$\frac{1}{V}\int_V \boldsymbol{\eta}\cdot\nabla\phi d^3\boldsymbol{\eta} \equiv 0$$

(symmetry), and

$$\mathfrak{I}\colon \nabla\nabla\Phi = \nabla^2\Phi,$$

it follows that

$$\nabla^2\Phi = \frac{10}{R^2}\left[\{\Phi(\boldsymbol{r})\}_{\text{av}} - \Phi(\boldsymbol{r})\right].$$

If Φ is harmonic, the average of Φ inside a sphere is equal to its value at the sphere's center. Writing

$$\{\phi(\boldsymbol{r})\}_{\text{av}} = \phi(\boldsymbol{r}) + \frac{1}{10}R^2\nabla^2\Phi(\boldsymbol{r}),$$

it appears that the average value of Φ in a small sphere equals its value at the center plus a *correction term due to spatial variation*. This correction term is governed by the Laplacian of Φ. Since $\nabla^2\Phi$ is very important in the differential equations of physics, Maxwell's interpretation enables us to attach a simple physico-geometrical meaning to many field equations in physics.

In the century that followed Maxwell, classical physics was divided into the following main categories:

- *Particle physics*, including the mechanics of a particle and systems of discrete particles and small bodies, kinetic theory of gases, and classical statistical mechanics. The unifying mathematical law is Newton's equation $m_i \ddot{q}_i(t) = F_i$ where m_i, $q_i(t)$, F_i are the respective masses, generalized coordinates and generalized forces. The mathematical vehicle is therefore the theory of systems of *ordinary differential equations* in the time variable.
- *Continuum physics*, including the classical physics of rigid bodies, elasticity, electromagnetism, fluid dynamics etc. Here we have functions, each of a finite number of variables, that vary continuously over a given domain, constituting a *field*. The state of the system is governed by field functions $F_i(x_1, x_2, x_3; t)$ [e.g. the velocity field in a fluid $\boldsymbol{V}(x_1, x_2, x_3; t)$]. The appropriate mathematical theory is that of *partial differential equations.*

Maxwell was born in Edinburgh, Scotland, the son of wealthy parents. His mother died when he was nine. At age 16 he entered the University of Edinburgh, and at 21 Trinity College, Cambridge. His class included such later celebrities as **Thomson** (Kelvin), **A. Cayley**, **Ferrers**, **Tait** and **Routh**.

In 1858 Maxwell married Katherine Mary Dewar, seven year his senior. They had no children. Earlier Maxwell had an emotional involvement with his cousin Elizabeth Cay, a girl of great beauty and intelligence, which they had to terminate because of the perils of consanguinity in a family already inbred. From 1860 to 1865, Maxwell served as a professor of natural history at King's College, London. In 1865 he retired from regular academic life to write his celebrated "*Treatise on Electricity and Magnetism*". In 1871 he was appointed professor of experimental physics at Cambridge, and planned and developed the Cavendish Laboratory. He died of abdominal cancer on 5 November, 1879.

The advance of physics during the two centuries following the publication of Newton's '*Principia*' was made possible largely due to two convictions:

- The elucidation of scientific laws was a prerequisite for the systematic ordering of empirical data, and reflected the fundamental order within the realm of objective reality.
- The basic concepts and processes reflected in these laws are mechanical in nature.

The usefulness of the second of these two pillars of the Newtonian clockwork world view reached its zenith, and the beginning of its end, with Maxwell's equations of the electromagnetic field. The aptitude with which he combined the experimental results of Faraday with his own mathematical intuition, serves as a quintessential example of the scientific method at work.

Maxwell unified electricity, magnetism and light: the electromagnetic spectrum runs the wavelength (and frequency) gamut from gamma rays, through X rays to ultraviolet light, to visible light to infrared light to radio waves, encompassing the technologies of radio, television and radar. His four equations unified the experimental results of **Oersted**, **Ampère** and **Faraday**. Light now appeared to behave as waves and to derive from electric and magnetic fields.

Maxwell has ushered in the age of modern physics – on his own, driven only by curiosity, costing the government almost nothing, himself unaware that he was laying the grounds for the next great revolutions in both science and technology.

Like most other great British scientist (Faraday, Darwin, Dirac and Crick), Maxwell was never knighted. Moreover, the communications media, the instrument of education and entertainment that Maxwell made possible, have never offered even a mini series on the life and thought of their benefactor and founder: he is almost forgotten in popular culture.

On Maxwell

"The greatest change in the axiomatic basis of physics — in other words, of our conception of the structure of reality — since Newton laid the foundation of theoretical physics, was brought about by Faraday's and Maxwell's work on electromagnetic phenomena.

Before Maxwell people conceived of physical reality as material points, whose changes consist exclusively of motions, which are subject to total differential equations. After Maxwell they conceived physical reality as represented by continuous fields, not mechanically explicable, which are subject to partial differential equations. This change in the conception of reality is the most

profound and fruitful one that has come to physics since Newton."

Albert Einstein[341] (1931)

"From a long view of the history of mankind there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics. Even the American Civil War, will pale into provincial insignificance before this more powerful event of the 1860's."

Richard Feynman (1964)

"Immortality of the soul, in its old religious sense, had been thoroughly discredited. But there is another and far nobler sense in which the soul truly was immortal. In living our lives, each of us makes some impression on the world, good or bad, and then dies; this impression goes on to affect future events for all times, so that part of us lives after us, diffused through all humanity, more or less, and all Nature. This is immortality of the soul."

Oliver Heaviside (1886)

Electromagnetic waves — from Oersted to Maxwell (1820–1864)

The course of physics up to about 1820 was a triumph of the Newtonian scientific program. The "forces" of nature – heat, light, electricity, magnetism, chemical action – were being progressively reduced to instantaneous attractions and repulsions between the particles of a series of fluids. Magnetism and static electricity were already known to obey inverse-square laws similar to the law of gravitation. The first 40 years of the 19th century witnessed a growing reaction against such division of phenomena in favor of some kind of "correlation of forces".

[341] Maxwell died in 1879, the year that Einstein was born.

Oersted's *discovery of electromagnetism (1820) was at once the first vindication and the most powerful stimulus of the new tendency, yet at the same time it was oddly disturbing; the action he observed between an electric current and a magnetic field* differed *from previous phenomena in two essential ways:*

- *It was developed by electricity* in motion;
- *The magnet was neither attracted to nor repelled by – but set* transversally *to the wire carrying the current.*

To such a strange phenomenon widely different reactions were possible. ***Faraday*** *took it as a new irreducible fact by which his other ideas were to be shaped. He was first to suggest that the force acting between two separate objects arises because of a* field, *created by the existence of electric charge. Faraday went on to a major discovery: Electric fields were not only created by charges but also by changing magnetic fields. The two heretofore different forces were thus connected in both directions. But the seminal breakthrough was still to be made.*

Ampère *and his followers sought to reconcile electromagnetism with existing views about instantaneous* action at a distance. *Indeed, shortly after Oersted's discovery, Ampère discovered that a force also exists between two electric currents and put forward the brilliant hypothesis that* all magnetism is electrical in origin. *In 1826 he established a formula which reduced the known magnetic and electromagnetic phenomena to an inverse-square force along the line joining two current elements $j\,\boldsymbol{dl}$, $j'\,\boldsymbol{dl'}$ separated by a distance r*

$$F_{jj'} = G\frac{jj'\,dl\,dl'}{r^2}, \tag{1}$$

where G is a geometrical factor involving the angles between r, $\boldsymbol{dl}$ and $\boldsymbol{dl'}$.

In 1845, ***F.E. Neumann*** *derived the potential function corresponding to Ampère's force and extended the theory to electromagnetic induction. Another extension developed by* ***W. Weber*** *was to combine Ampère's law with the law of electrostatics to form a new theory, which also accounted for electromagnetic induction, treating the electric current as the flow of two equal and opposite groups of charged particles, subject to a force whose direction was always along the line joining two particles e and e', but whose magnitude depended upon their relative velocity $\dot{\boldsymbol{r}}$ and relative acceleration $\ddot{\boldsymbol{r}}$ along that line:*

$$F_{ee'} = \frac{ee'}{r^2}[1 - \frac{1}{c^2}(\boldsymbol{r}^2 - 2\boldsymbol{r}\cdot\ddot{\boldsymbol{r}})], \tag{2}$$

c being a constant with dimensions of velocity.

In 1856, Weber and **F.W.G. Kohlrausch** *(1840–1910, Germany) determined c experimentally by measuring the ratio of electrostatic to electrodynamic forces. Its value in the special units of Weber's theory was about 2/3 of the velocity of light! Equations (1) and (2) and Neumann's potential theory provided the starting points for almost all the work done in Europe on electromagnetic theory until the 1870's.*

The determining influences on **Maxwell** *were* **Faraday** *and* **William Thomson***. He progressively extended their ideas about* lines of electric and magnetic force, *and merged it with the result of Kohlrausch and Weber.*

By 1863, Maxwell had found a link of a purely phenomenological kind between electromagnetic quantities and the velocity of light. His paper: "A Dynamical Theory of the Electromagnetic Field" (1865), clinched matters. It provided a new theoretical framework for the subject, based on experiment and a few general dynamic principles from which the propagation of electromagnetic waves through space followed without any special assumptions about molecular vortices or the forces between electric particles.

In 1865, Maxwell developed a group of 8 scalar equations describing the electromagnetic field. The principle they embody is that electromagnetic processes are transmitted by the separate and independent action of each charge (or magnetized body) on the surrounding space rather than by direct action at a distance. Formulas for the forces between moving charged bodies may indeed be derived from Maxwell's equations, but the action is not along the line joining them, *is not instantaneous, and can be reconciled with dynamical principles only by taking into account the* exchange of momentum (and energy and angular momentum) with the field.

Maxwell discovered that the data, formulated and reconciled by his mathematical equations, produced a permanent marriage of the electric and magnetic fields. Not only did changing magnetic fields produce electric fields but changing electric fields produced magnetic fields. This implied the existence of self-sustaining and moving electromagnetic waves. These waves were soon identified with light. The entire spectrum of electromagnetic waves now stretches from the ultra-low frequencies and hence long wavelengths (kilometers or even larger) through the infrared, the visible region, to the ultraviolet, to the ultra-short-wave, X- and γ-rays radiated by excited heavy atoms, nuclei and elementary particle collisions (10^{-9} m down to 10^{-15} m and smaller).

We now had two force fields capable of acting through great distances: gravitation and the electromagnetic field (gravitation was still described as action-at-a-distance until the advent of GTR). The unification by Maxwell of three historically diverse phenomena — electricity, magnetism and optics — led to a deeper understanding of the phenomena and was an inspiring lesson for what would come later (the theories of Relativity, quantum-mechanical

matter waves, quantum field theories, and the quest for further field unifications).

Maxwell's equations[342] *for stationary material media and slowly moving sources, expressed in Gaussian units (*$\boldsymbol{E}$*,* $\boldsymbol{D}$*,* ρ *in electrostatic units and* $\boldsymbol{H}$*,* $\boldsymbol{B}$*,* $\boldsymbol{J}$ *in electromagnetic units), are as follows:*

(1) $\operatorname{curl}\boldsymbol{E} = -(1/c)\partial\boldsymbol{B}/\partial t - (4\pi/c)\boldsymbol{J}_m,$ *Faraday's law;*

(2) $\operatorname{curl}\boldsymbol{H} = (1/c)\partial\boldsymbol{D}/\partial t + (4\pi/c)\boldsymbol{J}_e,$ *Ampère-Maxwell law;*

(3) $\operatorname{div}\boldsymbol{D} = 4\pi\rho_e,$ *Coulomb's law;*

(4) $\operatorname{div}\boldsymbol{B} = 4\pi\rho_m,$ *Gauss' law.*

Here c *is the velocity of electromagnetic waves in vacuum,* $\boldsymbol{E}$ and $\boldsymbol{B}$ *are the electric and magnetic induction vectors respectively,* $\boldsymbol{D}$ *is the electric displacement vector, and* $\boldsymbol{H}$ *is the magnetic-field vector. The entities* $\{\rho_e, \boldsymbol{J}_e\}$ *are the respective free electric charges and free current density, whereas* $\{\rho_m, \boldsymbol{J}_m\}$ *are the free magnetic charge and free current density.*

The current and charge densities are subject to the local conservation laws

(5) $\operatorname{div}\boldsymbol{J}_e = -\partial\rho_e/\partial t, \qquad \operatorname{div}\boldsymbol{J}_m = -\partial\rho_m/\partial t,$

which are easily verified upon taking the respective divergence of (1) and (2), using (3) and (4).

For linear, isotropic and non-conducting media, the constitutive relations

(6) $\boldsymbol{D} = \epsilon(\boldsymbol{r})\boldsymbol{E}, \qquad \boldsymbol{B} = \mu(\boldsymbol{r})\boldsymbol{H}$

are assumed, where the dimensionless point-functions ϵ*,* μ *are respectively the dielectric constant (permittivity) and magnetic permeability.*

The elimination of $\boldsymbol{H}$ *and* $\boldsymbol{D}$ *between (1), (2) and (6) leads to a wave equation in the electric field vector, in which the current densities are assumed to be known:*

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[342] To dig deeper, consult:

- Schwinger, Julian et al., *Classical Electrodynamics*, Perseus Books, 1998, 569 pp.
- Jackson, J.D., *Classical Electrodynamics*, John Wiley & Sons: New York, 1975, 848 pp.
- Landau, L.D. and E.M. Lifshitz, *Electrodynamics of Continuous Media*, Addison-Wesley, 1960, 417 pp.
- Ben-Menahem, A., "Green's Tensors and Associated Potentials for Electromagnetic Waves in Inhomogeneous Media", *Proc. Roy. Soc. (London) A* **426** 79–106, 1989.

(7) $\operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl}\boldsymbol{E}\right)+\frac{\epsilon}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2}=-\frac{4\pi}{c^2}\frac{\partial\boldsymbol{J}_e}{\partial t}-\frac{4\pi}{c}\operatorname{curl}\left(\frac{1}{\mu}\boldsymbol{J}_m\right).$

In the special case $\boldsymbol{J}_e=0,\ \boldsymbol{J}_m=0,\ \rho_e=0$ *and* $\mu=const.$ *(non-magnetic insulator with no free electric charges), equation (7) reduces to*

(8) $\nabla^2\boldsymbol{E}-(\epsilon\mu/c^2)\partial^2\boldsymbol{E}/\partial t^2=-\operatorname{grad}(\epsilon^{-1}\nabla\epsilon\cdot\boldsymbol{E}).$

The corresponding wave equation for the magnetic vector is obtained in a similar way,

(9) $\operatorname{curl}(\epsilon^{-1}\operatorname{curl}\boldsymbol{H})+\frac{\mu}{c^2}\frac{\partial^2\boldsymbol{H}}{\partial t^2}=-\frac{4\pi}{c^2}\frac{\partial\boldsymbol{J}_m}{\partial t}+\frac{4\pi}{c}\operatorname{curl}(\epsilon^{-1}\boldsymbol{J}_e).$

Although physically realizable electromagnetic sources can be described solely in terms of electric charges and currents, the use of equivalent magnetic current is sometimes a convenient artifice (for example, a magnetic line current element in an isotropic medium is equivalent to a circular electric current flowing around a path of vanishingly small radius in a plane normal to the element).

In the modern formulation of Maxwell's electrodynamics (both classic and quantum), $\rho_m=\boldsymbol{J}_m=0$*; all magnetic fields (whether due to macroscopic electricity flow or single molecules, atoms, ions and electrons) are due to electric currents; and the full electric current and charge in bulk media (which determine* $\boldsymbol{E}$ and $\boldsymbol{B}$ *through the* vacuum *Maxwell's equations) are given by*

(10) $$\rho_{total}=\rho_e-\operatorname{div}\boldsymbol{P}$$

(11) $$\boldsymbol{J}_{total}=\boldsymbol{J}_e+\frac{\partial\boldsymbol{P}}{\partial t}+c\operatorname{curl}\boldsymbol{M}$$

where

(12) $$\boldsymbol{P}=\frac{1}{4\pi}(\boldsymbol{D}-\boldsymbol{E})=\frac{\epsilon(\boldsymbol{r})-1}{4\pi}\,\boldsymbol{E}$$

is the electric-dipole-moment density, and

(13) $$\boldsymbol{M}=\frac{1}{4\pi}(\boldsymbol{B}-\boldsymbol{H})=\frac{\mu(\boldsymbol{r})-1}{4\pi}\,\boldsymbol{H}$$

is the effective magnetic-dipole-moment density, and $\{\rho_{total}-\rho_e,\ \boldsymbol{J}_{total}-\boldsymbol{J}_e\}$ *are the* bound *charge- and current-density, respectively.*

Maxwell's synthesis (1865) of the empirical laws of electricity *and* magnetism*, gathered over the previous 150 years – together with his new "displacement current" term (*$\frac{1}{c}\frac{\partial\boldsymbol{D}}{\partial t}$*) in equation (2), introduced for mathematical and aesthetic reasons – and the consequent explanation of the entire field of* optics *as an electromagnetic phenomenon – was the most far-reaching advance of the 19th century theoretical physics. These were previously considered to*

be unrelated phenomena; magnetism appeared with lodestones; current flowed in wires. Electricity had to do with rubbing amber rods to produce "charged" objects. These phenomena were known since **Thales**, *and many* optical *phenomena are so ubiquitous that they were known to most persons since the dawn of humanity. But now, mathematical descriptions of experimental results were available. Some of the experiments that provided the data were obtained by* **Coulomb**, **Cavendish**, **Orsted**, **Ampère** *and* **Faraday**.

The Maxwell theory also explained all the known data in geometrical and wave optics, and predicted all future results in radio wave generation, propagation, and reception. When coupled with STR and quantum mechanics, Maxwellian electrodynamics became 20th-century QED – a theory that explains to a very high accuracy all branches *of* natural science *except gravitation, cosmology and nuclear- and particle physics (in particular, it should be able to explain* life*). In the 1970's, QED was successfully unified with the* weak nuclear force, *and its* gauge principle, *in a generalized form, underlining the* entire *Standard Model of particle physics.*

The Kinetic Theory of Gases, or —
How fared the atoms of Democritos?

The basic ideas of the Greeks were transmitted via **Epicuros** *to the Roman philosopher* **Lucretius**, *and through that connection to the Renaissance scientists, including* **Galileo**. **Newton** *used these hard, massive atoms in his work on chemistry and optics.*

In about 1800, **Roger Boscovitch** *wrote presciently about the primary elements of matter as point-like entities, having no extension in space but acting upon one another by forces that he describes in detail: strong repulsion when the particles are close, attraction when they are more distant.*

Experimental techniques for studying matter were improved by chemists, such as **Lavoisier** *(1783) who began to clarify the idea of chemical elements.*

***Dalton**, in about 1800, summarized and advanced the idea that an atom is the simplest structure that contains all the properties of an element, and he recognized twenty elements. This number would grow to ninety by the end of the century. A suggestive step in the process of understanding atoms was made by **J.L. Meyer** and **Mendeleev** (1869) who discovered the chemical periodicity of their properties.*

***Prout** (1815), aware of the simple relation of atomic weights, suggested that all elements were made of hydrogen (recalling Thales and his contentious students). That a repetitive pattern in properties of elements would suggest a complex and similarly repetitive internal structure of the atoms corresponding to these elements, came much later. The model of atoms, hard, indivisible, but subject to Newton's laws, led to new ideas about heat and energy.*

*Two main ideas, each advocated by a different group of people, were beginning to emerge. Following the Newtonian track, **Daniel Bernoulli** (1733), **J. Herapath** (1820) and **T.J. Waterston** (1843) advanced the notion that gas molecules interact by elastic collisions between mutually-repulsive particles (atoms, molecules, ions) and that heat is a form of motion.*

They produced early kinetic gas models, but even as late as 1845, these models were persistently rejected by the scientific establishment, namely the London Royal Society.

*In another vein, the universal principle of conservation of energy was established by the experiments of **Rumford** (1798), **Carnot** (1824), **Joule** (1843) and **Helmholtz** (1847). Once this idea became accepted, the treatment of heat as a form of mechanical energy arising from the motion of polyatomic molecules made more sense. It was now up to **A.K. Krönig** (1822–1879) and **Rudolf Clausius** to make the final step in forging the kinetic theory[343] of gases as a connection between the physicists' thermodynamics and the chemists' 'atomic theory' through Avogadro's hypothesis. Clausius' partitioning of the total energy of a system between motions of translation, rotation and vibration, encouraged work on specific heats, while his introduction of the concept of average distance traversed by molecules before collision ('mean free path') stimulated the development of a statistical interpretation of thermodynamics and molecular physics, namely – statistical mechanics. Since heat was now viewed as the kinetic energy of restless atoms, and since there are too many atoms to keep track of via their individual Newtonian equations of motion, they must be treated actuarially.*

Here again, with the application of the idea that all matter is made of atoms, a second great synthesis took place. The jiggling and chattering of

[343] The term *'kinetic theory'* was popularized in the 1870's by **O.E. Meyer** (1834–1915) as a title of a textbook.

atoms bombarding vessel walls explained "pressure". The increase in temperature of a gas is simply an increase in average speed of the atoms (faster jiggling). Heated liquids evaporate because their atoms move fast enough to escape. An enormous collection of observations on the properties of solids, liquids and gases became understood by this kinetic theory, much of it developed by ***James Clerk Maxwell*** *(1859) and* ***Ludwig Boltzmann*** *(1866).*

Despite certain difficulties in deriving specific heat capacities (unresolved until quantum mechanics) most Victorian physicists regarded the kinetic theory as a triumphant example of the mathematization of physics.

History of the Theories of Light III

C. Rebirth of the wave theory *(1801–1888)*

The 19^{th} *century opened with a series of experimental and phenomenological studies which soon put the wave theory of light on a secure foundation, as a transverse propagating undulation of the elastic ether, explicable in mechanical terms.*

The first step toward this was the enunciation by ***T. Young*** *(1801) of the principle of interference, and the explanation of the colors in thin films. His views, however, were expressed largely in a qualitative manner and therefore did not gain general recognition. Young was also first to recognize (1817) that the wave motion of light was transverse. Earlier, in* 1808, *polarization of light by reflection* was discovered by ***Etienne Louis Malus*** *(1775–1812,* France), *who did not attempt an interpretation of this phenomena.*

In the meantime, the corpuscular theory had been developed further by ***P.S. de Laplace*** *and* ***J.B. Biot****, and under their influence the Paris Academy proposed the subject of diffraction for the prize question of* 1818, *in the expectation that a treatment of this subject would lead to the crowning triumph of the corpuscular theory.*

To their dismay, and in spite of strong opposition, the prize was awarded to ***A.J. Fresnel****, whose treatment was based on the wave theory. His work was the first of a succession of investigations, which, in the course of a few years, were to discredit the corpuscular theory completely. In his memoir*

Fresnel effected a synthesis of Huygens' envelope construction with Young's principle of interference. This was sufficient to explain diffraction phenomena. Fresnel calculated the diffraction caused by straight edges, small apertures, and screens. [He was advised by ***Francois Jean Arago*** *(1786–1853,* France) *to read the publications of Grimaldi and Young, but could not follow this advice because he could read neither English nor Latin.]*

Fresnel's theory predicted that in the center of the shadow of a small disc there should appear a bright spot. This counter-intuitive fact caused ***S.D. Poisson*** *to reject the theory. Fresnel was saved by Arago, who performed the experiment by himself and verified that Fresnel's theory was indeed correct. Poisson acquired his share of fame in the event: the spot became known as Poisson's spot!*

In 1818 *Fresnel developed his theory of the partial convection of the luminiferous ether by matter [a theory apparently confirmed in* 1851 *by the direct experiment carried out by* ***A.H.L. Fizeau****]. In* 1821 *Fresnel gave the first indication of the cause of* dispersion, *by taking into account the molecular structure of matter. The first terrestrial determination of the speed of light was performed by Fizeau in* 1849. *His colleague* ***J.B.L. Foucault*** *then followed suite and measured the speed of light in water (1851), finding it to be* less *than that in air.*

In contrast, the corpuscular theory *explained refraction in terms of attraction of the light-corpuscles at the boundary toward the optical denser medium, and this implies a greater velocity in the denser medium. On the other hand the wave theory demands, according to Huygens' construction, that a smaller velocity is obtained in the optically denser medium. Thus, the direct measurement of the velocity of light in air and water decided unambiguously in favor of the wave theory.*

In another vein, a major effort during 1821–1876 *was aimed at establishing the theory of the elastic ether. This theory persisted, in spite of many difficulties, for a long time and most of the great physicists of the* 19^{th} *century contributed to it. Among these were* ***Lord Kelvin***, ***Lord Rayleigh*** *and* ***G. Kirchhoff****.*

While all this was happening in optics, the study of electricity and magnetism was also bearing fruits, culminating in the discoveries of ***M. Faraday*** *(1839).* ***J.C. Maxwell*** *(1873) succeeded in synthesizing all previous experiences in this field in a system of equations, the most important consequence of which was to establish the possibility of electromagnetic waves.*

Maxwell was able to show, purely theoretically, that the electromagnetic field could propagate as a transverse wave in the luminiferous ether. Solving for the velocity of the wave, he arrived at an expression in terms of electric

*and magnetic properties of the vacuum ('ether'): $c = (\epsilon_0\mu_0)^{-1/2}$. Upon substituting empirically determined values for ϵ_0 and μ_0 [**Rudolph Kohlrausch** (1809–1858) and **Wilhelm Weber** (1804–1891) in 1856], Maxwell obtained a numerical result equal to the measured velocity of light. This led Maxwell to conjecture that light waves are electromagnetic waves.*

However, it soon became apparent that the new electromagnetic theory of light, while capable of explaining all phenomena associated with the propagation of light, failed to elucidate the processes of emission and absorption, in which the finer features of interaction between matter and light-radiation are manifested.

As often happens in the sciences, the limits of applicability of a theory exist in latent form long before the theory itself is demised. Indeed, a body of stubborn spectroscopical data had been accumulating since the early days of the 19^{th} century, which 100 years later turned the tide again in favor of a corpuscular aspect of light:

*In 1802, **William Hyde Whollaston** (1766–1828, England) made the earliest observations of the dark lines in the solar spectrum. Because of the slit-shaped aperture generally used in spectroscopes, the output consisted of narrow colored band of light, the so-called spectral lines. In 1814, **Joseph Fraunhofer** (1787–1826, Germany) independently rediscovered the dark lines in the solar spectrum, since named after him. These were interpreted in 1859 as absorption lines on the basis of experiments by **G. Kirchhoff** and **Robert Wilhelm Bunsen** (1811–1899, Germany), in the following way: The light of the continuous spectrum of the sun, passing through cooler gases of the sun's atmosphere, losses by absorption just those wavelengths which are emitted by the gases.*

This discovery was the beginning of spectrum analysis, which is based on the recognition that every gaseous chemical element has its own signature of a characteristic array of spectral lines. The problem of how light is produced or destroyed in atoms involves the mechanics of the atom itself, and the laws of spectral lines reveal not so much the nature of light as the structure of the emitting particles. But this story began to unfold only in the 20^{th} century.

History of Magnetism II (1600–1894 CE)

(a) Background

Although it was known (since 1581) that unlike magnetic poles attract and like poles repel, it was difficult to establish experimentally the law of magnetic force between poles. Thus, an exact determination of the mutual action could only be made under conditions which were in practice unattainable. The difficulty was finally overcome by ***Coulomb*** *(1785), who [by using very long and thin magnets, so arranged that the action of their distant poles was negligible] succeeded in establishing the law named after him. It stated that the force of attraction or repulsion exerted between two magnetic poles varies inversely as the square of the distance between them.*

Several previous attempts had been made to discover the law of force, with various results,some of which correctly indicated the inverse square law; in particular ***John Michell*** *(1750),* ***J. Tobias Mayer*** *(1760) and* ***Johann Heinrich Lambert*** *(1766) may fairly be credited with having anticipated the law which was afterwards more satisfactorily established by Coulomb*[344]*. The accuracy of this law was confirmed by* ***Gauss*** *(1832), who employed an indirect but more rigorous method than that of Coulomb.*

Gauss continued to work on terrestrial magnetism (1833)and other magnetic phenomena (1838).

H.C. Oersted *discovered (1819) that a magnet placed near a wire carrying an electric current tended to set itself at right angles to the wire, a phenomenon which indicated that the current was surrounded by a circulating magnetic field. The discovery constituted the foundation of electromagnetism*[345]*, and its publication (1820) was immediately followed by*

[344] **Joseph Priestley** showed (1767) that *electric charges* obeyed the Newtonian inverse-square force law. **John Michell** suspended a magnet by a thread and brought up another magnet, measuring the repulsive force between them by means of the twist imparted to the thread. **Coulomb** rediscovered Michell's torsion balance and with it, from 1785 to 1789, demonstrated the inverse-square law for *both* electrical and magnetic attractions and repulsions.

[345] The German natural philosophers, headed notably by **Friedrich Schelling** (1775–1859), believed that there was only one kind of power behind the development of nature, namely, that of the *World Spirit* (*weltgeist*). They thus held that light, electricity, magnetism, and chemical forces, were all interconnected; all where different aspects of the same thing. One of **Schelling's** disciples was

A.M. Ampère's *experimental and theoretical investigation of the mutual actions of electric currents (1820) and of the equivalence of a closed electric circuit to a polar magnet.*

In the same year ***D.F. Arago*** *(1820) succeeded in magnetizing a piece of iron by an electric current and in 1825,* ***W. Sturgeon*** *(1783–1850) exhibited the first actual electromagnet. The experiments of* ***Michael Faraday****, which ran from 1831 to 1895, established the phenomena of* electromagnetic induction, paramagnetism, diamagnetism *and* permeability, *describing electric and magnetic field in terms of 'lines of force'.*

The unification of all the known electric and magnetic phenomena was finally accomplished by ***James Clerk Maxwell*** *(1873), who translated Faraday's ideas into a mathematical form. Maxwell explained electric and magnetic forces, not by the action at a distance assumed by earlier mathematicians, but by stresses in a medium permeating all space, and possessing qualities like those attributed to the old luminiferous ether. In particular, he found that the calculated velocity with which it transmitted electromagnetic disturbance, was equal to the observed velocity of light. Hence he was led to believe, not only that his medium and the ether were one and the same, but, further, that light itself was an electromagnetic phenomenon.*

The hypothesis known as molecular theory of magnetism *originated with* ***Ampère*** *who proposed in 1823 that magnetism was due to electric currents circulating within matter. The idea was then developed further by* ***W.E. Weber*** *who suggested that the molecules of a ferromagnetic metal are small permanent magnets, randomly oriented under ordinary conditions. These notions were based upon two age old observations:*

- *An unmagnetized bar of steel can be made into a permanent magnet by stroking the bar with a loadstone. Careful investigation of the process reveals that nothing material has been imparted to the bar of steel. The presence of the loadstone apparently had only a* directing effect *on something already present in the steel bar.*

- *If we were to take a steel bar which has been converted into a permanent magnet and cut it in two, we would find two magnets, and, if the process were continued until molecular dimensions were approached, each resulting particle would prove to be a magnet. Thus, the steel bar, even in its unmagnetized condition, possesses magnetic particles of molecular or atomic dimensions distributed throughout the bar in perfectly random manner, so that the gross effect is zero magnetization. The presence of the loadstone with its magnetic*

Hans Christian Oersted (1777–1851), who announced (1807) that he was looking for the connection between magnetism and electricity.

field had the effect of aligning the elementary particles so that their magnetic axes are made parallel.

While the identification of the 'Amperian currents' with the motion of electrons had to await the discovery of the electron at the turn of the 20th century, Maxwell's electromagnetic field theory consolidated the understanding of the bulk (macroscopic) magnetic properties of isotropic matter that had already started with **Poisson**, **Gauss**, **Faraday**, **Weber**, **F.E. Neumann** and **Lord Kelvin**. *By 1873, the concepts of magnetic moment, magnetization, magnetic induction, magnetic susceptibility, paramagnetism, diamagnetism and permeability were in use both in theory and practice.*

(b) The Classical Theory

(i) Magnetostatics — the magnetic force

The expression for the magnetic force $\boldsymbol{F}$ *between two magnetic point-monopoles* m_1 *and* m_2 *at distance* r *apart (force on* m_2 *by* m_1*) is obtained from the Coulomb law*

$$\boldsymbol{F} = \frac{m_1 m_2}{\mu r^2}\boldsymbol{e}_r, \tag{1}$$

where $\boldsymbol{e}_r$ *is a unit vector directed from* m_1 *to* m_2 *and* μ *is a constant of proportionality, known as the permeability of the medium surrounding the magnet. It depends on the units chosen and also upon the medium between the poles. Using electromagnetic units (emu),* $\boldsymbol{F}$ *is in dynes and* r *in cm, and* $\mu = 1$ *for the vacuum.*

The poles themselves are a mathematical fiction, since they cannot exist isolated but only in pairs, and the force between two current loops is not even exactly a magnetic dipole-dipole force, except asymptotically for $r \gg$ *loop sizes. However, if we assume two very long bar magnets with two poles close together and the other two far apart, the situation is fulfilled in practice. The sign convention adopted is that a positive pole is one which is attracted towards the earth's north magnetic pole.*

(ii) Magnetic field-strength and the Lorentz force law

A pole of strength m is positioned at the origin of a coordinate system, embedded in an infinite medium of permeability μ. The force it exerts upon a unit pole at a position P, at position $\boldsymbol{r}$, defines the magnetic field-strength $\boldsymbol{H}$ at that point, namely

$$\boldsymbol{H} = \left(\frac{m}{\mu r^2}\right)\boldsymbol{e}_r, \tag{2}$$

with $\boldsymbol{e}_r = \frac{\boldsymbol{r}}{r}$, $r = |\boldsymbol{r}|$.

The force on a pole of m' units at P, will then be $\boldsymbol{F} = m'\boldsymbol{H}$. It is assumed that m' is not large enough to disturb the field $\boldsymbol{H}$ at the point of measurement, i.e., $m' \ll m$. In this way the notion of the field is decoupled from the source in the sense that it becomes a local property of space *(locally in space and time) divorced from the source that created it. In emu, $\boldsymbol{H}$ is in Oersteds = dynes per unit pole charge. The field $\boldsymbol{H}$ may also be generated by* current *flowing in a wire rather than by a pole or poles of magnetized material.*

According to the law of ***Biot-Savart*** *(1820), the magnetic field at position $\boldsymbol{r}$ due to an element $\boldsymbol{ds}$ of a straight line wire at the origin, through which a current J flows, is $|d\boldsymbol{H}| = \frac{J\,\boldsymbol{ds}\times\boldsymbol{e}_r}{4\pi r^2}$ with a corresponding force on a pole m' at $\boldsymbol{r}$ being $d\boldsymbol{F}' = m'\,d\boldsymbol{H}$. By Newton's law of action and reaction there must be a force on the current equal to*

$$d\boldsymbol{F} = -d\boldsymbol{F}' = J\,\boldsymbol{ds}\times\left(\frac{-m'\boldsymbol{e}_r}{4\pi r^2}\right) = J\,\boldsymbol{ds}\times\boldsymbol{B}(\boldsymbol{o}),$$

where $\boldsymbol{B}(\boldsymbol{o}) = \mu\boldsymbol{H}(\boldsymbol{o})$ and $\boldsymbol{H}(\boldsymbol{o})$ is the field strength at the current element due to m (according to Eq. (2)). This is a derivation of the Lorentz force law. *Note that Newton's third law can only be applied in the* magnetostatic *case ($\boldsymbol{J}$, $\boldsymbol{ds}$, m, $\boldsymbol{r}$ static), for otherwise the EM field can absorb momentum.*

Note that the molecular Amperian electric-current loops cause magnetic dipole moments, and spatial variations in the density $\mu\boldsymbol{M}$ of these magnetic dipole moments lead to additional atomic currents $\boldsymbol{j}_{\text{mag}} = c\,\text{curl}\,\boldsymbol{M}$. Accordingly we have from Maxwell's equations $4\pi(\boldsymbol{j} + \boldsymbol{j}_{\text{mag}}) = c\,\text{curl}\,\boldsymbol{B} - \frac{\partial\boldsymbol{D}}{\partial t}$, or

$$4\pi\boldsymbol{j} = c\,\text{curl}(\boldsymbol{B} - 4\pi\boldsymbol{M}) - \frac{\partial\boldsymbol{D}}{\partial t} = c\,\text{curl}\,\boldsymbol{H} - \frac{\partial\boldsymbol{D}}{\partial t}$$

with the new field $\boldsymbol{H} = \boldsymbol{B} - 4\pi\boldsymbol{M}$, analogous to the definition $\boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P}$, where $\boldsymbol{P}$ is the electric polarization – the density of electric *dipole moments.*

(iii) Magnetic moment, current loop, magnetic shell

Since the (fictional) magnetic poles always exist in pairs, the fundamental magnetic entity is the magnetic dipole; two poles of strength $+m$ and $-m$ separated by distance h. Then the magnetic moment is defined as $\boldsymbol{m} = m\,h\boldsymbol{e}$, the unit vector $\boldsymbol{e}$ extending from the negative pole towards the positive pole.

An example is a very short magnet in the form of a thin sheet with one face magnetized as an N-pole and the other as an S-pole.

Apart from permanent natural magnets (macroscopic, atomic or scales in between), a magnetic field is produced by an electric field and/or current: A small planar loop of wire with area ds and unit normal vector $\boldsymbol{n}$ carrying a steady current J in a counterclockwise direction about $\boldsymbol{n}$ behaves like a magnetic dipole of moment $d\boldsymbol{m}$ such that (in vacuo)

$$d\boldsymbol{m} = \lambda(J\,ds)\boldsymbol{n} = \lambda J\,d\boldsymbol{s} \tag{3}$$

Here λ is a constant of proportionality which can be set to $\lambda = 1$ by a proper choice of units[346].

If a current flows in a circuit C of finite area S (magnetic shell), it can be criss-crossed into a virtual net of tiny meshes, each with a current J flowing around it.

Each such mesh, at position $\boldsymbol{r}'$, has a magnetic moment $d\boldsymbol{m}$, with a magnetic potential $d\Omega = d\boldsymbol{m}\cdot\boldsymbol{\nabla}'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}$ where P is a point outside the circuit at position $\boldsymbol{r}$. The total potential is

$$\Omega(P) = \int_S d\boldsymbol{m}(\boldsymbol{r}')\cdot\boldsymbol{\nabla}'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = J\int_S \frac{\cos\theta}{|\boldsymbol{r}-\boldsymbol{r}'|^2}\,ds = J\int_S d\omega = J\omega, \tag{4}$$

where $\cos\theta = (\boldsymbol{r}-\boldsymbol{r}')\cdot\boldsymbol{n}$ and ω is the solid angle suspended at P by the loop C. The magnetic field of the circuit is given by $\boldsymbol{H} = -\nabla\Omega$. It is then shown that this magnetic field is given by the line integral $\boldsymbol{H}(\boldsymbol{r}) = J\oint_c \frac{d\boldsymbol{l}\times\boldsymbol{r}}{r^3}$, where $d\boldsymbol{l}$ is the vectorial loop element (Biot-Savart law), and the closed curve C follows the wire loop in the direction of positive current. The work done

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346 These are known as *electromagnetic units* e.m.u for short.
It can be shown that

$$1 \text{ e.m.u } = c \text{ e.s.u} \qquad c = 3\times 10^{10} \text{ cm/ sec (velocity of light in vacuum)}$$

$$1 \text{ ampere } = \frac{1}{10} \text{ e.m.u of current}$$

by the field $\boldsymbol{H}$ *can only be approximated by a dipole far from the loop, and furthermore* Ω *is not single valued: if a unit magnetic charge is carried in a closed loop* Γ *that links* C once *topologically (in a clockwise sense), the work done by the wire field is*

$$\Delta\Omega = \oint_\Gamma \boldsymbol{H} \cdot d\boldsymbol{l} = 4\pi J$$

(Ampere circuital theorem).

This result is compatible with (2) since $\Delta\omega = 4\pi$ *upon a complete traversal of* Γ*. For a current flowing in a straight wire, Ampere's theorem for* Γ *a circle of radius* r *about the wire, yields* $4\pi J = (2\pi r)H(r)$ – or $H(r) = \frac{2J}{r}$ *for the (circumferential) field at distance* r *from the axis.*

The magnetic moment of a particular volume *containing currents of density* $\boldsymbol{j}_m$ *is defined as*

$$\boldsymbol{m} = \frac{1}{2}\int (\boldsymbol{\xi} \times \boldsymbol{j}_m)\, dv \tag{5}$$

$$\boldsymbol{\xi} = \text{ coordinate vector inside the volume.}$$

If $\boldsymbol{j}_m = \rho\boldsymbol{u}$, $\rho =$ charge density moving with velocity $\boldsymbol{u}$,

$$\boldsymbol{m} = \frac{1}{2}\int \rho(\boldsymbol{\xi} \times \boldsymbol{u})\, dv \tag{6}$$

This is analogous to the expression for mechanical angular momentum $\boldsymbol{s}$ *in term of the velocity of a distribution of mass densities* ρ_m

$$\boldsymbol{s} = \int \rho_m(\boldsymbol{\xi} \times \boldsymbol{u})\, dv \tag{7}$$

It is convenient to define:

$$\Gamma = \frac{|\boldsymbol{m}|}{|\boldsymbol{s}|} = \text{ gyromagnetic ratio } = \frac{e}{2m} \text{ for an electron.}$$

This is the classical *result, and also holds for orbital electron motions inside atoms and molecules. For more complex structures* $\Gamma = g\frac{e}{2m}$*. Due to different masses and* g*-factors for different particles, atoms and molecules,* Γ *for a typical macroscopic bulk medium may be quite complicated to compute.*

For the intrinsic electron spin *contribution to material magnetic moments* $g = 2$*. For atomic nuclei, the* g *values are not well understood, though known empirically.*

(iv) Magnetic induction — flux density

If we imagine an isolated positive magnetic pole of m *units at the center of a sphere of radius* r*, then on the surface* $\boldsymbol{B} = \mu \boldsymbol{H}$ *points radially outwards and* $|\boldsymbol{B}| = \frac{m}{r^2}$*, where* μ *is the permeability of the medium.*

$\boldsymbol{B} \cdot \boldsymbol{n}\, ds$ *is the magnetic flux threading through a vectorial surface element* $d\boldsymbol{s} = \boldsymbol{n}\, ds$*, and* $\boldsymbol{B}(\boldsymbol{r})$ *is called the magnetic induction field. The unit of induction (in the unit system employed here) is called the gauss*[347]*. In air, for which* $\mu \approx 1$*,* B *and* H *are numerically equal.*

(v) Intensity of Magnetization: moment density; susceptibility

A magnetic body placed in an external magnetic field becomes magnetized by induction. The intensity of magnetization is proportional to the strength of the field and its direction in isotropic materials, is in the direction of that field. It is defined as the magnetic moment per unit volume, that is, $\boldsymbol{M} = \frac{\boldsymbol{m}}{v} = M\boldsymbol{e}$*. Practically, this magnetization by induction amounts to lining up the dipoles of the magnetic material: for this reason* $\boldsymbol{M}$ *is often referred to as the magnetic polarization. We write* $\boldsymbol{M} = k\boldsymbol{H}$*, where* k *is the magnetic susceptibility.*

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[347] *International Systems of units* (1974) are based on the following six basic entities:

Length = meter (m); *Mass* = kilogram (kg); *Time* = second (s); *Electric current* = ampere (A); *Temperature* = Kelvin (K); *Amount of substance* = mole (mol).

The derived units are:

Force = Newton (N) = $\frac{\text{kg}\cdot\text{m}}{\text{s}^2}$; *Energy, work* = Joule (J) = N · m;

Power = Watt (W) = J/s; *Electric charge* = Coulomb (C) = A · s;

Magnetic induction vector $\boldsymbol{B}$ whose unit is 1 Tesla = $\frac{\text{N}}{\text{A}\cdot\text{m}}$.

Another unit of this vector is the gauss (G) such that $1T = 10^4 G$;

also $1\gamma = 10^{-5} G = 10^{-9} T$.

Magnetic flux = $\int_S (\boldsymbol{B} \cdot \boldsymbol{n})\, dS$. Its unit is 1 Weber (Wb) = T · m^2.

Hence $\boldsymbol{B}$ has also the meaning of *magnetic flux density* with the unit 1 T = $\frac{1\ \text{Wb}}{\text{m}^2}$. Note that 1 Wb = Volt × sec.

Table 4.9: SUSCEPTIBILITY OF SOME COMMON SUBSTANCES

SUBSTANCE	SUSCEPTIBILITY, $k \times 10^6$
Cerium	100
Manganese	80
Chromium	26
Aluminum	1.7
Magnesium	1.2
Tin	0.2
Oxygen	0.15
Air	0.03
Silicon	-0.20
Water	-0.72
Copper	-0.80
NaCl	-1.0
Zinc	-1.0
Silver	-1.5
Mercury	-2.4
Gold	-3.1
Carbon	-8.0
Bismuth	-14.0

The value of k is positive for paramagnetic *substances and negative for* diamagnetic *substances. Most substances have a permeability which differs only very slightly from unity. Some examples are given in Table 4.9.*

Macroscopic permanent magnets (e.g. ferromagnets) may have a permanent component $\boldsymbol{M}_0$ to $\boldsymbol{M}$ but, unless saturated, also have a small-field susceptibility (which depends on $\boldsymbol{M}_0$):

$$\boldsymbol{M} - \boldsymbol{M}_0 = k\,\boldsymbol{P} + O(H^2).$$

In a non-isotropic medium, k should be replaced by a susceptibility tensor.

(vi) Diamagnetism

In 1778 ***Anton Brugmans*** *(1732–1789, Holland) observed that a small piece of bismuth was repelled when a magnet was brought up to it. When a ball of bismuth is suspended from a thread and brought between the poles of a powerful electromagnet, it is driven out of the field when the current is switched on.* ***Faraday*** *(1845) was the first to carry out systematic investigations of the magnetic properties of a range of substances.*

The behavior of Bismuth is due to its diamagnetic properties.

Diamagnetism is caused by the variation in the frequency of an electron circulating in an atom; it occurs upon a change in the magnetic induction due to the introduction of the atom into a magnetic field or during the ramping-up of a magnetic field (e.g. by an electromagnet). At the classical level, consider an undisturbed orbit in the xy plane and equate the centripetal force to some expression $f(r)$ *which is a function of the radius* r *only,* $m\omega_0^2 r = f(r)$. *If a uniform magnetic field* $\boldsymbol{B}$ *is now applied along the* z*-axis, this field will exert a force* $\boldsymbol{F} = e(\boldsymbol{v} \times \boldsymbol{B})$ *which is directed radially outwards if the electron revolves counterclockwise in the xy plane. The total force on each electron is therefore*

$$m\omega^2 r = f(r) \pm evB, \tag{8}$$

where $v = \omega r$. *For moderate fields* B, *the radius of the orbit will not change while the electron speed in its orbit will increase or decrease.*

Let $\omega = \omega_0 + \Delta\omega$. *Then*

$$mr(\omega^2 - \omega_0^2) = \pm e\omega r B$$

or approximately

$$\Delta\omega = \pm\frac{eB}{2m}, \tag{9}$$

independent of ω_0 *if* $|\omega - \omega_0| \ll \omega_0$.

Thus, an electron in a magnetic field acquires an additional angular velocity characterized by the frequency $\omega_L = |e|\frac{B}{2m}$, *known as the Larmor frequency. It can be shown that the angular-momentum vector assigned to any electron orbit precesses about the lines of force of an applied magnetic field where* ω_L *is the angular velocity of precession.*

Since the electron velocity in an atom placed in a magnetic field varies, its kinetic energy varies as well. On the other hand, since r *remains unchanged, the potential energy also does not change. But since the magnetic field is always perpendicular to the electron's velocity, the magnetic field does not*

perform any work. However, according to Faraday's Law of magnetic induction, a sudden change in $\boldsymbol{B}$ *gives rise to an* induced electric field *which must accelerate or decelerate the electron in its orbit.*

The angular momentum of the electron is $|\boldsymbol{L}| = |m(\boldsymbol{r} \times \boldsymbol{v})| = m\omega r^2$. *Its magnetic moment is*

$$|\boldsymbol{p}| = \pi r^2 \cdot \frac{e}{T} = \pi r^2 \cdot \frac{e\omega}{2\pi} = \frac{1}{2} e r^2 \omega = \frac{e}{2m} |\boldsymbol{L}|.$$

But the equation of motion of the electron is

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{N} = \text{ torque } = -\boldsymbol{p} \times \boldsymbol{B} = -\frac{e}{2m}(\boldsymbol{B} \times \boldsymbol{L}).$$

If we consider the electron's orbit as a perfectly rigid body rotating with angular velocity $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$, *its motion is given by the equation* $\frac{d\boldsymbol{L}}{dt} = \boldsymbol{\omega}_L \times \boldsymbol{L}$. *Therefore* $\boldsymbol{\omega}_L = -\frac{e}{2m}\boldsymbol{B}$, *and the whole atom precesses in a magnetic field like a gyroscope — the Larmor precession.*

Now, a circular current appearing as a result of the Larmor precession of each electron in an atom causes an additional magnetic induction due to this induction current which is directed against the magnetic vector of the external field. The magnetic moment of the atom, appearing as a result of the precession as well as the magnetization, is directed against the magnetic induction of the external field. *This is the basis if* diamagnetism. *It can be shown that the* diamagnetic susceptibility *is*

$$\chi_d = -\mu_0 \frac{ne^2}{2m} \overline{r^2} \qquad (10)$$

where n = *number of electron per unit volume,* $\chi_d = \frac{M}{H}$ *and* $\overline{r^2}$ = *mean square distance between the electrons and the nucleus.*

Typical values are $\overline{r^2} \approx 10^{-20}$ *meter,* $m = 9 \times 10^{-31}$ *kg,* $\mu_0 = 12.5 \times 10^{-7} \frac{\text{Weber}}{\text{Amp·meter}}$, $e = 1.6 \times 10^{-10}$ *Coul,* $n = 10^{28}/\text{meter}^3$, *so that* $\chi_d \sim 10^{-5}$.

The diamagnetic susceptibility is independent *of temperature since the thermal motion and collision of atoms are unable to draw them from the state of Larmor Precession for any appreciable time.*

(vii) Paramagnetism (Faraday, 1845)

Paramagnetic materials are substances whose molecules have a constant magnetic moment. The energy that the magnetic moment $\boldsymbol{p}$ *has in an external magnetic field is equal to* $W = -\boldsymbol{p} \cdot \boldsymbol{B}$. *The minimum value of energy is attained when* $\boldsymbol{p}$ *aligns with the direction of the magnetic field. In this case, when a paramagnetic material is introduced into a magnetic field, a preferred orientation of magnetic moments of paramagnetic atoms takes place in the direction of the magnetic induction in accordance with the Boltzmann distribution, and the body is accordingly magnetized.*

The additional induced field coincides with the direction of the external field and enhances it. The failure of the magnetic moments of individual atoms to fully align with the field results from collisions and interaction between atoms. The fraction that does line up depends on the ratio of the magnetic energy to the mean thermal energy, $f = \frac{pB}{3kT}$. *Here, the factor* $\frac{1}{3}$ *is geometrical, stemming from an average over dipole orientations. Thus* $M = npf = \frac{np^2B}{3kT} = \chi_p H$ *where* $\chi_p = \mu_0 \frac{np^2}{3kT}$ *(The Curie Law). At room temperature* $\chi_p \sim 10^{-3}$, *which is two orders of magnitude higher than the diamagnetic susceptibility.*

1859 CE **Raymond Gaston Planté** (1834–1889, France). Physicist. Invented the first practical *storage battery* (accumulator). In improved form, his invention has become the most wildly used *rechargeable battery*.

Planté was born in Orthez, France. His academic career began as a lecture assistant in physics at the Conservatory of Arts and Crafts in Paris (1854). He then became a professor of physics at the Association Polytechnique, Paris (1860).

Prior to Planté, primary cell batteries eventually lost all of their electricity when the chemical reactions were spent. Planté overcame this shortcoming by constructing his cell with two thin lead plates separated a rubber sheets. He rolled the combination up and immersed it in a dilute sulfuric acid solution. Initial capacity was extremely limited since the positive plate had little active material available for reaction. Therefore, Planté had positively charged one of his plates, making it lead oxide. The other was simply lead, which has a negative charge; The created flow of electrons from the negative plate was

taken out of the battery as electricity and then *fed back into the battery*, thus creating a rechargeable battery.

1859 CE **Jean Joseph Étienne Lenoir** (1822–1900, France). One of the first to build a practical internal-combustion engine. By 1860 he was producing reliable engines equipped with an electric ignition system and using coal gas as a fuel. Many hundreds of these engines were constructed; they were used throughout Paris. About 1863, Lenoir built one of the first *automobiles*[348] to use a gas engine. The engine was a one-cylinder unit of his own design. The efficiency of the engine was, however, very low and it traveled about 5 km per hour.

1859 CE **Richard Christopher Carrington** (1826–1875, England). Astronomer. Discovered the differential rotation of the sun about its axis, with equatorial period of about 25 days. This was measured in 1887 by means of a spectroscopic Doppler effect (light coming from receding and approaching limbs).

Galileo (1613) first demonstrated that the sun rotates on its axis, by recording the apparent motions of sun-spots as the turning sun carried them across its disc. He found that the rotation period was about four weeks. (A typical sunspot group lasts about two months, so it can be followed for two periods.)

Earlier, **Christoph Scheiner** (1573–1650, Germany) announced the discovery of sunspots (1611), although ancient Chinese astronomers recorded such sightings already in ca 1000 BCE.

Very sensitive Doppler shift measurements over the interval 1973–1977 showed an average period of 24.65 days at the equator and 35 days at latitude 80°.

The sun, with over 99.8 percent of the total mass of the solar system, contains only 2 percent of all its angular momentum. It rotates in the same direction as the planets revolve, about an axis inclined at an angle $82^\circ 49.5'$ to the ecliptic.

1859–1871 CE **Zénobe Théophile Gramme** (1826–1901, Belgium). Electrical engineer and inventor. Invented and built the first commercially practical generator for producing *alternating current* (1869) and *direct current* (1859). Opened factory (1871) to produce dynamos, armature-rings, etc.

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[348] Yet, no revolution in transportation occurred until four decades later because the public was content with horses and railroads!

1859–1874 CE **William Stanely Jevons** (1835–1882, England). Economist, logician and statistician. Pioneer of mathematical economics and the application of mathematics to political economy. His main contributions outside economics are in mathematical logic where he developed the 'logical piano', a machine with 21 keys for operations in equational logic. It has many features which were later incorporated into computer design.

Jevons was born in Liverpool and studied at University College, London (1851–1853, 1859–1862). He became a professor of political economy at Manchester (1866–1879) and University College (1876–1880). He drowned whilst bathing near Hastings at the age of 47.

Jevons developed *marginal utility theory of value*[349] (1862) and a *sunspot theory*[350] of business cycles.

1860 CE **Gustav Theodor Fechner** (1801–1887, Germany). Physicist, philosopher, psychologist. A founder of *psychophysics*[351]. Used the *Weber law*[352] of discriminants to *scale* responses to stimuli. The **Weber-Fechner** law states that in order that the intensity of sensation may increase in an arithmetical progression, the stimulus must increase in a geometrical progression. Hence, in general, if M denotes a suitable quantity for scaling sensation, we get

$$M = a \log s + b$$

where s is the magnitude of a measurable stimulus, and (a, b) are contents of a particular phenomenon.

For example, the subjective impression of loudness L (sensation) is related to the physical intensity of sound (stimulus) I, measured in Watt/m^2, via the experimental relation

$$L = a \log I + b.$$

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[349] It was not till after the publication of this work that Jevons became acquainted with the application of mathematics to political economy made by earlier writers, notably **Antoine Augustin Cournot** (1838) and **H.H. Gossen** (1854). The theory of utility was developed independently by **Carl Manger** in Austria and **M.E.L. Walras** in Switzerland (1870).

[350] Sunspots influence the weather, the weather affects the crops, and the crops affect business conditions. Although few economists accept this explanation, some even today devote their efforts to proving the connection between storms in the sky and storms in the business atmosphere.

[351] Usually this term is nowadays avoided because of metaphysical implications.

[352] *Weber's law*: "Noticeable differences in sensation occur when the increase of stimulus is a constant percentage (about 5) of the stimulus itself".

The constants a and b are determined in the following way: At a frequency of 1000 Hz, the threshold of audibility (lowest intensity that can be heard) is nearly $I_0 = 10^{-12}$ Watt/m^2. Then L is made equal to

$$L = 10(\log I - \log I_0) = 10\log(I/I_0).$$

The unit of L is called decibel[353]and abbreviated by dB. For $I = I_0$, L is zero decibel. Another application of the Weber-Fechner law is used in the dose-response relationship in biological assay, assuming that the *response of a chemical drug* (vitamin, hormone, poison etc) is linearly dependent on the logarithm of the dose.

Fechner discovered yet another important application of his law to astronomy[354]: the experienced *brightness* of a star is by no means proportional to the light energy received by the eye. Again, we have a linear relationship between *brightness* (sensation) and the logarithm of light *intensity* I (stimulus).

A standard formula is

$$m = x - 2.5\log I.$$

Here c is a constant determined by the unit in which I is measured, and m is called the *apparent magnitude* of a star [the magnitude of *Sirius*, the brightest star, is -1.6, *Vega* has magnitude 0.1, and Betelgeuse 0.9. The clumsiness of negative magnitude might have been avoided if the magnitude zero had been better placed].

Fechner was born at Gross-Särchen, near Muskau, in Lower Lusatia, where his father was a pastor. He was educated at the University of Leipzig, in which city he spent the rest of his life. Appointed professor of physics (1834), but in 1839 contracted an eye disease while studying the phenomena of color

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[353] 1 decibel= $\frac{1}{10}$Bel in honor of **Alexander Graham Bell** (1847–1922), the inventor of the telephone. For a tone of any frequency other than 1000 Hz the unit dB cannot be used for the human ear.

[354] Fechner found that the eye can distinguish two brightnesses if their ratio (not *difference* between them!) amounts to a definite and constant amount (the one at least about 5 percent greater than the other).
This explains the daytime disappearance of the stars: the difference in brightness between a star and its surrounding is always the same, but the ratio of the brightnesses in the *daytime* differs from that at night. As a rule, it may be said that our visual impressions are determined mainly by the brightness ratios. This aspect of our sense of vision is of the utmost importance for our daily life. Thanks to this, the objects around us remain definite, recognizable entities, even in changing conditions of illuminations.

and vision, and, after much suffering, resigned. Subsequently recovering, he turned to the study of the mind and the relations between body and mind. He set out to prove a universal parallelism expressible through a logarithmic function[355].

1860 CE Wet-plate photographs of the *sun's prominences* – the first significant astronomical result achieved by *photography* – taken during a total eclipse by **Warren de la Rue** (1815–1889).

The first astronomical photographs were taken by use of the daguerreotype process, and during the 1840s photographs were obtained of the *sun*, *moon* and *solar spectrum*. In 1850 the first successful *star photograph* was secured at the Harvard College Observatory. With the discovery of the wet collodion process (1851) more sensitive plates were made available though limited to an effective exposure time of ten minutes. During the 1870s and 1880s the wet plate was in turn supplanted by the *dry plate*, ushering in the modern era of astronomical photography since the exposure times of the dry plates could be extended almost indefinitely.

In the late 19th and early 20th centuries *photography* transformed the methods of astronomical investigation because instead of having to rely on visual observations astronomers were now able to record permanently the light from sources, inspecting the photographs at their leisure.

1860 CE **Stanislao Cannizzaro** (1826–1910, Italy). Chemist. Employed Avogadro's hypothesis in the determination of molecular weights of gaseous compounds by comparing the weight of a volume of gas to that of an equal volume of hydrogen. From molecular weights he proceeded to atomic weights, thus establishing the usefulness of atomic weights in determining the formulae for organic compounds.

1860–1861 CE **Johann Philipp Jacob Reis** (1834–1874, Germany). Physicist and teacher. Invented the first *electrical telephone*. It could transmit speech through a wire over 100 meters — a forerunner of Bell's telephone.

Reis was born to a Jewish family in Gelnhausen. He was a physics teacher at a private school near Frankfurt-am-Main, and designed and exhibited the telephone for the entertainment of his pupils. Although Reis lectured on and demonstrated his machine publicly, he was unable to realize its full potential; he died at age 40, after a long illness which eventually robbed him of his voice. Bell (1876) acknowledged that he drew upon Reis' ideas in the construction of his telephone.

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[355] Modern psycho-physics favors a *power form* of the law.

1860–1873 CE **Émile Léonard Mathieu** (1835–1890, France). Mathematician. Extended and developed the formulation and solution of PDE's for a wide range of physical problems. Discovered (1860, 1873) 5 transitive permutation groups: M_{12}, M_{11}, M_{24}, M_{23} and M_{22}, known as the *Mathieu groups*. These are simple groups with exceptional properties. In the context of the 20^{th} century classification of all finite groups, these are a subset of the *special groups*. The best-known of his achievements are the *Mathieu functions*, which arise in solving the two-dimensional wave equation for the motion of an elliptic membrane (1868).

Mathieu was born in Metz. He was a student at the École Polytechnique in Paris and took his D.Sc. in 1859. He worked as a private tutor until 1869, when he was appointed to a chair of mathematics at Besancon. He moved to Nancy in 1874, and remained as a professor there until his death.

Mathieu's shy and retiring nature have accounted, to some extent, for the lack of worldly success in his life and career.

1860–1889 CE **Enrico Betti** (1823–1892, Italy). Mathematician. Noted for his contributions to algebra and topology. Derived important theorems in the mathematical theory of elasticity (*Betti's relation*, *Betti's reciprocity theorem*). In 1871 he did pioneering work in topology (*Betti's numbers*)[356] and wrote the first rigorous exposition on the theory of equations developed by **E. Galois**. Betti thus made an important contribution to the transition from classical to modern algebra. He showed (1854) that the quintic equation could be solved in term of integrals resulting in elliptic functions.

Betti was born near Pistoia, Tuscany. He studied at the University of Pisa where he rose to the rank of professor of mathematics in 1857. Under his leadership the Scuola Normale Superiore in Pisa became the leading Italian center for mathematical research education. Along with **Brioschi** and **Casorati** he visited mathematical centers in Europe (Göttingen, Berlin, Paris) making many important mathematical contacts. His work in the theory of elasticity was inspired by **Bernhard Riemann** who had visited Pisa in 1863. In 1874 he served for a short time as undersecretary of state for public education. He died in Pisa.

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[356] The (first) *Betti number* of a surface is the largest number of cross cuts which can be made without dividing the surface into more than one piece. The concept was extended by Poincaré to n-dimensional manifolds, where $n+1$ Betti numbers are defined; b_i is the number of independent, boundary-less i-dimensional submanifolds that are not themselves boundaries of any $(i+1)$-dimensional submanifold. One has $b_0 = b_n = 1$ and $b_i = b_{n-i}$, and the alternating sum of Betti numbers is the Euler characteristic of the whole manifold. Thus for $n = 2$ (a surface), $b_0 = b_2 = 1$, $b_1 = 2h$ (number of handles), and $b_0 - b_1 + b_2 = 2(1-h) = \chi$, the Euler characteristic.

Time's Arrow

The fundamental laws of motion (both classical and *relativistic), optics and electromagnetism that deal with macroscopic physics are governed by second order partial and ordinary differential equations in space and time. These equations remain equally valid when the direction of time is reversed. All phenomena described by these equations are therefore* reversible *in time.*

However, in spite of the fact that these equations permit two symmetrical solutions, natural macroscopic processes governed by these equations are for the most part irreversible.

*Thus, electromagnetic theory (using Maxwell's equations) is as compatible with the outflow of light from stars as with its inflow into them. (*retarded *potential solutions vs.* advanced *potential solutions). Yet we never see light flow* into *a star; nature seems to reject the advanced potential solution. This circumstance creates asymmetry between time and space coordinates, and also between past and future along the time axis itself.*

Physics had, therefore, to devise yet another kind of law *to account for the unidirectional trend of events in the universe – the* arrow of time. *It turned out that the observed irreversibility of natural phenomena emerges almost automatically when we begin to consider the coarse-grained stochastic evolution of large* aggregates *of particles, events or processes — despite of the fact that the underlying microscopic processes (even at the quantum-mechanical level) are* individually reversible[357]. *What Newton's, Maxwell's and Einstein's macroscopic laws and the governing equations of quantum mechanics had completely ignored, is accounted for by the science of* statistical mechanics. *Its laws are* statistical, *the laws of* crowds of events. *A crowd of individually reversible processes becomes irreversible in the bulk.*

By visualizing any macroscopic object as consisting of a very large, but finite, *number of atoms, molecules and/or other fundamental, few-degrees-of-freedom constituents, one can derive the behavior of substances in thermal equilibrium (or even away from equilibrium in some cases) by application of statistics. In particular, the non-occurrence of large fluctuations in the temperature distribution of a system in equilibrium can be understood through considerations of probabilities. Indeed, if one simplifies the description of a gas in thermal equilibrium as consisting of a number of fast molecules and an equal number of slow molecules in random motion within a box, one* expects *to*

[357] Except for a sub-variety of weak nuclear force, of negligible relevance in most situations where a statistical time asymmetry arises.

find about equal ratios of fast and slow molecules in any given sizable portion of the box.

Even if a very large number of observations were made, it would be very improbable to ever find a state realized in which all fast molecules are accidentally in one half of the box and the slow molecules in the other half. Quite generally, the probability of finding a particular macroscopically-described state of the gas will be proportional to the number of microscopic *states of realizing that coarse-grained state, and the gas tends towards macroscopic states of ever-increasing probabilities. The entropy of a state should be connected in a quantitative way with N, the number of possibilities of realizing that state; this connection would then furnish a statistical-mechanical explanation of the 2^{nd} thermodynamical law (non-decrease in entropy of a closed system).*

To find that connection, consider two separate systems, each in thermal equilibrium, so that the numbers of possibilities of realizing the respective states are N_1 and N_2, and the entropies of the respective systems are S_1 and S_2. If the two systems are completely independent of each other and boundary effects are neglected, the entropies will simply add: $S = S_1 + S_2$. The number of possibilities of realizing the combined state, N, however, is equal to the product $N = N_1N_2$.

Thus, if there is any connection between S and N at all, S must be proportional to the logarithm of N, because no other function $f(x)$ satisfies the functional relation $f(x_1x_2) = f(x_1) + f(x_2)$. One thus has the famous relation, due to ***Boltzmann****:*

$$S = k \log N,$$

where k is a positive proportionality factor depending on the units in which S is measured. In the mks (SI) system of units k has the value $k = 1.38 \times 10^{-23}$ J/°K.

Unfortunately, these considerations prove to be of little help when one tries to answer two questions, arising from the existence of irreversible processes in nature:

(1) Why is the part of the actual world in which we find ourselves in a state that appears to be very far from a state of thermal equilibrium? [Indeed, if we were not living in a part of the universe very far from statistical equilibrium we would not be here to speculate about this question, since living organisms and their habitats, including the earth as a whole, are of necessity open, far-from-equilibrium thermodynamical systems.]

(2) *Why does one observe in irreversible bulk processes, starting from initial states that are not produced by extremely unlikely statistical fluctuation, an increase and never a decrease in entropy? [No one has ever observed measurable entropy decrease in a closed macroscopic system.]*

Neither of these two questions would pose a conundrum if the present actual state of our (observable) part of the universe were *the result of a statistical fluctuation.*

Indeed: if the highly ordered state we notice about us – at the terrestrial, astronomical and cosmological scales – is the result of a statistical fluctuation, then there is an overwhelming probability that the parts of the universe we have not yet looked at should be in a state of higher entropy, and appear less ordered, than the part we are seeing now.

Judging by past experience, when the part of the universe accessible to observation was much smaller, we venture to predict, however, that every new advance in the art of telescopy will reveal, as it has time and again in the past, new distant parts of the universe that are at least as far away from thermal equilibrium as we are.

All astronomic experience indicates that the universe as a whole is in a state far away from equilibrium, and has been in such states for a long time. It should be admitted that a satisfactory answer to question (1) has not yet been found.

In some authors' opinion the universe should not be expected to attain a state resembling thermal equilibrium, on the grounds that the universe is not an isolated system in a constant environment — because the all-pervasive gravitational field and the cosmological expansion furnish an environment that cannot, in principle, remain constant in time.

But even if one could resolve the problem posed by question (1), there would remain the puzzle of question (2). Practically all initial states that lead to observation of irreversible processes (for example melting of an ice cube) are states that were prepared *by us or by natural processes and are just* not *picked at random from an ensemble of possible states. Apparently, an initial state (whether of an ice cube in this epoch, or of the entire universe at the time of the Big Bang) carries within itself the template for its development, when left alone, into states of higher entropy.*

This apparent "miracle" of the arising of an arrow of time in macroscopic systems is then ultimately traceable to the assumption that the initial conditions of the universe are microscopically *random. As time wears on, the interactions in the universe (or in a closed subsystem) gradually manifest*

this randomless macroscopically, which we perceive as an increase in disorder (entropy).

This is despite the fact that the laws of mechanics (even quantum mechanics) allow in principle the existence of prepared initial states which, when left alone, would evolve into states of lesser entropy.

In some authors' opinion the arrow of time observed in laboratory experiments is tied up with the arrow of time of the universe as a whole; in such a view there is a mechanism, perhaps gravitation, which makes it in principle impossible to truly isolate a system, and which imprints the arrow of time of the entire universe on all its parts.

Others, reluctant to accept the implication that what happens here on earth inside an isolated box, containing water and an ice cube, should be tied up with what happens inside another isolated box, with water and ice cube, on some planet in some distant galaxy, feel it ought to be possible to find sufficient reason for the arrow of time by considering only relatively small isolated systems. Thus question (2) is still open.

An individual molecule has no way of distinguishing between the two directions of time, and its behavior (apart from some tiny components of the weak nuclear force) is described by the basic time-symmetric laws of nature. Nothing prevents us from speculating about whether individual particles might be able to travel backwards in time, since there are no laws of physics that forbid motion backward in time[358].

A theory of this sort was proposed in 1949 *by* ***Richard Phillips Feynman*** *(1918–1988, U.S.A.) to explain pair-creation and pair-annihilation (transformation of one or more γ-ray photons into an electron and positron, or conversely the collision of the latter two to produce γ-ray photons and/or other quanta) by regarding a positron as a negative-energy electron traveling backwards in time (time-reversal compensating for charge and energy reversal). Thus, both of the above processes can be described by a single particle performing occasional "reflections" into the past*[359].

Though the arrow of time can disappear on the subatomic level, large fluctuations are extremely unlikely on the macroscopic or even mesoscopic

358 If we allow such motion in a theory of physics, however, care must be taken to avoid paradoxes, such as a person traveling back in time and killing one of his grandmothers in time to render his own existence paradoxical! Such care must be exercised at the microscopic, as well as the macroscopic, level.

359 Feynman's theory is causal, i.e., it allows the future (*or* past) quantum states to be determined from an arbitrary initial (or final) state, so it leads to no paradoxes. Recently, however, the possibility that Einstein's theory of gravity might allow time paradoxes, has aroused renewed interest.

level. [Ice cubes are not created by chance fluctuations — they are made in refrigerators; wineglasses are not created by chance — they are manufactured.] This guarantees that the arrow of time will not disappear in the world around us, and although it depends on statistical averages, it is very real.

Moreover, the arrow of time exists also in the universe as a whole, i.e. the direction of time is not a local phenomenon. When astronomers look at the universe, they see low entropy in the past, and they are justified in expecting that there will be higher entropy in the future. The cosmos, unlike the electron, is subject to the arrow of time.

The Rise of the New World,[360] *II*
The immigrants (1860–1924)

The first U.S.A. census in 1790 *recorded a population of almost* 4 *million, of which ca* 700,000 *were Negro slaves.*

Almost $\frac{1}{4}$ *of the white population was of non British ancestry: Swedish, Polish, German, Italian and Dutch, French Huguenots, Jews from Spain and Portugal, Scotch-Irish Presbyterians and others.*

The first great wave of the 19^{th} *century immigration (1830–1860) brought* 2 *million people, half of which were Catholic Irish. Thousands of English, Scottish, Welsh, French, Dutch and Swiss settlers assimilated with relative ease into American communities established by their predecessors. The abortive* 1848 *revolution in Germany brought thousands of its exiled leaders, students, intellectuals and artisans. By* 1900 *some* 5 *million more Germans (mostly peasants, but also tradesmen and craftsmen) followed.*

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360 For further reading, see:

- *The Story of America*, Reader's Digest Association: New York, 1975, 527 pp.

Beginning in 1825, “American fever” also swept Scandinavia and Finland, where political turmoil, overcrowding, feudalism and a series of poor harvests spurred a peasant exodus, that by 1920 *would bring to the United States a million Swedes and another million Norwegians, Danes and Finns.*

The dizzying rush of industrialization following the Civil War, the Homestead Act of 1862 *and the growth of a railroad network that made virtually every region accessible to settlement — all these provided new allures for Europe’s poor. Agents for U.S.A. railroads, shipping companies and other industrial concerns fanned out across Europe.*

This contributed to history’s most massive immigrant tide: 31 million people arrived in America in the period 1860–1924. *The big wave of the 1880’s was made up largely of Germans, English, Scandinavians, and Canadians. During the peak years of* 1900–1920, *more than* 3 *million immigrants came from Italy alone. Another* 3 *million came from the heart of the crumbling Austro-Hungarian Empire. An additional* 3 *million arrived from Russia and Poland, mostly persecuted Jews. [In 1907, the record year for immigration to the United States, the newcomers came primarily from Russia, Central Europe, and Italy.] During the same two decades, well over* 5 *million more people came from Britain, Scandinavia, Germany, France, Portugal, Greece, Armenia, Canada, Mexico and other Latin American countries.*

By 1890 *the population shot up to* 63 *million and by* 1924 *to a total of* 115 *million. Altogether,* 40 *million people immigrated to the U.S.A. since its birth in* 1776.

Most of the immigrants were uneducated, unskilled, poorly clothed, destitute or nearly so. The vast majority settled in the industrial Northeast: most of the rest went to the industrial regions of the North Central States. By 1912, *for example, there were men and women from* 25 *nations, speaking* 45 *tongues, manning the looms of Lawrence, Massachusetts.*

The isolationist fervor that gripped the nation after WWI, *compelled congress to pass a law in* 1921, *that for the first time restricted the number of immigrants.*

1860–1871 CE **Hermann Hankel** (1839–1873, Germany). Mathematician. Contributed to theory of functions, special functions and the history of mathematics.

Hankel was born at Halle. His father was a professor of physics at Halle and Leipzig. Hankel acquired a considerable knowledge in Greek at the Leipzig Gymnasium and improved upon it by reading the ancient mathematicians in the original. He studied in the University of Leipzig under **Möbius** and at Göttingen under **Riemann** (1860). The following year he studied in Berlin with **K. Weierstrass** and **L. Kronecker**, and in 1862 received his doctorate at Leipzig. In 1867 he became a full professor at Erlangen.

In 1869 he was called to Tübingen, where he spent the last four years of his life. His most important contribution to mathematics was his development of the theory of Bessel functions (1869), mainly integral representations and asymptotic expansions. In honor of Hankel, **Nielsen** denoted the linear combinations $J_\nu(z) \pm iY_\nu(z)$ by the symbol H_ν (1904), and they are known today as "Bessel functions of the third kind" or simply *Hankel functions*.

1860–1867 CE **Benjamin Peirce** (1809–1880, U.S.A.). Mathematician. Developed the theory of linear associative algebra, a classification of all complex associative algebras of dimension less than seven. He used the, now familiar, tools of idempotent and nilpotent elements. Defined mathematics as "*the science which draws necessary conclusions*". He worked on a wide range of mathematical topics from celestial mechanics and geodesy on the applied side to algebra and number theory on the pure side.

Peire graduated from Harvard and became a professor there. He established the Harvard Astronomical Observatory and helped determine the orbit of Neptune.

1861 CE *Massachusetts Institute of Technology* (M.I.T.) founded in Boston. It moved to its present location on the Charles River in Cambridge, Mass., in 1916.

1861 CE **Julius Weingarten** 1836–1910, Germany). Mathematician. Worked on the fundamental equations of differential geometry. *Weingarten equations* and *Weingarten surfaces* are named after him. Received his Ph.D. from Halle (1864).

1861 CE **Pierre-Paul Broca** (1824–1880, France). Surgeon and anthropologist. Discovered seat of motor control of speech in the brain, now referred to as 'Broca's area'.

Broca was born in the township of Saint-Foy-La-Grande and studied medicine in Paris.

1861 CE The interpretation of **R.W. Bunsen** and **G. Kirchhoff** of the *Fraunhofer lines* in the solar spectrum marked the beginning of modern spectroscopy, and provided the first observation that led in 1913 to the Bohr model of the atom.

1861–1865 CE *The American Civil War*[361]. A tragic conflict between northern and southern states over the issue of black slavery and also due to economic rivalry between the industrial north and the agricultural south. It took more American lives than any other war in American history. It ended the southern way of life that depended on slave labor in the cotton and tobacco fields and cemented the union of states. It granted freedom to the American black people but not equality.

In 1860, Abraham Lincoln, a Republican whose party wanted to limit slavery, was elected president. Afraid of being outnumbered by non-slave states, 11 southern states separated from the union (23 states) into a new nation, the *Confederate States of America*. Lincoln refused to recognize this secession, and fighting broke in April 1861.

The Civil War was expected to be a brief conflict in which the immense advantages of the North in resources and manpower would prove decisive. But the South, having the advantage of fighting on its own soil and superior commanders, put up a valiant fight, and in the early part of the war won some brilliant victories.

Nobody would have predicted that the war would last four years and would turn into one of the most costly military ventures up to that time. The confederacy finally surrendered to the Union forces in April 1865; ca 618,000 had died (out of a total fighting force of some 2.5 million soldiers).

The total cost of the war is estimated at 15 billion dollars (1976). Many southern cities and towns were destroyed and the economy of the South almost completely collapsed. The victory of the North was achieved mostly due to the military competence of Ulysses S. Grant. It was the first war to apply new warfare technological such as telegraphy, photography, balloon reconnaissance, repeated rifles, trenches, railroad transportation, wire entanglement, submarines and armored vessels.

1861–1879 CE **William Crookes** (1832–1919, England). Physicist, chemist, and inventor. Discovered the element *thallium* (1861), invented the *radiometer* (1873), and investigated passage of electrical discharge through

[361] Johnson, Paul, *A History of the American People*, Harper Collins, 1998, 1088 pp.

highly rarefied gases. He also pioneered investigations of *cathode rays* in high-vacuum tubes of his own design (1878), now known as *Crookes' tubes.* This latter work led directly to the discovery of the electron by **J.J. Thomson**.

Crooks was born in London and studied at the Royal College of Chemistry. He set up (1856) a private laboratory in London and made his living as a chemical consultant and editor of scientific journals. In 1861, while conducting a spectroscopic examination of the residue left in the manufacture of sulphuric acid, he observed a bright green line which has not been noticed previously. He then succeeded in isolating a new element, *thallium.* He served as president of the Royal Society (1913–1915).

Between 1874 and 1876 the scientific world has been stirred by Crookes' experiments with the *radiometer.* This device is composed of partially evacuated chamber containing a paddle wheel with vanes blackened on one side and silvered on the other, which spins rapidly when radiant heat impinges on it.

As the invention of the radiometer came shortly after the publication of Maxwell's *Treatise*, some persons (Maxwell included) though that the motion of the wheels can be ascribed to *light pressure*, but the forces were much greater than predicted from the electromagnetic theory, and in the wrong direction. It soon became evident that the effect is due to the residual gas. The observed rotation occurs because the light heats up the black faces more than the white ones. Molecules in the residual gas that drift up against a black (hotter) side of a vane therefore get a stronger kick than from a white vane, and the corresponding stronger recoil drive the rotation with the black sides receding[362].

1861–1884 CE **Carl von Voit** (1831–1908, Germany). Physiologist. Conducted, with the assistance of **Max von Pettenkofer** (1818–1901, Germany) pioneering experiments in animal and human *metabolism*, making first measurements of *energy* requirements and determinations of oxygen and nutrients utilization. Professor at Münich (1863–1908).

von Voit was influenced by the conceptions of energy that had become dominant in physics and chemistry at that time. He was a trained physician, but studied chemistry under Liebig. He began his physiological work

[362] The detailed quantitative behavior of Crookes' radiometer was much investigated during 1873–1930. Many theoretical papers were written, including important ones by **J.C. Maxwell** (1879), and **A. Einstein** (1924). Many experiments were performed [e.g. **H. Marsh** et al. (1925). A detailed discussion with many references is given in **I.B. Loeb**, *The Kinetic Theory of Gases* (1934, pp 364–388) and in **R.W. Wood** *Physical Optics* (1934, p 794).

by establishing the fact that healthy adult animals are normally in *nitrogen equilibrium* (excepting as much nitrogen as they take in).

In 1865 he showed that combination with oxygen was *not* the first step in energy production, but that a large number of *intermediary substances* were formed from the original food before the final union with oxygen occurred. Not all these intermediates were necessarily oxidized completely. Hence, oxygen did not cause metabolism. Much of modern biochemistry consists of the search for these products of *intermediate metabolism.*

1862 CE Astronomers **Alvan Clark** (1808–1887, USA) and his son **Alvan Graham Clark** (1832–1897, USA) discovered *Sirius B*, a *white dwarf.* Alvan Sr. was also a lens maker; his firm, Alvan Clark and Sons made the 66 cm refraction telescope[363] for U.S. Naval observatory and the 91 cm telescope for the Lick Observatory. His son discovered 16 double stars and made the 102 cm lens for the Yerkes telescope (1897).

Sirius consists of a pair of stars for which individual masses can be determined. The brighter component had been known since 1844, to be moving across the sky in a sinuous path, and it was rightly surmised that this sinuous motion was the result of orbital motion around an unseen companion. Its magnitude is +8.7, some 10,000 times fainter than its bright companion. This makes it very difficult to observe except when the two stars are farthest apart.

1862–1871 CE **Ernst Felix Immanuel Hoppe-Seyler** (1825–1895, Germany). Physiologist and chemist. A founder of *physiological chemistry.* Identified[364] and isolated (1862) *hemoglobin* as the oxygen-carrying substance. Prepared a crystalline form of hemoglobin (1862) and was able to show

[363] A *refracting telescope* consists of a large long-focus-length objective lens and a small, short-focus-length eyepiece that magnifies the image formed by the focus of the objective lens. The magnification, or *magnifying power* of a refracting telescope is equal to the focal length of the objective divided by the focal length of the eyepiece lens. *Chromatic aberration* is the most severe of a host of optical problems that must be solved in designing a high-quality refracting telescope. During the 19th century, master opticians devoted their lives to overcoming those problems.

Modern astronomers in the 20th century lost interest in this type of telescope, since all its technical shortcomings can be avoided by using *mirrors* instead of lenses. There are now 14 refractors around the world with objective lenses larger than 65 cm in diameter.

[364] This was independently done at about the same time by **Otto Funke** (1828–1879, Germany.

that hemoglobin contains a compound called 'heme' as part of its structure. ['Heme' was not an amino acid, but a rather complex atom grouping containing an *iron* atom.] Measured metabolic phenomena in isolated tissues. Discovered (1871) the enzyme *Invertase*, that speeds up conversion of sucrose into glucose and fructose.

Hoppe-Seyler was born in Freiburg-an der-Unstrut. He was a professor at Tübingen (1861–1872) and Strasbourg (1872–1895).

1862–1887 CE **Julius (von) Sachs** (1832–1897, Germany). Botanist and plant physiologist. The creator of experimental botany. Contributed to all branches of botany and his name will always be associated with the great development of plant physiology which marked the latter half of the 19th century. Under his general and enthusiastic leadership, Wuerzburg became an international center of plant physiology where some of Europe's most eminent botanists were trained.

Sachs discovered that:

- *photosynthesis* occurs in the chloroplasts (the structure in the plant cell containing the green pigment *chlorophyll*) and produces oxygen. Specifically: chlorophyll is the key component that forms CO_2+water into starch while releasing oxygen (1832).
- *Starch* in chloroplasts results from absorption of CO_2; a simple iodine test can be used to show the existence of starch in a whole leaf. Starch is then translocated from the leaf in the form of sugar.
- *Light* is necessary for the synthesis of chlorophyll.

Sachs also pioneered in studies of the nutritional requirements of plants: he published the first formula for a standard culture solution, a necessary basis for identifying the mineral elements essential for growth.

Sachs was born to a poor Jewish family in Breslau, Silesia (also Wroclaw, Poland). Left high-school early because of the death of his parents. He managed to find a job as an assistant to the physiologist **Johannes Purkinje** in Prague (1850) and was later able to complete his schooling. He attended Prague University, graduating with PhD (1856). Became a professor of botany at Wuerzburg (1868), remaining there for nearly 30 years.

Science and Musicology

"Music is a secret arithmetical exercise and the person who indulges in it does not realize that he is manipulating numbers".

Gottfried Wilhelm von Leibniz (1646–1716)

Although music played an important role in the cults of many ancient civilizations[365]*, it was the Greeks who first discovered the mathematical basis of music, and thus laid the foundations of the scientific development and evolution of this discipline.*

The Greek $\mu o \upsilon \sigma \iota \kappa' \eta$, from which music *is derived, was used very widely to embrace all those arts over which the Nine Muses ($\mu o \upsilon \sigma \alpha \iota$) were held to preside. Contrasted with gymnastics it included those branches of education concerned with the development of the mind as opposed to the body. Thus, such widely different arts and sciences as* mathematics*,* astronomy*,* poetry*,* literature*, and even reading and writing would all fall under $\mu o \upsilon \sigma \iota \kappa' \eta$, besides the singing and setting of lyric poetry. The philosophers placed special emphasis on the educational value of music in the formation of character, and this attitude affected their aesthetic analysis. $A \rho \mu o \nu \iota \alpha$ (harmony) was the name given by the Greeks to the art of arranging sounds for the purpose of creating a definite aesthetic impression.*

[365] In ancient *Egypt*, priests trained choirs in singing ritual music, and court musicians sang and played reed pipes and string instruments such as lyres, lutes, and harps as early as 4000 BCE. In *Babylonia*, court musicians played ornate instruments in the 2600's BCE. The Bible contains the words of many *Hebrew* songs and chants, such as the Psalms. It mentions harps, drums, trumpets, cymbals, and other instruments. The music in Solomon's Temple at Jerusalem in the 900's BCE included trumpets and choral singing to the accompaniment of stringed instruments [*I Chron* **25**, mentions a total of 228 skilled musicians in the service of the temple]. The ancient kingdoms of the Mediterranean recognized dance and music as integral forms of celebration. The early *Chinese* thought that music reflected the order in the universe (Chinese music used the five-tone scale. It had no half-steps, and sounded somewhat like the 5 black keys of the piano). Musical traditions in *India* go back to the 1200's BCE. Hindus believed that music was directly related to the fundamental processes of human life. They worked out musical theories by about 300 BCE. Their music was not, however, based on a system of whole steps and half steps, like the diatonic scale.

The Greeks were first to use letters of the alphabet to represent musical tones. They grouped the tones in tetrachords *(succession of 4 tones). The first and fourth tones have a relationship somewhat like that between C on the piano and the next F above. By combining these tetrachords in various ways, the Greeks created groups of tones called* modes. *Modes were the forerunners of the more modern* major *and* minor *scales.*

Trepander of Lesbos *(710–670 BCE, Sparta) is known as the Father of Greek music. He founded at Sparta the earliest musical school of Europe. He improved the* cithara *(7-chord), by adding a note at the top of the scale and leaving out the third from the top, so as to attain an octave with one note of the scale omitted.*

Pythagoras *(fl. 532 BCE) founded the mathematical theory of acoustics and music. He established the 7 note* diatonic musical scale *based on the primes* $\{2, 3, 5\}$*; the ancient Greeks, with their abundance of string instruments, discovered that when a string (or a flute) is shortened to half its length, the resulting tone, when played with the original one, resulted in a pleasant musical sound. Similar experiments with length combinations of* $3\colon 2$ *(fifth),* $4\colon 3$ *(fourth), and* $5\colon 4$ *(third) also resulted in luminous sounds. This led Pythagoras and his followers to believe that music and mathematics provide keys to the secrets of the world.*

Hippasos of Metapontium *(ca 500 BCE) developed the theory of the* harmonic mean. *To the three consonant intervals: octave, fifth and fourth he added the double octave and the fifth beyond the octave.* ***Archytas of Tarentum*** *(fl. ca 390 BCE) was a mathematician, mechanician, statesman and Pythagorean philosopher. He was a friend of Plato and founder of theoretical mechanics. Ptolemy called him the most important theoretician of music of the Pythagorean school. He calculated numerical ratios for new musical scales by means of arithmetical harmonic means.*

Plato *(in* Timaeus, *ca 380 BCE) believed that the 7 notes of the musical scale also embodied the intervals between the 7 known planets as viewed from an earth-centered perspective (Mercury, Venus, Mars, Jupiter, Saturn,*

the sun, and the moon), which he later referred to (in the Republic*) as the "harmony of the Spheres"*[366].

Aristoxenes of Tarentum *(fl. ca 320* BCE*) was a pupil of Aristotle, philosopher and mathematician, and the greatest theoretician of music in ancient times.* **Nicomachos** *(fl. ca 100 CE) wrote a manual on harmony which is our oldest source on Pythagorean music.* **Theon of Smyrna** *(fl. 127–132 CE) was a Platonic philosopher who developed the mathematical theory of music.*

The Italian monk ***Guido d'Arezzo****, in ca* 1025*, laid the basis for modern musical notation: the four-line staff, the F clef and the notes ut (do), re, mi, fa, sol,* and *la.*

The theory of music was considered a part of mathematics almost until modern times. It was one of the main ingredients of medieval education.

The greatest explorer of musical physics *in recent times was* ***Hermann von Helmholtz*** *(1821–1894), a German physicist who who wrote* The Sensations of Tone *(1862), the foundation book on sound as it is made and heard. He was also a talented pianist with a thorough practicing knowledge of his subject from several points of view. In an address given in* 1878*, the great Scottish physicist* ***James Clerk Maxwell*** *said: "Helmholtz, by a series of daring strides, has effected a passage for himself over that untrodden wild between acoustics and music — that Serbonian*[367] *bog where whole armies of scientific musicians and musical men of science have sunk without filling it up".*

The science of music developed due to the combined efforts of physicists and musicians, who complemented each other. Thus, we can place opposite Helmholtz, the musical scientist*, the* scientific musician ***Theobald Boehm***

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[366] These connections deeply influenced the *neoplatonists* of the Renaissance who felt that, as a result of this connection, the soul must have some kind of ingrained mathematical structure. Plato's emphasis on the importance of ratio of small integers had the greatest influence on Renaissance architecture.

[367] Maxwell chose his words with skill: Herodotos described the engulfing of armies in the mud of Lake Serbonis, a lake in the northern Sinai peninsula, which has since dried up. Many early musical instruments were made of the kind of cane which grows to this day in swampy places around the Mediterranean.

(1794–1881, Germany). The latter was trained as a goldsmith in the family business, but very quickly showed his ability on the flute. He made solo concert tours for several years before beginning to feel that many limitations of the flute of his day could be remedied.

Between 1830 *and* 1850 *he performed an extensive series of carefully chosen experiments, guided by what little acoustic theory was available to him, and in the end produced an instrument which was essentially the same as the modern flute. Not only does this instrument show the excellence of his researches into sound, but it also reveals Boehm as a first-class engineer. Most of the improvements which took place in the construction of other woodwinds during the last half of the* 19^{th} *century are directly descended from his ideas and from the stimulus of his success.*

The effect of a musical sound upon our ears depends first and foremost upon its frequency (pitch), i.e., the number of vibrations per second of the body emitting the sound. To say that we hear middle C*, is to say that our ears register* 256 *vibrations per second. Therefore a number is associated with each sound, and conversely, with each number, is associated a sound. For musical purposes however, we are led to employ only a limited number of sounds (frequencies), in each octave (if a string produces a certain note, half the length will produce the octave). Among the* 300 *discernible sounds in an octave, one must choose a series of monotonically increasing frequencies (scale)*[368].

Once a scale was adopted, it became practically impossible to change it; the continuity of musical life requires the continuing use of the same scale or those with practically negligible differences — which is in fact what has happened in the course of the history of Western music; nearly 2500 *years have passed and the present day scales are in fact only variations of the Greek scale.*

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368 *Colors*, too, are differentiated by their frequencies. Indeed our eyes and ears are analyzers of frequencies, but while a painter can put colors of any frequency whatever on his canvas, a composer cannot place sounds of arbitrary pitch in his work. Why is this? First, he must write his music and for this he needs a discrete alphabet, or else he would need an infinite number of symbols; deciphering such notation would be very slow in any case. Second, music is made to be played, and the large majority of our instruments can produce only a limited number of sounds. Finally, our ear is incapable of discerning two sounds that are too close, which makes it fruitless to use all frequencies. It is agreed that a practical ear can distinguish about 300 sounds in one octave; this is still too much for musical notation and for the capacity of the instruments (a concert piano of 8 octaves would have 2400 keys and a total length of some 100 meters, exceeding by far the hand span of a human pianist).

The reasons for this consistency are rooted in the nature of the human auditory system (anatomy of the ear and the signal processing in the brain); when two notes are played together or in succession, the resulting sound is generally more harmonious to the ear when the corresponding frequencies are in simple ratios, and much music takes advantage of this fact: particular intervals sounding especially harmonious are those with frequency ratios of $2\colon 1$ *(octave),* $3\colon 2$ *(perfect fifth),* $4\colon 3$ *(perfect fourth), and* $5\colon 4$ *(major third).*

In general, the aesthetic effect of a chord depends almost exclusively on the ratio of frequencies. *The whole question of harmony is therefore a question of the choice of* ratios[369]. *Furthermore, two sounds will be more agreeable to the ear, especially if heard* simultaneously, *to the extent that they offer harmonics in common (thus, unpleasant sound consists of incommensurable frequencies). That is why the A produced by a musical instrument such as a violin sounds richer and more pleasant than the rather mechanical sound of a pure A from a tuning fork. When two or more notes are sounded together from different instruments, the ear is pleased if their various overtones have a harmonic relationship. The more combinations there are of overtones that are the same as each other, or 2 or 4 times the frequency of each other, the better the ear likes it.*

The Greek diatonic *scale consists of 7 tones within its* octave *(Latin, for* 8^{th}*). If we normalize the lowest frequency to unity, the scale will consist of the numbers* $\big\{1, \frac{9}{8}$ *(major interval),* $\frac{5}{4}$ *(major third),* $\frac{4}{3}$ *(fourth),* $\frac{3}{2}$ *(fifth),* $\frac{5}{3}$ *(sixth),* $\frac{15}{8}$ *(seventh), 2 (octave)*$\big\}$*. In the* Just C Major *scale, the absolute frequencies corresponding to these notes are* $\{264, 297, 330, 352, 396, 440, 495, 528\ Hz\}$.

These are denoted by the letters $\{C_4, D_4, E_4, F_4, G_4, A_4, B_4, C_5\}$ *respectively, or by the names* $\{$*do-re-mi-fa-sol-la-ti-do*$\}$*. These notes are produced by the* white keys *of the piano keyboard. The sequence of frequencies of the above scale is neither arithmetic nor geometric, but the subset* $\{264, 330, 396\}$ *is geometric (with ratios of* $4\colon 5\colon 6$*) and known as a* major triad. *Since these notes have many harmonic overtones that match each other, tonal clusters which contain them are most pleasant to the ear.*

Note that the subsets $\{352, 440, 528\}$ *and* $\{396, 495, 594\}$ *are also major triads. If it were not for the preference of our ears, it might seen logical to divide the octave into 8 equally spaced tones. However, this is not the way the standard scale works. As one goes up the scale one meets 3 different ratio's of*

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[369] For that reason, an *interval* between two musical notes is understood as the *ratio* of the frequencies of these two notes.

frequencies: $\frac{9}{8} = 1.125$ *(known as a whole tone or major whole),* $\frac{10}{9} = 1.111$ *(minor whole), and* $\frac{16}{15} = 1.067$ *(half tone or major semitone).*

For instance, the ratio of frequencies between D and C (re and do) is $\frac{9}{8}$*. The next step up to mi is almost as large,* $\frac{10}{9}$*. As one goes on to fa, the step is only half a tone (as re has a frequency about 12% larger than the do, whereas the frequency of the fa is only* $6\frac{1}{2}\%$ *higher than the mi). As we proceed on to sol, la and ti, we go up whole tones, but between ti and do there is only a half tone again*[370].

The tones of the diatonic scale make up a specific pattern of whole steps $(C-D, D-E, F-G, G-A, A-B)$ *and half steps* $(E-F, B-C)$*. The first tone of the scale gives the scale its name. For example, the D major scale will start with* $D = 297$ *Hz and proceeds with the same intervals (ratios) as the C scale over the frequency set* $(297, \overline{334}, \overline{371}, 396, \overline{445}, 495, \overline{557}, 594,$ *where the bars indicate rounding-off to the nearest integer).*

Likewise, the E major scale proceeds over the set

$$\{330, 372, \overline{413}, 440, 495, \overline{550}, \overline{618}, 658\ Hz\}.$$

The Greeks also used the Minor diatonic scale, in which the intervals are the same as in the Major scale, but in a different order. Thus, for the C minor scale, the normalized set is $\left\{1, \frac{9}{8}, \frac{6}{5}, \frac{4}{3}, \frac{3}{2}, \frac{8}{5}, \frac{9}{5}, 2\right\}$*, with the corresponding frequencies* $\{264, 297, \overline{317}, 352, 396, \overline{422}, \overline{475}, 528\ Hz\}$*. This combination of frequencies includes three subsets with frequency ratios* $10\colon 12\colon 15$*, known as minor triads. They are:* $\{264, \overline{317}, 396\}$*,* $\{352, \overline{422}, 528\}$*,* $\{396, \overline{475}, 594\}$*. The matching harmonics of this scale also sound pleasant to the ear, but western culture interprets the result as a sad sound.*

If we wish to play more than one scale (say, the above C Major, D Major, E Major and C Minor) on a single musical instrument, certain complications arise; changing from one key to another introduces new tones slightly different from the corresponding tones of the former key, while preserving the ratios. If a person is playing an instrument with continuous tuning like a slide trombone or a violin, and that person is very skillful, he can make the slight adjustment so that the frequency ratios of the notes are exactly right.

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[370] Note also that $D = \frac{1}{2}(C+E)$, $E = \frac{1}{2}(C+G)$, $F = \frac{2C_4C_5}{C_4+C_5}$ (harmonic mean), $G = \frac{1}{2}(C_4+C_5)$. With the piano, the C major scale resides in the *white keys*. If one wishes to test the pleasantness of a scale made exclusively of whole tones, one will have to use some of the *black keys*, which does not sound better!

With valve instruments, however, that is harder to do, and with a piano it is impossible. If we had to have a different set of keys and strings on the piano for every key that might be played, the piano could not fit into a living room and no human would be able to play it. The most common tuning system alleviating this problem is called the equally tempered scale[371]*: the octave is divided into twelve equal intervals* $\frac{1}{12}$ *octave apart.*

An interval with a frequency ratio of $2^{1/12} = 1.0595$ *is called a half step and corresponds approximately to a semitone* $\frac{16}{15} = 1.0666$.

Any two half steps approximate a major interval
$\left(2^{2/12} = 1.1225 \approx \frac{9}{8} = 1.1250\right)$,
any four a major third $\left(2^{4/12} = 1.2599 \approx \frac{5}{4} = 1.2500\right)$,
any five a fourth $\left(2^{5/12} = 1.3348 \approx \frac{4}{3} = 1.333\right)$,
any seven a fifth[372] $\left(2^{7/12} = 1.4893 \approx \frac{3}{2} = 1.500\right)$,

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[371] The first mention of temperament is found in 1496, in the treatise *Practica musica* by the Italian theorist **Franchino Gafori**.

[372] The relation $2^{7/12} \approx \frac{3}{2}$ was known to the Greeks; the Pythagoreans asked themselves whether an integral numbers of octaves could be constructed from the fifth alone by repeated application of the simple frequency ratio $\frac{3}{2}$.
In mathematical notation, the Greeks sought a solution of the equation $\left(\frac{3}{2}\right)^n = 2^m$ in positive integers n and m. But the equation $3^n = 2^k$ has *no* integer solutions for $n > 0$. However, the Greeks were not discouraged and, by trial and error, found the excellent approximation $\left(\frac{3}{2}\right)^{12} \approx 2^7$, which is based on the near equality of $3^{1/19}$ and $2^{1/12}$. A systematic way of finding such near-coincidences is based on the continued fraction expansion (Daniel Shanks, *A Logarithm Algorithm*, *Mathematical Tables and Other Aids to Computation* **8**, 60–64, 1954)

$$\log_{a_0} a_1 = \frac{\log a_1}{\log a_0} = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \ddots}}},$$

where $\{n_1, n_2, n_3, \ldots\}$ is a sequence of positive integers, and $a_0 > a_1 > 1$. The n_i are determined by the relations

$$a_i^{n_i} < a_{i-1} < a_i^{n_i+1} \qquad \text{and} \qquad a_{i+1} = a_{i-1}/a_i^{n_i}, \qquad i = 0, 1, 2, \ldots.$$

Thus, for $a_0 = 3$, $a_1 = 2$ the n_i sequence $[1, 1, 1, 2, 2, 3, 1, \ldots]$ yields an excellent approximation for the *musical fifth*: $\log 2/\log 3 \approx 12/19$, from which the Greek result, $\left(\frac{3}{2}\right)^{12} \approx 2^7$, follows!

any nine a sixth $\left(2^{9/12} = 1.6818 \approx \frac{5}{3} = 1.6667\right)$,
and any eleven a seventh $\left(2^{11/12} = 1.8877 \approx \frac{15}{8} = 1.8750\right)$.

A piano keyboard has 7 white keys and 5 black keys (12 in all) per octave and can be tuned with such a scheme. Insofar as the human ear cannot

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Other results obtained through the above algorithm are

$$\log 3/\log 5 \approx 13/19 \quad \text{or} \quad 3^{1/13} \approx 5^{1/19},$$

$$\log 3/\log 7 \approx \frac{13}{23} \quad \text{or} \quad 3^{1/13} \approx 7^{1/23}.$$

These relations find application in musical scales with frequency ratio 3: 1 (called *tritave*) for which the basic frequency ratio of a tempered scale is $3^{1/13} = 1.088\ldots$.

Note that $\frac{p_k}{q_k}$, the k^{th} convergent to $\frac{\log a_1}{\log a_0}$, is given by

$$p_k = n_k p_{k-1} + p_{k-2}, \quad q_k = n_k q_{k-1} + q_{k-2}$$

$$(n_0 = 0, \quad p_{-2} = 0, \quad p_{-1} = 1, \quad p_0 = 0, \quad p_1 = 1;$$

$$q_{-2} = 0, \quad q_{-1} = 0, \quad q_0 = 1),$$

where $\{p_k, q_k\}$ are the respective *continuants*:

$$p_k = \begin{vmatrix} n_2 & 1 & & & & & \\ -1 & n_3 & 1 & & & & \\ & -1 & n_4 & 1 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & & n_{k-1} & 1 \\ & & & & & -1 & n_k \end{vmatrix}, \qquad k = 2, 3, \ldots;$$

$$q_k = \begin{vmatrix} n_1 & 1 & & & & & & \\ -1 & n_2 & 1 & & & & & \\ & -1 & n_3 & 1 & & & & \\ & & -1 & n_4 & 1 & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & & n_{k-1} & 1 \\ & & & & & & -1 & n_k \end{vmatrix}, \qquad k = 1, 2, 3, \ldots.$$

perceive the discords caused by the deviations of the tempered ratios for fifths and fourths from their ideal values, the scheme is satisfactory, although to some trained listeners the discord in the major third is on the limit of unpleasantness.

Note that since the fundamental interval is an irrational number, the tempered scale does not possess any simple interval (ratio), a fact which would have driven Pythagoras to despair! Moreover, the notes of which it is composed have no harmonics in common, which is very far from the physicist's conception of the affinity of sounds. Thus the tempered scale is clearly based upon a more complicated mathematical conception than the diatonic scale and could not have been conceived before the invention of logarithms, when nobody could calculate $2^{1/2}$*!*

With this system only 12 *notes are needed to play all the tones and half tones of a full scale. The frequencies of the* 4^{th} *octave are taken as*

$$\{262, 277, 294, 311, 330, 349, 370, 392, 415, 440, 466, 494, 523\ Hz\}$$

with the corresponding notation

$$\{C, C_{\#} = D_b, D, D_{\#} = E_b, E, F, F_{\#} = G_b, G, G_{\#} = A_b, A, A_{\#} = B_b, B, C\}.$$

The compromise in intonation is rarely greater than 1% for all scales, but the tones of the well-tempered scale are harsh when compared with tones corresponding to the ratio of small integers of the Just scale. A close rational approximation to the well-tempered scale is

$$\left\{1, \frac{16}{15}, \frac{8}{9}, \frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{45}{32}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \frac{7}{4}, \frac{15}{8}, 2\right\},$$

where the circles denote black keys.

Bela Bartok *(1881–1945, Hungary and USA) based the entire structure of his music on the* golden mean $\left[\frac{1+\sqrt{5}}{2} = 1.6180339\ldots\right]$ *and* Fibonacci series. *His interest in folk music led him to realize that most Eastern European folk music lies outside the traditional major-minor system. He therefore formed his own type of harmonic system, one which could accommodate melodies not based on a major-minor tonality.*

Bartok used the pentatonic scale *which is perhaps the most ancient human sound system and lies at the basis of the oldest folk melodies and the simplest nursery songs. It rests on the Fibonacci sequence* $\{2, 3, 5, 8\}$, *where the numbers are interpreted as the* number of semitone intervals *separating a*

note from the fundamental tone in the 12*-tone chromatic (well-tempered) scale. The black keys on the piano make up a pentatonic scale (successions of* 2 *and* 3 *halftones are the intervals between the black notes).*

Bartok also used Fibonacci numbers in an another way. Roughly speaking, the fabric of his music may be imagined to be built up of cells 2, 3, 5, 8 *and* 13 *in size, i.e., the minor second (2 halftones), minor third (3 halftones), fourth (5 halftones), minor sixth (8 halftones), and augmented octave (13 halftones).*

In musical notation the notes are: D, E flat, F, A flat, C sharp or $\{\frac{9}{8}, \frac{6}{5}, \frac{4}{3}, \frac{8}{5}\}$. *Bartok contrasted this Fibonacci scale with a scale obtained by subtracting his halftone series from the* 12*-tone chromatic scale. The result is (with the exception of one term, the major second* $\frac{9}{8}$, *and with accuracy of 2%) the arithmetic series* $\{1, \frac{9}{8}, \frac{10}{8}, \frac{11}{8}, \frac{12}{8}, \frac{13}{8}, \frac{14}{8}, \frac{15}{8}, \frac{16}{8}\}$. *We may consider this scale as being based on the overtones of the fundamental note. Thus, the chromatic scale can be separated into Fibonacci and overtone scales, each being a part of the whole and neither of which can exist apart from the other.*

The revolution in orchestral textures (which was one of the most characteristic achievements of the Romantic composers) was only possible because of the intense period of mechanical invention in the early decades of the 19^{th} *century, which led to radical changes in the wind section of the orchestra. Indeed, during the relatively short interval* 1820–1847, *the flute, clarinet, oboe, and bassoon were improved by means of various mechanical and structural innovations*[373].

There appeared new instruments such as the 4*-octaves saxophone (1840,* ***Adolph Joseph Sax****, 1814–1894, Belgium), the harmonica (1829,* ***Charles Wheatstone****, 1802–1875, England), the modern orchestral xylophone (1840), the accordion (1822,* ***Friedrich L. Buschmann****, Germany), and the celesta (1886,* ***Auguste Mustel****, France). The revolutionary development in the brass instrument was the introduction of the valve by* ***Heinrich Stölzel*** *(1818, Germany).*

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[373] *Clarinet*: Originated ca 2700 BCE in Egypt. Modernized in 1690 CE by **Johann Christoph Denner** (Germany). The orchestral flute was introduced in 1843 by **H. Klose** and **A. Buffet** in France; Range — 2 octaves.

Oboe: Originated ca 4000 BCE in Egypt. Its final orchestral form was shaped in 1876 in France; Range — 3 octaves.

Bassoon: Originated ca 1500 by **Afranio**, canon of Ferrara. The double bassoon was introduced in 1620 by **Hans Schreiber** in Germany. The Contra Bassoon was introduced in 1739 by **Stanesby** (England). Improved for orchestral use by the physicist **G. Weber** (Germany) in 1825; Range — $3\frac{1}{2}$ octaves.

Flute: Orchestral version introduced in 1847 by **Theobald Boehm**, with improved acoustics and mechanical hole control; Range — 3 octaves.

Every musical instrument consists of some source of vibrations and some arrangement for transmitting the energy to the air with reasonable efficiency. The basic vibration is rarely sinusoidal, but instead consists of a series of pulses of various shapes. The vibrating system is part of or is connected to a resonating system with its own pattern of preferred frequencies of oscillation.

The resonating system responds to the fundamental and overtones of the driving pulse and radiates these into the air with varying efficiency (spectral amplitude response) depending on the frequency of each and the shape of the resonator. The initial vibrator may be a stretched string, or vibrating lips, or vocal chords, or turbulent air in a constricted channel. The resonating system may be a carefully shaped wooden box that can vibrate in response to a whole range of frequencies or it may be a column of air enclosed in a pipe that will respond only to certain harmonic frequencies[374].

Thus, an open end[375] *instrument like a* bugle *of effective length* $L = 1.86\ m$ *produces the fundamental mode* $f_0 = 92\ Hz$ *(taking* $c = 343\ \frac{\mathrm{m}}{\mathrm{s}}$ *for the speed of sound), and its harmonics* $f_n = 184, 276, 368, 460, 553$ *corresponding to the five notes* $F_{\#}(185)$, $C_{\#}(278)$, $F_{\#}(370)$, $B_{\#}(476)$ *and* $C_{\#}(555)$. *A bugler can sound these different pitches (the fundamental sounds like a* nonmusical *growl, and is called the* pedal note*). The F sharp is the lowest* musical *pitch, and comprises two pressure pulses at a time in the tube.*

Since most wind instruments act like pipes open at both ends (with the driving vibration occurring at one end), the fundamental wavelength is approximately twice the length of the pipe, and the frequency of the lowest note that can be used is usually twice that of the pedal note. For that note, 2 different pulses are introduced into the instrument during the time for one round trip.

The length of the pipe is changed by pressing down valves in the case of the trumpet*, by sliding out a length of pipe in the case of the* trombone*, or by opening and closing holes in the side of the tube, which effectively changes its length, in the case of the* flute *or* clarinet*. With the violin, the lowest note produced by the G string is about 4 times the length of the case.*

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[374] In the *mechanical siren*, for example, compressed air blows through holes in a rapidly revolving disc. The frequency is strictly determined by the number of holes passing the air blast each second. Invented in 1820 by the physicist **Charles Cagniard de La Tour** (France).

[375] A *closed end pipe* of length L accommodates a series of harmonics with wavelength $\lambda_n = \frac{4L}{2n+1}$ $(n = 0, 1, 2, 3, \ldots)$, where $\lambda = 4L$ for the fundamental mode. An *open end pipe* of length L has $\lambda_n = \frac{2L}{n+1}$ $(n = 0, 1, 2, 3, \ldots)$ with $\lambda = 2L$ for its fundamental mode.

The fundamental frequency corresponding to the lowest frequency for the human singing voice is around 60 *Hz for a low base, while for the highest pitch it is about* 1300 *Hz for a very high soprano voice. The range of any voice, however, rarely exceeds* 2 *octaves, although there are Iranian singers capable of* 3 *or* 4 *octaves and there is at least one opera singer (**Yma Sumac**[376], b. 1924) capable of a possible range of* 5 *octaves.*

The frequency ranges for musical instruments are much wider than for voice. The frequency corresponding to the lowest pitch that the ear recognizes as sound is about 30 *Hz, yet the piano goes down to a frequency of* 27 *Hz, and some organs descend to* 8 *Hz. At the other extreme, the piano can get up to a frequency of* 4186 *Hz* ($7\frac{1}{2}$ *octaves). There are organs with pipes* 18 *mm long that can go to* 8372 *Hz (the frequencies referred to are those of the fundamental tones, not the higher overtones). In orchestral instruments, the lowest frequency is carried by the harp (32 Hz) and bass viola (41 Hz). The piccolo at* 3729 *Hz has the highest fundamental frequency of the orchestral instruments. Overtones that accompany the fundamentals can go beyond the limits of hearing (20,000 Hz).*

The practical invention of the pianoforte (Hammerklavier) *is due to the Italian **Bartolomeo di Francesco Cristofori** (1655–1731,* Padua) *who in* 1709 *had the idea of combining in one the qualities of two keyboard instruments then in use: the clavichord and the harpsichord. Of the former he borrowed the action (struck string), but replaced the metal blade which set the strings in vibration by wooden hammers whose heads were covered with leather. Of the second, the harpsichord, he retained a row of jacks which, fitted with cloth, formed the dampers.*

Although all the mechanical elements of the modern pianoforte are found in Cristofori's early specimens, the pianoforte did not have much initial success; its wide dynamic range did not quite compensate for the dullness of its tone in comparison with that of the harpsichord. However, during the 18^{th} *century its mechanism was further perfected and by the* 1770*'s **J.S. Bach**, **Haydn** and **Mozart** were writing for it.*

*The London cabinet-maker **John Broadwood** invented (1783) the sustaining and damper pedals. By* 1802 *the harpsichord no longer appears in the titles of Beethoven's works and the piano was established in the concert hall. The piano soon became the instrument of the Romantic composers and performers like Chopin, Liszt, the Schumanns, Mendelssohn and others.*

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[376] Known as the '*Voice of the Incas*'; Allegedly born in a village in the Peruvian High Andes. The combination of her extraordinary voice, exotic looks and stage personality, made her a hit with American audiences.

Vibrations of pianoforte strings: The quantitative laws governing the vibrational frequencies have been known since 1713, *and can be summarized as follows: the frequency of a string rises proportionally with the square root of the tension, varies inversely with its length, and for fixed tension and length, is inversely proportional to the string's diameter.*

If all strings are to have the same diameter and tension, it turns out that a string-length of 5 cm *for the highest* C *entails a length of ca* 7 *meters for the lowest* C*! It was also found that one cannot vary the tension parameter much without affecting low tone quality. If, to avoid these difficulties, one tries to increase the diameter of the strings at the lower end of the scale, one finds that the lowest "wires" would have to be little steel bars nearly as thick as a pencil, with a resulting horrible tone. In addition, a wire of this thickness, length, and tension vibrates at a frequency considerably higher than the simple formula* $f_0 = \frac{1}{Ld}\sqrt{\frac{F}{\pi\rho}}$ *(ideal string of length* L*, diameter* d*, mass per unit volume* ρ*, stretching force* F*;* ***Brook Taylor****, 1713) would lead us to expect.*

A real wire*, on the other hand, has some stiffness in an amount that increases with the diameter. Such a wire vibrates under the influence of two sets of forces: one set arising from the string tension, and the other from its stiffness. As a result, the vibrational shapes and frequencies of all the various modes of oscillation are of a sort of intermediate between those of flexible string under tension and those of a stiff but unstretched bar whose ends are clamped. The equation of motion is*

$$\rho\frac{\partial^2 y}{\partial t^2} = \frac{F}{S}\frac{\partial^2 y}{\partial x^2} - Qk^2\frac{\partial^4 y}{\partial x^4}$$

where S *is the area cross-section,* k *its radius of gyration, and* ρ *and* Q *are the density and modulus of elasticity of the material, respectively.*

The usual boundary conditions, corresponding to a wire clamped at both ends, lead to the approximation for the fundamental mode $f = f_0(1+\epsilon)$*, where* $\epsilon^2 = 4\frac{QSk^2}{L^2F}$ *and* $\epsilon \ll 1$*. To get a good* musical string *(in which harmonics are whole-number multiples of the lowest frequency) the piano maker wants to use as thin and flexible strings as possible that can stand the highest possible tension.*

Most of the difficulties mentioned above were resolved by practical men over the years without help of formal technical knowledge. In general, the piano makers have had to back away from the ideal in order to get an instrument of practical size, and have arranged things so that the heavily used middle of the piano is good, while the tone quality at the high and low ends is gradually spoiled; down to an octave below the middle C*, the strings are*

lengthened by a factor of 1.94 *instead of* 2 *per octave, their diameter is increased by 9.3% per octave, and the tension is reduced to the proper amount to bring the string into tune.*

Below this point the wires lengthen very little, and the pitch is lowered by using strings wound closely with copper wire, in such a way as to add the mass without increasing the stiffness unbearably. The diameter, and therefore the stiffness, of the last unwound string is chosen to match the stiffness of the first wound one, so that an even-sounding scale is obtained.

Around middle C the notes of almost any honestly built piano can sound musical and clear, because the string proportions have not been compromised very much. At the bass end of the scale, however, anyone can hear the difference between the noble sounds issuing from a first-class concert grand and the clumping noises from shrunken pianos sold as pieces of furniture. The reason is that the bass strings of a concert grand are about twice as long as they are in a small piano, so that they can be pulled up to four times tension if the thicknesses are the same.

On a concert grand the tension may go as high as 200 *kg per string, and the total pull distributed over the frame is about* 20 *tons! With forces like these to contend with, a piano designer must be a good mechanical engineer as well as a capable vibration physicist.*

A piano string is set into forceful vibration by means of a hammer blow. The place where the hammer strikes, the fact that the hammer may be soft or hard, wide or narrow, and the strength of the blow are among the prime factors that determine the spectral content of the pulse around the center frequency of the struck key.

***Thomas Young** discovered (1800) that no possible mode of vibration can be set up that has a node at the position at which the disturbance is applied. **Helmholtz** (1862) advised to place the hammer strokes* $\frac{1}{7}$ *to* $\frac{1}{9}$ *along the length of the string because that was the distance that would make the* 7^{th} *and* 9^{th} *harmonics weak and since these are less consonant, the tone would be more pleasant without them. Others have found that placing the hammer close to, but not at* $\frac{1}{8}$ *of the length, enhances the fundamental and gives a strong and full tone.*

The quality of the tone is due to the strength and number of the harmonics, and these change with the speed of which the hammer strikes the string. However, when notes are struck simultaneously and in succession, then the time intervals between them may generally determine the quality of the tone produced. In general, the duration of a harmonic depends on its amplitude and because of this, weak harmonics die out sooner than strong ones. The

duration may be extremely short and a fraction of a second difference between the onset of two notes can alter the total picture of the number of harmonics that are present at any instant. It is probably the variation of these extremely small time sequences between succeeding notes that is meant by touch.

The motion of violin strings: The violin is a peculiarly shaped box on which 4 strings are stretched so that they run over a bridge that couples their vibrations to the box and its enclosed air. Sounds are brought out of this device by rubbing the strings with a bow, and the player chooses different pitches by pressing down with his fingers on one or the other of the bowed strings in such a way as to shorten its vibrating length. The 4 strings of the violin are tuned a fifth apart at G_3, D_4, A_4 *and* E_5.

The highest fundamental that can be played is B_7 *(3951 Hz). The G-string is wound with a spiral of fine aluminum or silver wire to provide it with sufficient mass to have the desired low fundamental. The violin string is excited by the friction of the moving bow*[377]*. The number of bow hairs in*

[377] The motion of the bow-driven string with friction, moving between two rigid supports, is governed by the equation:

$$\frac{\partial^2 y}{\partial t^2} + 2k\frac{\partial y}{\partial t} - c^2\frac{\partial^2 y}{\partial x^2} = f(x,t),$$

where y is the string's lateral displacement, T the fixed tension of the string (given), $c = \sqrt{\frac{T}{\mu}}$, $k = \frac{R}{2\mu}$, $f(x,t)$ is the applied force per unit mass of the string [with $f(x,t) = 0$ for $t < 0$], R is the frictional force of the medium per unit length per unit velocity, and μ is the string's mass per unit length. The solution can be given in the form

$$y(x,t) = \int_a^b G(x, x_0; t)dx_0,$$

where $G(x, x_0; t)$ is a solution of the above equation for a point force at $x = x_0$, namely

$$\frac{\partial^2 G}{\partial t^2} + 2k\frac{\partial G}{\partial t} - c^2\frac{\partial^2 G}{\partial x^2} = \delta(x - x_0)f(x_0, t),$$

and (a, b) is that part of the string over which the force is applied. It can be shown that if

$$f(x_0, t) = f(t) = \frac{1}{2}P_0 a e^{-a|t|},$$

then for $t > 0$,

$$y(x, x_0; t) = \frac{2P_0 a^2 c^2}{L}\sum_{n=1}^{\infty}\left\{\frac{\sin(\pi n x/L)\sin(\pi n x_0/L)}{\omega_n W_n^2}\right\} e^{-kt}\sin(\omega_n t - 2\phi_n)$$

contact with the string is changed by varying the angle and the plane that the bow makes with the strings. Studies on bowing have shown:

(1) increased bow pressure tends to increase the intensity of the Fourier components above the fundamental;

(2) intensity of tone depends on the speed of bowing and the number of bow hairs in contact with the string;

(3) the closer the bow is to the bridge, the more prominent are the higher Fourier components.

The air in the box is capable of vibrating at any of a large number of resonant frequencies. If the motion of the wooden enclosure is at one or another of these cavity resonance frequencies, the air will radiate strongly into the room by way of the f-holes cut in the belly of the violin. The top of the bridge vibrates back and forth on its slim waist when it is driven at certain frequencies, and thus alters the forces which it passes on to the violin belly in a way strongly depending on frequency. Thus, the sound radiated from the violin's body is controlled by the bow-string-bridge system. Physicists have obtained the response of a violin to a single exciting frequency (not the same as that for a bowed violin):

The results show a peak at about 300 Hz (D_4), which is due to the air resonating in the body of the violin, and another due to the vibration of the body of the violin at about 440 Hz (A_4). These peaks may be up or down by at least a semitone in various different violins. A comparison between good and poor violins shows that fine instruments seem to have frequency-response

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$$+\frac{P_0 a e^{-at}}{2k'_a}\begin{cases}\dfrac{\operatorname{sh}(k'_a x)\operatorname{sh}[k'_a(\ell-x_0)]}{\operatorname{sh}(k'_a L)}, & x<x_0\\[2ex] \dfrac{\operatorname{sh}(k'_a x_0)\operatorname{sh}[k'_a(\ell-x)]}{\operatorname{sh}(k'_a L)}, & x>x_0\end{cases}$$

where

$$k'_a=\tfrac{a}{c}\sqrt{1-\tfrac{2k}{a}},\qquad \omega_n=\tfrac{\pi nc}{L}\sqrt{1-\left(\tfrac{kL}{\pi nc}\right)^2},$$

$$W_n^2=[\omega_n^2-k^2+a^2]^2+4k^2\omega_n^2,\qquad 2\phi_n=\operatorname{tg}^{-1}\tfrac{2\omega_n^2}{W_n}.$$

The response consists of a term having the behavior of the forcing function plus a series representing the free vibration of the system caused by the discontinuity of the forcing function at time $t=0$.

curves that are fairly uniform over the range from 300 *to* 4000 *Hz, and fall almost linearly from* 4000 *to* 6000 *Hz.*

If one compares response curves of a violin made by ***Antonio Stradivarius*** *(1644–1737, Italy; made* 1116 *violins during* 1700–1725 *of which very few genuine specimens remain) with those of modern scientifically constructed violins, one is struck by the similarity of the resonance curves.*

Organ: A keyboard musical instrument. While a piano makes sounds by causing steel strings to vibrate, a pipe organ *creates sound by forcing air through metal or wooden tubes called pipes. It is the largest and most powerful of all musical instruments, and may have more than* 5000 *pipes, each producing a different frequency. The longest pipes, producing the lowest notes, may be more than* 9 *meters long and* 30 *cm in diameter. The smallest pipes, which produce the highest notes, are only* 18 *cm long and less than* 6 *mm in diameter. Some organs have* 6 *keyboards. Organ pipes are designed to play only the* fundamental frequency*. Consequently, there is only one pressure pulse at a time in the pipe (with other wind instruments, however, the driving vibrator can be controlled to send in one, two, three, or more pulses into the pipe before the first one has returned).*

In ca 250 BCE, ***Ctesibios of Alexandria****, a Greek engineer and inventor, built an organ that used water power to force air into the pipes. The major features of the modern organ were developed during* 200–1600 *CE. By* 1900*, interest in the organ had declined among composers and performers, and the instrument was played regularly only in churches as part of religious services. In 1934,* ***Laurens Hammond*** *(U.S.A.) patented the first commercially practical electronic organ.*

Leonard Bernstein on Music

"The genius of Johan Sebastian Bach was to balance so delicately and so justly the two forces of chromaticism and diatonicism that were equally powerful and presumably contradictory.

What makes music dramatic? Contrast; duality of two tones, two contrasting ideas or emotions within a single movement. Contrast makes drama — black against white, good against evil, day and night, grief and joy.

Bach represents the last stand against the dualistic concept. Any single movement is always concerned with one single idea. He clung to the older concept of one thing at a time — grief or joy, day or night: once the theme

is stated at the beginning, the main event is over. The rest of the movement will be a constant elaboration, recitation, and discussion of that main event. But if you are expecting any change in mood – say, a sudden yielding to sentimentality or lyricism – you are not going to get it. Contrast is there all right, but it is restricted to laud and soft, or key change, or different instrument grouping, but the dramatic contrast of themes is not there.

Consider the vast catalogue of Bach's output: songs, dances, suites, partitas, sonatas, tocatas, preludes, fugues, cantatas, oratorios, masses, passions, fantasies, concertos, chorales, variations, motets, passacalias – the creation of fifty years. What is that holds all these pages together, that makes it all inevitably the product of one man — the religious spirit.

For Bach, all music was religion; writing was an act of faith; and performing was an act of worship. Every note was dedicated to God and nothing else. This is the spine of Bach's work: simple faith. Otherwise, how could he have ever turned out all that glorious stuff to order, meeting deadlines, playing the organ, directed a choir, taught school, instructed his army of children, attended board meetings and keeping his eye out for better-paying jobs.

Bach was a man, after all, not a god; but he was a man of God, and his godliness informs his music from first to last."

* * *

"In the opera, the basic human emotions are pinpointed and magnified way beyond life size so that you can't miss them. Each emotion comes at you gigantically, in a clear direct, uncluttered, full-blown way. One of the chief reasons for the direct power of opera is that it is sung: Indeed, among of all the different instruments in the orchestra, there is none that can compete in any way with the expressivity of the human voice. And when such a voice, or several, or many together, carry the weight of a drama, then there is nothing in all the theater to compare with it for shear immediacy of impact. Now these emotions are not merely presented to us; they are hurled at us. You see, music is something very special. It does not have to pass through the censor of the brain before it can reach the heart; it goes directly to the heart."

1862–1889 CE **Eugenio Beltrami** (1835–1899, Italy). Distinguished differential-geometer. Professor of mathematics at the Universities of Pisa, Pavia and Rome. Placed *hyperbolic geometry* on a firm foundation, developed the Riemannian calculus of n-dimensional manifolds, and discovered the so-called 'differential parameters' that bear his name. Contributed to the mathematical theory of elasticity, optics and thermodynamics.

He brought Riemann's work into connection with non-Euclidean geometry. In his work "*An attempt to interpret the non-Euclidean geometry*" (1868), he demonstrated that the plane geometry of Lobachevsky-Bolyai holds on surfaces of constant negative curvature embedded in Euclidean space, straight lines being replaced by geodesics. Such surfaces are capable of a conformal representation on a plane, in which geodesics are represented by straight lines[378]. Interest in hyperbolic geometry was rekindled in the 1860's when unpublished work of **Gauss** (d. 1855), came to light. Learning that Gauss has taken hyperbolic geometry seriously, mathematicians became more receptive to non-Euclidean ideas.

The works of **Lobachevsky**, **Bolyai** and **Minding** were rescued from obscurity and, approaching them from the viewpoint of differential geometry, Beltrami was able to give them the concrete explanation that had eluded all his predecessors. He was interested in the geometry of surfaces and had found the surfaces which could be mapped onto the plane in such a way that their geodesics went to straight lines. They turned out to be just the surfaces of constant curvature. In the case of positive curvature (the sphere), such a mapping is a central projection onto a tangent plane, though this maps only half of the sphere onto the whole plane. The mapping of surfaces of constant negative curvature, on the other hand, take the whole surface onto only part of the plane.

He derived (1892) the so-called *Bertrami-Michell compatibility equation* in linear elasticity theory, serving as *integrability conditions* in terms of the components of the stress tensor and the applied forces.

Beltrami was born in Cremona to an aristocratic family. He was educated at the University of Pavia under Brioschi. During 1856–1861 he was a secretary to a railroad engineer, but in 1862 returned to the academia as a professor of rational mechanics in Bologna (1862–1864), Pisa (1864–1866), Bologna (1866–1873), Rome (1873–1876), Pavia (1876–1891) and again Rome (1891–1899).

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[378] Beltrami's method allows us to map only a *part* of the Lobachevsky plane on a *part* of a surface of negative curvature. **Hilbert** has shown that it is impossible to continue an analytical surface of constant negative curvature indefinitely without meeting singular lines when this surface is embedded in ordinary Euclidean space.

1862–1899 CE **Ernst Heinrich (Philipp August) Haeckel** (1834–1919, Germany). Biologist, physician, eugenicist and philosopher. Popularized Darwinism in Central Europe and applied it to some of the oldest problems of philosophy and religion. Outlined the essential elements of modern zoological classification (1864). Hypothesized that the nucleus of a cell contains *hereditary information* (1866). First to use the term *ecology* (1866) to describe the study of living organisms and their interaction with other organisms and with their environment. Haeckel's attempt to describe human evolution in racial terms later became a part of the pseudo-scientific basis for *Nazism*.

He was born at Potsdam and studied medicine and science at Würzburg, Berlin and Vienna. Graduated at Berlin as M.D. (1857). At the wish of his father he began to practice as a doctor in that city, but his patients were few in number, and after a short time he turned to more congenial pursuits. He came to the University of Jena in 1861 and was later appointed to the chair of zoology (1865–1909). He was on scientific expeditions to the Canary Islands (1866–1867), Red Sea (1873), Ceylon (1881–1882), and Java (1900–1901).

It happened that just when he was beginning his scientific career Darwin's *Origin of the Species* was published (1859), and such was the influence it exercised over him that he became the apostle of Darwinism in Germany. He therefore gave a wholehearted adherence to the doctrine of organic evolution and treated it as the cardinal conception of modern biology. He was first to draw up a genealogical tree relating all the various orders of animals, showing the supposed relationship of the various animal groups.

Haeckel tried to discover (1862) the symmetry of crystallization in Radiolarians (one-celled sea animals). Promoted the theory of *recapitulation* which states that the embryo repeats the evolutionary changes that its ancestors underwent.[379]

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[379] Haeckel believed that the development of the embryo imitated an organism's entire evolution as a species. He supported his theory with embryo drawings that have since been shown to *deliberately faked* to get more support for his ideas (almost every biology book for the past century has included pictures of vertebrate embryos made by him, purportedly demonstrating the amazing similarity of fish, chickens, and humans in the womb).

In the 1990's, British embryologist **Michael Richardson** was looking at vertebrate embryos through a microscope and noticed that they look nothing at all like Haeckel's drawings. Richardson and his team of researchers examined vertebrate embryos and published actual photos of the embryos in the August 1997 issue of the journal *Anatomy and Embryology.* It turned out that Haeckel had used the same woodcuts for some of the embryos and doctored others to make sure that the embryos looked alike. Indeed, Haeckel's drawings turned out to be one of the most famous fakes in biology. It turned out that this has

His integrated views on the philosophical implications of the theory of evolution were published under the title *Die Welträtsel* (1899), which in 1901 appeared in English as the *Riddle of the Universe*. In this book, adopting an uncompromising *monistic* attitude, he proposed that all nature is a unity with life originating in crystals and evolving to man; matter alone is the one fundamental reality (material monism); the mind depends upon the body, and hence does not survive after it; animals with a central nervous system possess consciousness. He believed in the singularity of essence of both the organic and the inorganic, and rejected religions and their ideas of God.

According to his "carbon theory", the chemico-physical properties of carbon in its complex albuminoid compounds are the sole and the mechanical cause of the specific phenomena of movement which distinguishes organic from inorganic substances, and the first development of living protoplasm arose from such nitrogenous carbon compounds by a process of spontaneous generation.

He regarded psychology as merely a branch of physiology, and psychical activity as a group of vital phenomena which depend solely on physiological actions and material changes taking place in the protoplasm of the organism in which it is manifested. Every living cell has psychic properties, and the psychic life of multicellular organisms is the sum-total of the psychic functions of the cells of which they are composed.

Moreover, just as the highest animals have been evolved from the simplest forms of life, so the highest faculties of the human mind have been evolved from the soul of the brute beasts, and more remotely from the simple cell-soul of the unicellular Protozoa. As a consequence of these views Haeckel was led to deny the immortality of the soul, the freedom of will, and the existence of a personal God[380].

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been known for a century!!. **Stephen Jay Gould** responded in the March 2000 issue of *Natural History* magazine, saying he had known all along. But Darwinists kept mum because Haeckel's crackpot theory constituted one of the main pieces of evidence in support of evolution.

Oddly enough, in 2005, the *New York Times* reported that biology textbooks were still running Haeckel's fake drawings. The *Times* specially singled out the third edition of *Molecular Biology of the Cell*, the bedrock text of the field, as one of the culprits.

380 **D'Arcy Wentworth Thompson** reacted to these views in his *On Growth and Form* (1917): "*Many a beautiful protozoan form has lent itself to easy physico-mathematical explanations; others, no less simple and no more beautiful, prove harder to explain. Nature keeps some of her secrets longer than others*".

1862–1921 CE **Josef Popper-Lynkeus** (1838–1921, Austria). Inventor, scientist, social thinker and humanist. Held in high esteem by **Mach**, **Freud** and **Einstein**. Foreshadowed some of the fundamental ideas of aerodynamics, electric power transmission, relativity and quantum physics that were later formulated by others. Attempted to enunciate a general science of energetics. Anticipated Freud's essential characteristics and most significant part of *dream theory* (the reduction of dream-distortion to an inner conflict). Rejected, years ahead of **L.E.J. Brouwer**, the logical *principle of the excluded middle*.

Josef Popper was born in the Jewish ghetto at Kohlin, Bohemia and lived therein up to his 15^{th} year. There he attended the elementary school where he was educated in a devoutly religious environment. In 1854 he was admitted to the Polytechnikum in Prague, where for three years he majored in physics, mathematics and engineering. He continued his studies at the University of Vienna but being a Jew he could not obtain an academic position[381] (although recommended by his teachers!). However, his income from his invention royalties enabled him to live humbly and pursue his writings. His major achievements were:

- Conducted pioneering experiments and proposed ways of transmission of electric power from its natural sources [1862; *Die Physikalischen Grundsätze der elektrischen Kraftübertragung*, 1883); (Physical Principles of Electric Power Transmission)].
- Suggested a connection between the laws of conservation of *mass and energy* (1883).
- Suggested an experiment to establish the existence of an *energy-quantum* through which one could interpret the periodic table of the elements. (In a letter to Mach, 1884.)
- Elaborated on the principles of *heavier-than-air flight* (*Flugtechnik*, 1888; *Der Machinen und Vogelflug*, 1911 = Mechanical Aviation and the Flight of Birds).
- Proposed a social reform plan in which he viewed the state as no *more* than a utilitarian association to assure security of existence for individuals living on a common soil, and to lighten the burden of their lot on this earth. He rejected compulsory military service; no one should be compelled to kill or to be killed. Even the freedom of criminals should be curtailed no *more* than the protection of society requires.

[381] Unlike Karl Marx and many others he was a proud Jew and refused to convert to Christianity for the sake of a university position.

> However, every man should give a decade of his lifetime to work for the state in order to assure (through organized production of all truly necessary means) a secure existence to each individual for the rest of his life. All other economic endeavor should be completely free. [*Das Recht zu leben und die Pflicht zu sterben* (The Right to Live and the Duty to Die), 1878; *Das Individum und die Bewertung menschlicher Existenzen* (The Individual and Evaluation of Human Existences), 1910; *Die allgemeine Nährpflicht als lösung der sozialen Frage* (The Obligation of Securing a Guaranteed Subsistence for all as the Solution to the Social Problem), 1912; *Krieg, Wherpflicht und Staadsverfassung* (War, Military Service and the State Constitution), 1921.]

Sigmund Frued touched upon the hidden background that linked him with Popper (1899):

"*A special feeling of sympathy drew me to him, since he too had clearly had painful experience of the bitterness of the life of a Jew and of the hollowness of the ideals of present-day civilization*".

Albert Einstein elucidated the unique phenomena of a scientist as a humanist and moralist (1954):

"*Popper-Lynkeus was a prophetic and saint person, and at the same time a thoroughly modern man. In love with the natural sciences and modern technology, he remained throughout a long, strenuous, and difficult life steadfastly true and dedicated to the aim he set for himself — that of contributing to the improvement of the lot of mankind and to their moral advancement. He affirmed passionately the technological advancement of our age as a liberator from soul-destroying physical labor and as the originator of cognition and creativeness which he loved for their own sake, for their own beauty*".

1863 CE **Luigi** (**Antonio Gaudenzio Giuseppe**) **Cremona** (1830–1903, Italy). Mathematician. Known for his work in projective geometry. The birational *Cremona Transformation*[382] is named after him.

Cremona was born to Jewish parents. He was a professor in Bologna (1860), Milan (1866) and Rome (1873). He became senator (1879) and Minister of Education (1898).

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[382] A transformation of a plane curve in a plane: (x, y) goes into

$$(R_1(x, y), R_2(x, y)),$$

where R_1 and R_2 are rational algebraic functions. He later generalized it to a rational transformation in 3-dimensional space.

1863–1898 CE *Underground railway*: The growing congestion of 19th century urban traffic prompted **Charles Pearson** (1843) to suggest the building of underground tunnels in London through which railway lines could be laid. The project was approved in 1853 and construction began in 1860. The project was successful, for it transported some 10 million passengers in its first year of service (1863). The London network expanded and became electrified (1890). Similar projects followed in Glasgow (1886), Boston, Budapest and Paris (1898).

1863 CE **Pietro Angelo Secchi** (1818–1878, Italy). Astronomer. First to classify stars into four major classes according to the general arrangement of the dark lines in their spectra. Proved that prominences seen during solar eclipses are features of the sun itself. This marks the nascence of *stellar spectroscopy.*

During 1814–1823, **Joseph von Fraunhofer** compared the spectra of the sun and the stars, but the first fairly comprehensive attempt at classification was undertaken by Secchi "to see if the composition of the stars is as varied as the stars are innumerable". He noticed that while the stars are innumerable, their spectra can be grouped in certain distinct groups. He observed the spectra visually by attaching a spectroscope to his telescope and pointing it toward the stars.

In those days, the nature and cause of spectral lines were not well understood, and astronomers classified each star by assigning a letter from A through P, depending on the strength of the hydrogen Balmer lines (**Johann Jakob Balmer**, 1825–1898, Switzerland) in the star's spectrum. The A stars have the strongest Balmer lines and the P stars have the weakest. Secchi was unable to see the fainter lines, which were not observed until the application of photography to this study.

Secchi was born at Reggio in Lombardy and entered the Society of Jesus at an early age. In 1849 he was appointed director of the Vatican Observatory. There he devoted himself with great perseverance to researches in physical astronomy and meteorology. He completed the first spectroscopic survey of stars, cataloging the spectrograms of about 4000 stars (1868).

1863–1864 CE **Arminius Vambery** (Herman Wamberger, 1832–1913, Hungary). Orientalist and explorer. The first European to explore Turkestan, Uzbekistan and Afghanistan.

He was born of poor orthodox Jewish parents at Duna-Szerdahely, a village on the island of Shütt, on the Danube near Pozsony. The name of the family was originally Bamberger. He got an orthodox Jewish education, and studied the Talmud in the village school until the age of 12. His mother destined her

son, who was lame, for the trade of a dressmaker. After being for a short time apprentice to a ladies' tailor, he became a tutor to an innkeeper's son. With the aid of friends he was enabled to enter the Gymnasium during the difficult and unsettled time of the revolution of 1848, but lack of resources forced him to quit school and go to Slavonia as a tutor. From then on he educated himself and at 20 gained knowledge in 15 European and ancient languages.

Later he studied at Vienna and Budapest and turned his attention to the study of Turkish and Arabic. In 1854 he migrated to Constantinople, where he worked as a tutor. During his 6-year stay there he acquired some 20 oriental languages and Turco-Tartar dialects, and published a Turkish-German dictionary. He became a secretary to Fuad Pasha and, for all practical purposes, a Muslim.

In 1861, the Hungarian Academy sponsored his journey to Central Asia. Under the name of Reshid Effendi, and in the guise of a Sunnite dervish, he traveled with Muslim pilgrims across the Turkestan desert to Khiva, Bokhara and Samarkand. He experienced hardships rarely sustained by a European before, braving the risk of being detected and put to death by the offended Muslims. He left the pilgrims and continued to travel to Herat in Afghanistan.

In November 1863 Vambery left Herat for Meshed, having joined a caravan of pilgrims and merchants. This was the first journey of its kind undertaken by a European; and since it was necessary to avoid suspicion, Vambery could not take even fragmentary notes except by stealth. He returned to Europe in 1864, and in the next year received the appointment of professor of oriental languages in the University of Budapest, retiring therefrom in 1905.

Earlier he became an adherent of the Protestant faith. On several occasions he carried out diplomatic missions for Great Britain in the Near East. He became a personal friend of the Prince of Wales, later King Edward VII. Sultan Abdul Hamid consulted him on problems of foreign policy, and **Theodor Herzl** (1860–1904) enlisted Vambery's aid in his negotiations with the sultan on behalf of his Zionist movement.

1863–1873 CE **John Tyndall** (1820–1893, Ireland). Experimental physicist, educator, pioneer researcher of the physics of the atmosphere, science writer and alpinist. Discovered[383] (1869) the *Tyndall effect* – the scattering of light by invisibly small colloidal particles in solution, thus making the light

[383] **Leonardo da Vinci** understood the basic phenomenon around 1500 CE. In particular, his experiments with the scattering of sunlight by wood smoke observed against a dark background [see: *The Notebooks of Leonardo da Vinci*, Dover edition]. The phenomenon was confirmed (1871) by theoretical studies of **Lord Rayleigh** and known as the *Rayleigh scattering*.

It is the incoherent scattering of electromagnetic radiation (light) by gas mole-

beam visible when viewed from the side. Suggested that the blue color of the sky is due to greater scattering of the shorter wavelength blue light by the colloidal particles of dust and water vapor in the atmosphere [beyond the atmosphere the sky is black!]

Tyndall was born in Leighlin-Bridge, County Carlow, Ireland. Self educated. Employed as a railway engineer (1844) and college teacher (1847) before studying physics (1848–1851) at the University of Marburg (under **Bunsen**) and Berlin (under **Magnus**), obtaining his PhD in 1851.

He became professor of natural philosophy at the Royal Institution (1854) and the Royal School of Mines in London (1859–1868). Tyndall toured the US 1872–1873. He died from accidental poisoning with chloral. Apart from the *Tyndall effect*, his other investigations and discoveries are:

- confirmed Pasteur's claim of the fallacy of the doctrine of *spontaneous generation* by showing that air contains living organisms (1881);
- carried out experimental work on the absorption and transmission of heat by water vapor and atmospheric gases which was important in the development of *meteorology*;
- the first scientist to describe the *greenhouse effect* for water vapor (1863);
- discovered that the transmission of sound is affected by variation of density in the atmosphere;

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cules or other randomly distributed electric dipole scatterers in accordance with the $\{\sin^4\theta/\lambda^4\}$ distribution of the flux density of an oscillating dipole field [$\theta =$ scattering angle, $\lambda =$ wavelength]. This explains the blueness of the sky, the redness of sunrise and sunset, and the scattering of radar waves by droplets of ice crystals.

The theory correctly describes scattering by *molecules*, as well as tiny spherical particles whose radius is smaller than about $(0.03)\lambda$. When the diameter is comparable to the wavelength of the incident radiation, *Mie scattering* theory takes into account the particle size and the dipole model is inadequate. The Mie theory predicts weak dependence of the scattered field on wavelength. This is significantly different from Rayleigh scattering, and because of this the *clouds* are white and the sky is blue.

Rayleigh scattering is also accompanied by *polarization*. Unpolarized incident light that is scattered through 90°, is linearly polarized in the direction perpendicular to the plane of incidence. Thus, unpolarized sunlight becomes almost completely polarized upon scattering through an angle of 90° by air molecules.

- made the first demonstration (1870) of the guiding of light by internal total reflection. In front of an audience of the Royal Academy of London, he demonstrated that light illuminating the top surface of water in a pail can be guided along a semi-arc of water streaming out through a hole in the side of the pail — a precursor of *fiber optics*.

1863–1875 CE **William Huggins** (1824–1910, England). Astronomer. First to identify some of the lines of stellar spectra with those of known terrestrial elements (1863). Huggins made the first radial-velocity determination of a star in 1868. He observed the Doppler red-shift[384] in one of the hydrogen-lines in the spectrum of Sirius, and found that the star was receding from the sun at a velocity of about 45 km/sec.

Huggins was born in London. He built in 1856 a private observatory in the south of London. Kirchhoff's discoveries in spectrum analysis turned his attention to the problem of the internal constitution of stars. The advent of photographic astronomy led him in 1875 to adopt and adapt the *gelatin dry plate*, enabling him to obtain stellar spectrograms with exposures of any desired length and thus produce permanent accurate pictures of celestial objects so faint as to be completely invisible to the eye, even when aided by a

[384] The radial or "line-of sight" velocity of a star can be determined from the Doppler shift of the lines of its spectrum, using the formula

$$\frac{\Delta\lambda}{\lambda} = \sqrt{\frac{1+v/c}{1-v/c}} - 1 \approx \frac{v}{c}$$

(for small v/c). Here c is the speed of light in vacuum, λ is the wavelength emitted by the *source*, and $\Delta\lambda$ the difference between λ and the wavelength measured by the *observer*. v is the relative line-of-sight velocity of the observer and the source, which is counted as positive if the velocity is one of recession and negative if it is one of approach. If a star approaches (recedes) from us, the wavelengths of light in its continuous spectrum appear shortened (lengthened), as well as those of the dark lines. However, unless its speed is tens of thousands of km/sec, the star does not appear noticeably bluer or redder than normal. The Doppler-shift is thus not easily detected in a continuous spectrum (except for very remote galaxies) and cannot be measured accurately in such a spectrum. On the other hand, the wavelengths of the absorption lines can be measured accurately, and their Doppler-shift is relatively simple to detect. The known wavelengths of the bright lines in the spectrum of a laboratory source (such as the bright lines in the spectrum of the arc lamp) serve as standards against which the wavelengths of the dark lines in the star's spectrum can be accurately measured.

powerful telescope. [His results were, however, affected by serious systematic errors. During 1888–1891, German astronomers at the Potsdam Observatory reduced the average probable error in the radial velocity measurements to only 2.6 km/sec.]

In 1900 Huggins, commenting on the future of radial-velocity determinations, concluded: "*This method of work will doubtless be very prominent in the astronomy of the near future, and to it probably we shall have to look for the more important discoveries in sidereal astronomy which will be made during the coming century*".

1863–1875 CE Global attack of *cholera*. In 1866, ca 300,000 died in Europe. Worldwide deaths were in the millions.

1863–1892 CE **Francis Galton** (1822–1911, England). Scientist. A cousin of Charles Darwin. Pioneer in modern meteorology and statistics. The father of racist 'eugenics', to be later embraced by Hitler in his *Mein Kampf*.

Invented the *teletype printer* (1850). Introduced modern weather-mapping techniques and established the existence and theory of *anticyclones* (high-pressure areas of the atmosphere) in his book *Meteorographica* (1863); it was the first serious attempt to chart the weather on an extensive scale.

Noted the uniqueness of each individual's *fingerprints* and worked out a system of classifying them (1885). His work became important for law enforcement through *fingerprint identification*.

Introduced the concept of *correlation* (the measure of interdependence of two sets of variables) and defined the useful *correlation coefficients* (1888–1889). Those he incorporated in his statistical techniques related to *genetics* and his pioneer use of statistics in psychological measurements[385].

His name, however, is most closely associated with studies in anthropology and especially *heredity*, which he expounded in *Hereditary Genius* (1869), *Human Faculty* (1884) and *Natural Inheritance* (1889).

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[385] He studied, for example, the familial tendency to inherit brilliance, or the related problem of the extinction of family surnames. In this connection he raised the question [known as the *Galton problem* (1873)]: If each male in a population has a family with x sons, where the random variable x has the distribution $F(s) = \sum_{x=0}^{\infty} p(x)s^x, \quad |s| \ll 1;$ and so on for the next generation, what is the probability $p(x)$ of any particular male-line dying out?

The first solution of this problem was given by H.A. Watson (1874), and the complete solution by Steffensen (1930).

Galton (1883) coined the word '*eugenics*' (from the Greek 'good in birth' or 'noble in heredity'). Eugenics was defined as the science of improving the human stock through systematic selective breeding, by checking the birthrate of the *unfit* (paupers, insane, physically impaired and feeble-minded) and furthering the production of the *fit* (talented, gifted, healthy, beautiful, etc.). Galton thus endeavored to improve or impair the racial qualities of further generations, either physically or mentally. He thus recommended forced sterilization of "unfit" humans, saying they could not be persuaded to stop breeding on their own. Eugenics, he said, "must be introduced into the national consciousness as a new religion". Hitler's worldview was based on the evolutionary ethic of Galton and his followers.

For Galton, science and progress were almost inseparable. Men could be improved by scientific methods, in the same way that plant and horse breeders improve their stock. Would it not, he wondered, be "*quite practicable to produce a highly gifted race of men by judicious marriages during several consecutive generations*?" The scientific assumptions behind this were explicit: most human attributes are inherited. His program was derived from ideas about natural selection and evolution. Not only was talent perceived of as being inherited, so too were pauperism, insanity and any kind of perceived feeble-mindness.

Galton graduated in medicine from Cambridge (1844). After inheriting ample fortune he was able to abandon his medical career, holding no scientific or teaching posts. Instead, he set out to see the world, traveling in Europe, Asia Minor, the Holy Land and southwest Africa.

Eugenic ideas may be detected as early as **Plato** (427–347 BCE), but eugenic became significant only after the publication of **Charles Darwin**'s (1809–1882) *Origin of the Species* (1859), which implied that man was the outcome of a natural process of evolution. Galton's campaign on behalf of eugenic breeding stimulated a popular social movement (from 1900) and the formation of centers of eugenic study in Britain, America, the Soviet Union and Nazi Germany[386].

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[386] The ideas of Galton, amplified by Karl Pearson from University College London, received support from a variety of sources, which included Fabians such as Bernard Shaw and psychologists like **Havelock Ellis**.

In the United States, **Charles Davenport** (1904) came to believe that certain races were feeble-minded. To this end he favored a selective immigration policy coupled to the prevention of reproduction of the genetically defective. The list of distinguished scientists that initially gave eugenics positive support is impressive: **Ronald Aylmer Fisher** (1890–1962), **J.B.S. Haldane** (1892–1964), **J.S. Huxley** (1887–1975), **W.E. Castle** (1867–1962), **T.H. Morgan** (1866–1945).

1864–1875 CE **James Croll** (1821–1890, Scotland). Geophysicist. Presented an astronomical theory of the ice ages caused by periodical changes in the earth's *orbital eccentricity* (100,000 year cycle), and the *precession of the equinoxes* (22,000 year cycle). Croll noticed the relevance of the periodic variation in the tilt of the earth's axis to the insulation and heating of the polar regions. But since he depended on the calculations of **LeVerrier**[387], he did not follow the consequences of this important line of reasoning, save the qualitative notion that ice ages would be more likely to occur during periods when the axis is closer to vertical, for then the polar regions receive a smaller amount of heat.

Croll plotted orbital changes during the past 3 million years, and found cyclical changes with long intervals of low eccentricity and long intervals of high eccentricity. He then concluded that ice ages occurred during periods of *high eccentricity*, alternating from the Northern to the Southern Hemisphere in response to the 22,000 year precession cycle.

Wrongly believing the crucial factor to be minimum winter solar radiation, Croll postulated that when the eccentricity is high, the hemisphere whose winter occurs at the time of the earth's farthest distance from the sun will

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However, in the 1930's, Huxley, Haldane, **Hogben** and other biologists at last began to react against many of the wilder claims for eugenics. But it was too late, for the ideas had permeated into mainland Europe, and especially into the ideology of the German National Socialists. They claimed that there is a biological basis for the diversity of mankind: what makes a Jew a Jew, a Gypsy a Gypsy is in their blood, that is to say in their genes — all this based on the genetic ideas of the eugenic movement. Thus it is quite easy to see the direct line from the eugenic movement to the statement by the animal behaviorist **Konrad Lorenz** (1935; Nobel prize, 1973): "*It must be the duty of racial hygiene to be attentive to a more severe elimination of morally inferior human beings than is the case today... . This role must be assumed by a human organization; otherwise humanity will be annihilated by the degenerative phenomena that accompany domestication*".

Another metaphor from Lorenz is the '*analogy between bodies and malignant tumors on the one hand, and a nation and individuals within it who have become asocial because of their defective constitution*'.

In 1933, the Nazi Cabinet promulgated a *Eugenic Sterilization Law* which can be considered as leading to the atrocities by doctors and others in the Nazi concentration camps.

[387] In 1843 **LeVerrier** used perturbation theory to show that in the past 100,000 years, the earth's orbital eccentricity has varied from a low near zero to a high about 6%. He calculated that the tilt of the earth's axis fluctuates within the range $23\frac{1}{2}^\circ \pm 1\frac{1}{2}^\circ$, but did not determine the period of this motion.

experience an ice age. Nevertheless, he was the first scientist to develop the idea now referred to as *positive feedback*[388].

James Croll was born in a peasant family at Little Whitefield in Perthshire. Lacking any formal education, he drifted from one occupation to another: mechanic, millwright, carpenter, shopkeeper, hotel-keeper, salesman and janitor. Finally in 1864, at the age of 43, he came across **Adhemar**'s book *Revolutions of the Sea* (1842). Although he realized that the French mathematician was wrong in believing that a change in the length of warm and cold seasons could cause an ice age, Croll was convinced that some other astronomical mechanism must lie behind these geological phenomena.

Following the publication of his theory, he received an appointment in the Scottish Geological Survey (1867), and for 13 years he took charge of the Edinburgh office. He has been compelled by ill-health to withdraw from public service in 1880. He was elected Fellow of the Royal Society in 1876.

Croll's theory created an immediate and profound impression on the world of science. Here, at last, was a plausible theory of the ice age that could be tested by comparing its predictions with the known geological record. Over the next 30 years Croll's ideas were widely and hotly debated: scientific expeditions were organized to dig for facts in drift deposits all over the world; articles and scientific journals probed the details of Croll's theory; and arguments pro and con filled many pages in geological textbooks.

As time went on, however, many geologists in Europe and America became more and more dissatisfied with Croll's theory, which maintained that the last Glacial Epich began about 250,000 years ago and ended some 80,000 years ago. The new evidence showed that the last ice age ended 10,000 years ago — at variance with Croll's results. Moreover, theoretical arguments were advanced against the theory by meteorologists who calculated that the variations in solar heating described by Croll were too small to have any noticeable effect on climate. By the end of the 19^{th} century, the tide of scientific opinion had turned against Croll, and his astronomical theory came to be treated as an historical curiosity, interesting but no longer valid. Eventually it was almost forgotten.

Almost, but not quite, for it would be picked up years later by a Yugoslavian astronomer. But in 1890, when James Croll lay on his deathbed in

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388 Croll reasoned that a decrease in the amount of sunlight received during *winter* favors the accumulation of snow. Any small initial increase in the size of the area covered by snow must result in an additional loss of heat by reflecting more sunlight back into space. Therefore, any astronomically induced change in solar radiation (however small) would be amplified.

Scotland, **Milutin Milankovich** was only 11 years old, quite unaware of the task that the goddess of science had destined for him.

1864–1877 CE **Siegfried Marcus** (1831–1897, Germany). Engineer and inventor. Inventor of the automobile[389]. Built the first horseless carriage (1864) and in 1875 the second, which was the first 4-stroke engine, petrol-driven vehicle to function. This he drove about the streets of Vienna, amid general astonishment.[390]His automobile patents were registered in Germany (1882) and it was not until four years later (1886), that the first Daimler motor-car was built.

Marcus was born in Malchin, Mecklenburg, to Jewish parents and settled in Vienna (1853). He first began to study medicine, but later turned to electrotechnics and chemistry. Apart from his theoretical studies, he did practical work and thus acquired a sound knowledge of mechanics. His research work in chemistry drew his attention to the problem of fuel. In 1864, Marcus built his first model which he improved in 1875.

This vehicle (now kept in the Technical Museum of Vienna) contained all the essential parts of todays motorcar. The engine was driven by petrol supplied and mixed with air by a carburetor. This mixture entered the cylinder through a conic valve operated by a camshaft. The ignition spark was provided by a magneto at the moment when the piston arrived at the top dead center. The exhaust gas escaped through an outlet valve. The engine-power was transmitted by a conical clutch and two belt pulleys to the rear axle. The body, which looked like a horse-carriage was equipped with shock absorbers in the form of rubber buffers placed between the body and the rear axle. Two half-elliptical springs were fitted above the front axle. The steering box was of the worm gear in use today.

In 1975, on the occasion of a festival in honor of the inventor, Siegfried Marcus' car was put to use; the vehicle was ready after minor repairs. Experts were surprised at the ease with which the car ran.

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[389] Whilst history records the German **Otto** as the inventor of the 4-stroke engine (1877), the German **Daimler** as the first to use petrol to drive on engine (1885) and the Frenchman **Levassor** (1887) as the first to utilize a petrol-engine to drive a vehicle — in fact, all these inventions had been made previously by one man, Siegfried Marcus.

[390] The first traffic report to the first driver in the first motorcar was made by a Viennese policeman, when he stopped Siegfried Marcus' car and forbade him to continue his journey because of the noise he was making (1875). In doing so, this policeman had delayed the development of an invention which was destined to change the face of the earth.

1864–1880 CE **Cato Maximilian Guldberg** (1836–1902, Norway), chemist and mathematician, and **Peter Waage** (1833–1900, Norway), chemist, formulated the *law of mass action*[391] (1864–1867), the basic law of chemical kinetics. In it simplest form, it states that in a state of equilibrium, the fraction of molecules of one kind changing into molecules of another kind is a time independent constant[392].

Guldberg became a professor at the University of Christiania (Oslo) in 1869. With **Henrik Mohn** (1835–1916, Norway) he published a book on the circulation of the atmosphere, providing the theoretical foundation for *dynamic meteorology* (1876–1880). In their study, they incorporated the Coriolis deflection and the friction between the earth and the atmosphere.

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[391] **Berthollet** (1803) recognized that the concentrations of the reacting compounds influence the reaction but failed to render a general mathematical formulation.

[392] In general, reactants with concentrations $A, B, C, \ldots$ may combine in various well-defined proportions to give products with concentrations $G, H, \ldots$ according to the stoichiometric equation

$$\nu A + \mu B + \eta C + \cdots = \gamma G + \delta H + \cdots,$$

where the integers ν, μ, η, γ, δ are called *stoichiometric coefficients* (namely, the numbers of molecules that partake in the reaction). The law of mass-action states that at equilibrium

$$\frac{G^{\gamma} H^{\delta} \cdots}{A^{\nu} B^{\mu} C^{\eta} \cdots} = K(T, P),$$

where K is a constant independent of time. Large values of K indicate the formation of large quantities of new products.
The law of mass action results in a natural way from the differential equation governing the reaction kinetics. Consider for example the symbolic reaction equation $A + X \rightleftharpoons B + Y$, which means that whenever a molecule of component A encounters a molecule of X, there is a certain probability a reaction will take place and a molecule of B and a molecule of Y will be produced. Likewise, the collision between molecules of Y and B can set off the opposite reaction.
The total variations in concentrations of the chemicals is given by the balance between the forward and the reverse reaction. Consequently

$$\frac{dX}{dt} = \frac{dA}{dt} = -\frac{dY}{dt} = -\frac{dB}{dt} = -kAX + k'BY.$$

In the state of equilibrium the forward and reverse reactions compensate one another statistically so that there is no longer any overall variation in the concentrations $\left(\frac{dX}{dt} = 0\right)$. This compensation implies that the ratio between equilibrium concentrations is given by $\frac{AX}{YB} = \frac{k'}{k} = K$.

1864–1884 CE **Julius Friedrich Cohnheim** (1839–1884, Germany). Physician. Pioneer of pathological anatomy. Revolutionized medical thought and practice when he advanced the basic theory of *inflammation* (1864–1867) and pus formation. Devised techniques of freezing tissue samples before sectioning. Author of classical textbooks in general pathology (1877–1880).

Cohnheim was born in Demmin, Pommerania, to Jewish parents. Converted to Christianity to advance his career, he studied in Berlin, where he graduated in medicine and studied for a year with the cellular pathologist **Rudolf Virchow**. Professor of Kiel (1868), Breslau (1872), Leipzig (1878).

Cohnheim devised new ways of looking at specimens of human tissue under the microscope and worked out many of the early cellular events that occur in inflammation. He showed by experiments how the blood cells vessels respond in the early stages of inflammation, and proving that the white cells (leukocytes) passed through capillary walls where inflammation was occurring, later degenerating to become pus corpuscles.[393]

Cohnheim worked on a whole range of diseases, including tuberculosis, myocardial infarct and cancer. **Metchnikov** was among many later workers who confirmed and extended his early studies.

1864–1886 CE **Edmond Nicolas Laguerre** (1834–1886, France). Mathematician. Although most of his research efforts were in the field of geometry, this part of his output [foci of algebraic curves (1853), geometric interpretation of homogeneous forms and their invariants, curves mapped onto themselves by inversion, 4^{th} order curves, studies of curvature and geodesies and pioneering investigations of the complex projective plane] has been largely absorbed by later theories or has passed into the general body of geometry without acknowledgment[394].

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393 The essence of the inflammatory response is the migration of white blood cells to a wound. The inflammatory response results in an increased flow of blood to the site of injury and an increased permeability of the endothelium (the tissue that comprises the walls of capillaries). Both of these effects assist the migration of phagocytic (germ engulfing) white blood cells from the blood to the interstitial fluid. Here the white blood cells can begin engulfing debris and any infecting microorganisms.

394 e.g. his work on differential invariants is included in the more comprehensive *Lie group* theory. He was also one of the first to point out that a distance function (metric) can be imposed on the coordinate plane of analytic geometry in more than one way.

Laguerre's current reputation rests on his discovery (1879) of the *Laguerre differential equation* and its polynomial solutions (*Laguerre polynomials*). These functions have wide use in mathematical physics and applied mathematics — for example, in the solution of the Schrödinger equation for hydrogen-like atoms and in the study of electrical networks and dynamical systems. The 1879 memoir of Laguerre is significant not only because of the discovery of the Laguerre equations and polynomials, but also because it contains one of the earliest infinite continued fractions which are known to be convergent.[395]

Laguerre was born in Bar-le-Duc. His education was completed at the École Polytechnique in Paris. In 1854 he left school and accepted a commission as an artillery officer (1854–1864), where he published nothing. Upon his return to Paris he became a tutor at the École Polytechnique and in 1874 was appointed *examinateur*. In 1883 he accepted, concurrently, the chair of mathematical physics at the Collège de France. In 1886 his continually poor health broke down and he returned to Bar-le-Duc, where he died.

In evaluating the life-work of Laguerre one encounters a phenomena common to many brilliant and innovative minds: his name is little known and his work so seldom cited because *he was primarily occupied with details and did*

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[395] The polynomials

$$\begin{aligned} L_n(x) &= e^x \frac{d^n}{dx^n}(x^n e^{-x}) \\ &= n!\left[1 - \binom{n}{1} x + \binom{n}{2}\frac{x^2}{2!} - \binom{n}{3}\frac{x^2}{3!} + \cdots + (-)^n \frac{x^n}{n!}\right] \end{aligned}$$

satisfies the differential equation $xy'' + (1-x)y' - ny = 0 \quad (n = 0, 1, 2, \ldots)$. The *associated Laguerre polynomials* $L_n^m(x) = (-)^m \frac{d^m}{dx^m} L_{n+m}(x)$, satisfy the generalized equation $xy'' + (m+1-x)y' + ny = 0$. Laguerre arrived at his equation through an investigation of the exponential-integral function $\int_x^\infty \frac{e^{-u}}{u} du$. for which he obtained the continued-fraction representation

$$\int_x^\infty \frac{e^{-u}}{u} du = \cfrac{e^{-x}}{x + 1 - \cfrac{1}{x + 3 - \cfrac{4}{x + 5 - \cfrac{9}{x + 7 - \cfrac{16}{x + 9 - \cdots}}}}}.$$

Laguerre proved that the m^{th} convergent of the fraction could be written as $e^{-x}\left[\varphi_m(x)/f_m(x)\right]$, where $f_m(x)$ is the Laguerre polynomial of degree m, $L_m(-x)$, and thus demonstrated that a divergent power series can be converted into a convergent continued fraction.

not step back to draw together various pieces and put them into a single theory. The result is that his work has mostly come down as various interesting special cases of more general theories discovered by others.

1865 CE **Eugene Charles Catalan** (1814–1894, Belgium). Mathematician. Contributed to the theory of continued fractions and number theory. The constant[396] $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915, 965, 594\dots$ is named after him.

Catalan was in Liouville's class at Ecole Polytechnique (1833) but was expelled the following year. Allowed to resume his studies in 1835. With Liouville's help he obtained a lectureship in descriptive geometry at the Ecole Polytechnique (1838) but his career was damaged by being politically active with strong left-wing views.

1865 CE The London Mathematical Society founded[397].

1865 CE **Gregor Johann Mendel** (1822–1884, Austria). Botanist. Discovered the mathematical principles of heredity. Observing the contrasting characteristics of different pea plant species, he grew successive generations of such plants and studied how these characteristics were inherited.

Mendel was born in Heinzendorf, Austria. He became interested in plants while a youth on his father's farm. In 1843 he entered the Augustinian

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[396] It is a special case of the *Dirichlet series* $\beta(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^z}$. Catalan's constant has many series, integral and continued fraction representations. It is unknown whether $G = \beta(2)$ is irrational. **Ramanujan** showed that $G = \frac{\pi}{4}\ {}_3F_2(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2})$ and that

$$2G = 2 - \cfrac{1}{3 + \cfrac{2^2}{1 + \cfrac{2^2}{3 + \cfrac{4^2}{1 + \cfrac{4^2}{3 + \dots}}}}}$$

[397] The following *mathematical societies* were established in the indicated order: *France* (1872), *Edinburgh* (1883), *Palermo* (1884), *New York* (1888), *Germany* (1890), *India* (1907), *Spain* (1911), *U.S.A.* (1915).

The first seven International Congresses were held at: *Paris* (1889), *Chicago* (1893), *Zürich* (1897), *Paris* (1950), *Heidelberg* (1904), *Rome* (1908) and *Cambridge* (1914).

The number of *mathematical periodicals* increased as follows: 1700 (17), 1800 (210), 1900 (950), 1990 (ca 3800). This roughly fit the curve
$N(t) = 17(1+t)^{1/4}e^{t/75}$, where $t = 0$ corresponds to the year 1700.

About 200,000 new mathematical theorems are being proved each year, since 1990.

Monastery in Brünn. Except for his education at the University of Vienna and short periods of teaching natural history at nearby schools, Mendel spent his life in the monastery.

In 1865, Gregor Mendel read his paper before the Brünn Society for the Study of Natural Science. The records of the society state that there were neither questions nor a discussion following his presentation. Like a stone dropped down a well, Mendel's work disappeared from view of the scientific community — without causing so much as a ripple.

It was not until 1900, sixteen years after his death, that biologists came to appreciate what he had accomplished. At that year his work was rediscovered by three distinguished biologists: **Hugo de Vries** (1848–1935), **Carl Correns** (1864–1933) and **Erich Tschermak** (1871–1962). Thereafter, Mendel's ideas have steadily gained ground, and came to exert upon biology an influence not less than that associated with the name of Darwin.

1865 CE **Hermann Johann Philipp Sprengel** (1834–1906, England). Chemist, physicist and inventor. Invented the high vacuum pump which had far reaching effects: for example. it made possible **Crooke's** investigations of radiation in a high vacuum, leading eventually to the discovery of the electron by **J.J. Thomson** (1895).

Sprengel was born in Schillerslage, near Hanover, and educated at the universities of Göttingen and Heidelberg. He moved to England (1859) and carried out research at Oxford and in the laboratories of several institutions in London. He mechanized the pump devised by **Heinrich Geissler** (1858), making the action of the pump much swifter and more efficient.

1865 CE **Immanuel Lazarus Fuchs** (1833–1902, Germany). Mathematician. One of the creators of the modern theory of differential equations.

Fuchs was born to Jewish parents in Moschin, near Posen and studied at Berlin with **Kummer** and **Weierstrass**. He became professor at the University of Berlin (1884), after converting to Christianity. In 1865 he combined two methods in the study of linear differential equations with complex functions as coefficients. One, using power series, as elaborated by **A.L. Cauchy**; the other method uses the hypergeometric series as has been done by **G.F.B. Riemann**. A special type of linear ordinary differential equations bear his name.

One of his most able students, **Zvi Hermann Shapira** (1840–1898), became a professor of mathematics at Heidelberg (1887–1898) and contributed to the theory of co-functions. He reissued and annotated (1880, Leipzig) the medieval mathematical treatise of **Avraham bar Hiyya**. Shapira was also active in the Zionist movement and suggested the idea of the Jewish National Fund (1897).

1865–1877 CE **Heinrich Anton de Bary** (1831–1888, Germany). Botanist. Founder of science of mycology and of plant pathology. First to work out life histories, morphology and physiology of many *fungi*, esp. parasitic fungi; first to demonstrate *heteroecism*. Demonstrated symbiotic nature of lichens.

Born at Frankfurt, Germany. Professor at Freiburg (1855–1866), Halle (1867–1872), Strasbourg (1872–1888).

1865–1881 CE **Carl Gottfried Neumann** (1832–1925, Germany). Mathematician and theoretical physicist. Pioneered in boundary value problems of potential theory and contributed to the theory of Bessel functions. He coined the term '*logarithmic potential*' (1870). Neumann was born in Königsberg. His father **Franz Ernst Neumann** (1798–1895) was a known mineralogist and physicist, and his mother was a sister-in-law of the astronomer F.W. Bessel. From 1868 until 1911 he was a professor at the University of Leipzig.

1866–1882 CE **Camille Marie Ennemond Jordan** (1838–1922, France). Mathematician. Known for his important contributions to algebra, topology and group theory. Gave a generalization of the Serret-Frenet formulae for a curve in an R^n space, and also established the existence of principal directions for any subspace of such a manifold. Introduced with **Giuseppe Peano** (1858–1932) the concept of 'Riemann content' in measure theory. Made significant contributions to topology ('*Jordan curve theorem*'), group theory and measure theory. Showed that algebraic equations of *any* degree can be solved in terms of *modular functions*.

Jordan studied mathematics at the Ecole Polytechnique and from 1873 taught there and at the College de France. He introduced important topological concepts (1866) such as *homotopy* and defined a homotopy group of a surface without explicitly using group terminology [he was aware of Riemann's work but not of the work of Möbius]. His introduction of group concepts into geometry (1869) was motivated by studies of crystal structure. Defined the *normal form* for matrices (1870) over a finite field, and brought *permutation groups* to a central role in mathematics. Originated the concept of *functions of bounded variation* and is known especially for his definition of the length of a curve (1882). He also generalized the criteria for convergence of *Fourier series*.

Two of Jordan's students, **Sophus Lie** and **Felix Klein**, drew upon his studies to produce their own theories of continuous and discontinuous groups.

1866–1896 CE **Robert Whitehead** (1828–1905, England). Engineer and inventor. Father of the modern *torpedo*. Designed and built the first unmanned, self-propelled torpedo. It was propelled by a compressed-air engine and carried 9 kg of dynamite. Its most important feature was a self-regulating device which kept it at a constant preset depth. Many of the basic component parts used in his early prototypes were, in fact, still in use during the Second World War and the overall form of the torpedo has been retained to the present day. He was first to use the *gyroscope* in military equipment (1896).

Whitehead was born near Bolton, Lancaster, UK and came from a family of engineers. After a long apprenticeship with a company in Manchester he left in 1840 to seek his fortune abroad. In 1864 he began to work for the Austrian Navy and undertook to build for them an unmanned, self-propelled surface boat packed with explosives which could be directed at blockading ships. In 1870 he brought two of his weapons to England for trial with the Royal Navy. The larger was $4\frac{1}{2}$ m long, diameter 40 cm, charged with 9 kg of dynamite and having a range of ca 1000 m. The Royal Navy were very impressed and bought the manufacturing rights for 15,000 sterling in 1871. In 1896, Whitehead used the gyroscope to steady the motion of his torpedo.

Whitehead was clearly one of the greatest British inventors of the 19th century. The type of torpedo that he invented exerted more influence over the tactics of naval warfare than all the world's top admirals and naval architects put together. Yet although he was honored by many other nations, he received minimal recognition from his country of birth. Even today, apart from current and past members of the Royal Navy, his name remains virtually unknown. (You will not find his name mentioned in any of the Britannica editions prior to 1975!).

1866 CE **Georges Leclanché** (1839–1882, France). Engineer. Invented the first *dry cell* (Zinc-Carbon cell), where the electrolyte is moist. [It is called "dry" in comparison with cells like the Daniell cell, in which the electrolytes are *aqueous solutions*.]

The container of this dry cell, made of zinc, also serves as one of the electrodes. The other electrode is a carbon rod in the center of the cell.

The zinc container is lined with porous paper to separate it from the other materials of the cell. The rest of the cell is filled with a moist mixture (the cell is not really dry) of ammonium chloride (NH_4Cl), manganese oxide ($Mn(IV)O_2$), zinc chloride ($ZnCl_2$), and a porous, inert filler. Dry cells are sealed to keep the moisture from evaporating. As the cell operates (the electrodes must be connected externally), the metallic Zn is oxidized to Zn^{2+},

and the liberated electrons flow along the container to the external circuit. Thus, the zinc electrode is the anode (negative electrode).

$$Zn \rightarrow Zn^{2+} + 2e^- \qquad \text{(oxidation, anode)}$$

The carbon rod is the cathode, at which ammonium ions are reduced.

$$2NH_4^+ + 2e^- \rightarrow 2NH_3 + H_2 \qquad \text{(reduction, cathode)}$$

Addition of the half-reactions gives the overall cell reaction

$$Zn + 2NH_4^+ \rightarrow Zn^{2+} + 2NH_3 + H_2, \qquad E_{\text{cell}} = 1.6V$$

As H_2 is formed, it is oxidized by MnO_2 in the cell, while the Mn is reduced. This prevents collection of H_2 gas on the cathode, which would stop the reaction.

$$H_2 + 2Mn(IV)O_2 \rightarrow 2Mn(III)O(OH)$$

The ammonia produced at the cathode combines with zinc ions and forms a soluble compound containing the complex ions, $[Zn(NH_3)_4]^{2+}$:

$$Zn^{2+} + 4NH_3 \rightarrow [Zn(NH_3)_4]^{2+}$$

Leclanche's invention, which was quite heavy and prone to breakage, was steadily improved over the years. The idea of encapsulating both the negative electrode and porous pot into a zinc cup was first patented by **J.A. Thiebaut** in 1881. But it was **Carl Gassner** of Mainz who is credited as constructing the first commercially successful "dry" cell. Variations followed. By 1889 there were at least six well-known dry batteries in circulation. Later *battery* manufacturing produced smaller, lighter batteries, and the application of the tungsten filament in 1909 created the impetus to develop batteries for use in torches (flashlights).

Leclanché was born in Parmain, France, and educated in England. After completing a technical education in Paris (1860) he began to work as an engineer.

1866 CE **Gabriel Auguste Daubrée** (1814–1896, France). Geologist and mineralogist. Suggested that the center of the earth is a core of iron and nickel.

He was born at Metz and educated at the École Polytechnique in Paris. Qualified as a mining engineer, he was put in charge of the mines in Alsace (1838) and was subsequently a professor of mineralogy and geology at Strasbourg (1852). In 1861 he was appointed professor of geology at the museum of natural history in Paris. The minerals *daubreeite* and *daubreelite* are named for him.

1866–1867 CE **Daniel Kirkwood** (1814–1895, U.S.A.). Astronomer. First drew attention to gaps in the distribution of asteroids' mean distances from the sun, known today as *Kirkwood gaps*. He noticed that very few asteroids have orbits whose orbital periods correspond to simple fractions (such as $\frac{1}{2}$, $\frac{3}{7}$, $\frac{2}{5}$, $\frac{1}{3}$) of Jupiter's orbital period. *Resonant gravitational perturbations* due to the repeated alignments with Jupiter have deflected asteroids away from these orbits and prevented the formation of a planet between Mars and Jupiter[398]. Kirkwood pointed out that the divisions in the ring structure of *Saturn* may have similar origin (the *Cassini divisions*, 1675), created by gravitational perturbations of the Saturnian moons on the icy fragments of the rings.

The Planet that Failed to Form[399]

The giant planet Jupiter orbits the sun in an ellipse that is 5 *times larger and somewhat more eccentric* ($e = 0.048$) *than the orbit of the earth. It comes to perihelion at* 740,558,340 *km and recedes to* 815,602,000 *km at aphelion, so its distance from the sun varies by some* 75 *million km. Its mean orbital speed of* 13.1 *km/sec carries it once around its orbit (sidereal period) in nearly* 12 *years. Since its synodic period*[400] *is* 399 *days, it comes to opposition every*

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398 The inner planets formed when swarms of planetesimals a few kilometers in size collided at velocities low enough to permit bodies to grow larger by accretion. Numerous resonances from the rapidly growing and massive planet Jupiter probably permeated the region between 2 and 4 AUs. These resonances may have pumped up the orbital eccentricities of the planetesimals there, accelerating the objects to velocities so high that successful accretion on a planetary scale was impossible. Today the asteroids remain in an environment dominated by collisions, encountering one another at about 5 km/sec.

399 For further reading, see:

- Gallant, R.A., *Our Universe*, National Geographic Society, 1994, 284 pp.
- Moore, P., *Atlas of the Universe*, Philips, 2005, 288 pp.
- Caprara, G. (ed.), *The Solar System*, Firefly Books, 2003, 255 pp.

400 The time taken by the earth to catch up with Jupiter by one lap.

13 months. Its rapid rotation with a period of only 9^h50^m *has produced a noticeable flattening at its poles (0.062).*

The mass of Jupiter (as determined from the motions of its inner satellites and the perturbation it produces on the motion of asteroids) is 318 *times the mass of the earth, and it is nearly* 11 *times the earth in diameter. In both volume and mass, it is larger than all the other planets put together*[401]*. Jupiter's gravitation perturbs the motion of the sun and the other planets, and holds its own satellites in orbit.*

During the 18^{th} *century, when post-Newtonian astronomers began to seek for law and order in the solar system, they noticed that all the ratios of distances of neighboring planets from the sun lie between* 1.3 *and* 2.0 except *the Jupiter-Mars ratio which came to* 3.4*. Thus, the gap between these two planets is twice as great as it might be expected to be. It is almost as though a planet ought to exist between Mars and Jupiter, and doesn't.*

Toward the end of the 1700*'s, astronomers were thinking along these lines and were beginning to plan a telescopic sweep of the sky in order to see if such a missing planet could be spotted. Between* 1800 *and* 1845*,* 5 *minor planets were found.* ***Herschel*** *named them* asteroids*. By* 1866*, enough asteroids had been discovered so that one could see that the average distances were spread out fairly evenly between the orbits of Mars and Jupiter — but not entirely evenly!*

The modern theory of the formation of the solar system has it beginning in a huge cloud of dust and gas. Slowly this cloud turned and came together under its own gravitational pull. As the cloud condensed into a smaller and smaller object, it turned faster and faster. Eventually, the central part of it condensed into the sun, while some of it at its midsection was kept in the outskirts by the centrifugal force, like a large equatorial bulge. The thinner cloud of dust and gas that spread out beyond the sun's midsection formed larger and larger objects that kept colliding until the planets were formed, all circling more or less in the equatorial plane of the sun.

Most of the total angular momentum of the solar system lodges in the orbital motion of the massive outer planets such as Jupiter and Saturn

401 *The center of gravity* of the solar system shifts in a complicated fashion as the planets circle the sun, but most of the time it is about 45,000 km above the sun's surface in the general direction of Jupiter, causing the sun to wobble slightly — making one complete wobble in about 11.86 years, close to the orbital period of Jupiter.

($J_{\rm Jup} \approx 16 J_\odot$)[402]. *The part of the dust cloud lying between the orbits of Mars and Jupiter has collected into small solid bodies of various sizes, but could not take the final step of coalescing into a single large body, because the gravitational influence of Jupiter kept stirring up the asteroids, preventing them from coming together. Attempts to explain the Kirkwood gaps go all the way back to Kirkwood. Chief among them is the so-called gravitational hypothesis; it suggests that asteroids drift away from the commensurable orbits under the influence of Jupiter's gravitational perturbation alone, needing no help from collisions.*

The mechanism of this interaction is as follows: Every time the asteroid wheels into that part of its orbit which happens to be near Jupiter's position at the time, it feels Jupiter's pull particularly strongly. If Jupiter happens to be a little ahead of the asteroid at the time of closest approach, it will pull the asteroid forward. If Jupiter happens to be a little behind, it will pull the asteroid backward. On the average, the forward and backward pulls will cancel each other and, in the long run, the asteroid's orbit will remain unchanged.

If, however, the period of revolution of an asteroid is some simple fraction of the period of revolution of Jupiter, their relative position will be repeated periodically every T years; the perturbations will not balance out but tend to accumulate in a preferred direction, with the consequence that the asteroid will regularly be pushed out of its orbit closer or farther from the sun. A gap in the asteroid belt will form at a series of distances which are simple fractions of the period of revolution of Jupiter.

The asteroids all revolve about the sun in the same direction as the principal planets (from west to east), and most of them have orbits that lie near the plane of the earth's orbit. The main asteroid belt contains minor planets with orbits of semimajor axes in the range 2.2 to 3.3 AU, with corresponding periods of orbital revolution about the sun from 3.3 to 6 years.

Calculations show that perturbations by Jupiter of asteroids near or in the Kirkwood gaps can result in ejecting asteroids to the part of the solar system occupied by the earth.

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[402] The orbital angular momentum of a satellite of mass m in a circular orbit of radius r around a gravitating center of mass M is proportional to the square root of r. This follows from Newton's law of gravitation which requires the satellite's centripetal acceleration to have the value $v^2/r = GM/r^2$, so that $vr = \sqrt{GMr}$ and therefore the orbital angular momentum is $J = mvr = m\sqrt{GMr}$. One may thus compare the orbital angular momentum of the planet Jupiter ($m \approx 10^{-3}M$; $M \approx 2 \times 10^{30}$ kg; $r \approx 8 \times 10^{11}$ m) with the spin angular momentum of the sun $J_\odot = \frac{2}{5}MR^2\omega$ obtained if one treats the sun approximately as a rigid sphere ($R = 7 \times 10^8$ m) rotating with a period of about 25 days (i.e., $\omega \approx 3 \times 10^{-6}$ sec^{-1}).

In recent years (1985–1992), celestial-mechanics theorists brought to bear a new outlook upon the formation mechanism of the Kirkwood gaps: given enough time, it seems an asteroid with a period commensurable with that of Jupiter will experience a chaotic burst of eccentricities, high enough to put it in an orbit where it is likely, sooner or later, to have a close encounter with Mars. The modern study of chaos deals with the onset of wild and unpredictable fluctuations in a system governed by simple deterministic equations from which one would naively expect nothing but good behavior.

When one speaks of chaos, especially in an essentially conservative, Hamiltonian system like the solar system, one does not mean the unpredictability inherent in intrinsically disorderly phenomena such as thermal noise or innumerable random collisions. What is meant here is a dependence on initial conditions so hypersensitive that it thoroughly destroys predictability, despite the simple, deterministic equations that govern the system. After a few hundred thousand years, two asteroids that were initially traveling together in one of the Kirkwood gaps will become completely uncorrelated.

During 1866–1981, *nobody was able to present an analytically tractable solution that offer a detailed explanation of the Kirkwood gaps. The explicit numerical integration of Newton's equations without radical approximations consume so much computer time that the orbits of the nine planets, including their mutual interactions, have never been explicitly calculated beyond* 5×10^5 *years into the past and future. A thousand fold increase in computing speed is gained by a method developed by* **Boris Chirikov** *(1979) and applied by* **Jack Wisdom** *(1981) to the study of transition to chaos in the solar system.*

Through this method one replaces the full differential equation describing the behavior of the system by an algebraic mapping that carries the system over a sequence of discrete time intervals. Then one looks at the system only at stroboscopic intervals corresponding to the orbital period of Jupiter. If there were no longer-term variations in the problem, the mapping point would remain fixed in phase space from one strobe time to the next. The movement of the map-point describes only variations slower than the annual revolution. The thousandfold increase in computing speed is gained because the mapping algorithm obviates the need to integrate the differential equations over many smaller time intervals within the 12*-year strobe step; and because the mapping rule is algebraic rather than differential, one has better digital accuracy.*

When the mapping was applied to the $3\colon 1$ *Kirkwood gap, the eccentricity variation suddenly shot up chaotically to fluctuations reaching 35% after behaving itself for* $20,000$ *years, sufficient for the asteroid to cross the orbit of Mars.*

The Kirkwood gaps now had a plausible origin. Close encounters (not necessarily collisions) with Mars would eventually perturb these high-eccentricity orbits out of the commensurable band.

There is a class of asteroids whose orbits come close to or cross that of the earth. They are divided into 3 groups: The Atens have orbits that cross the orbit of the earth, but lie wholly within the orbit of Mars. The Apollos are objects that cross both the orbits of the earth and Mars. The Amors cross the orbit of Mars but do not, at present (1993), quite come as close as the earth's orbit. Some earth-crossing asteroids have been observed at their near-earth passes: of these, Hermes passed very close to earth in 1937. *Icarus missed our planet by only* 6.4 *million km on June* 14, 1968, *and Geographos by only* 10 *million km in 1969*[403].

Thus, the missing planet never had a chance to form. What remains today in the gap between the orbits of Jupiter and Mars appears to be simply a remnant of the scattered debris from the original solar nebula that elsewhere accreted into planets. Effects of resonances, or locations where the orbital period of a body is some exact integer ratio of Jupiter's orbital period — are clearly visible.

1866–1884 CE **Ludwig Eduard Boltzmann**[404] (1844–1906, Austria). One of the greatest physicists of the 19^{th} century. Opened the door to an

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[403] Asteroid fragments (usually called *meteoroids*) have in the past collided with our planet. It resulted in *impact crater* whose diameter depends on both the mass and the speed of the impinging object. One of the most impressive and best preserved impact craters is the *Barringer Crater* near Winslow, Arizona. It measures 1200 meters across and is 200 meters deep. The crater was formed some 25,000 years ago when an iron-rich object measuring 50 meters across struck the ground with a speed estimated at 11 km/sec. The resulting blast, was mechanically equivalent to the detonation of a 20-megaton hydrogen bomb.

[404] For further reading, see:

- Broda, E., *Ludwig Boltzmann: Man. Physicist. Philosopher*, Ox Bow Press, 1983, 169 pp.
- Harris, S., *An Introduction to the Theory of the Boltzmann Equation*, Holt, Rinehart and Winston, 1971, 221 pp.
- Rumer, Yu.B. and M.S.Ryvkin, *Thermodynamics, Statistical Physics and Kinetics*, Mir Publishers, Moscow, 1980, 600 pp.

understanding of the macroscopic systems in a manner consistent with their reversible microscopic *molecular dynamics*. Among the founders of classical statistical mechanics. The originality of Boltzmann's ideas made them difficult for some of his contemporaries to grasp. His important achievements are:

- Molecular kinetic gas theory: *Boltzmann transport equation*[405] (non-linear integro-differential equation for the phase-space distribution function), Maxwell-Boltzmann distribution, the *H-theorem*. The probabilistic interpretation of *entropy* (1866).
- Stress-strain relation for a most general linear viscoelastic solid (1876), known as the *Boltzmann superposition principle*[406].
- *Ergodic hypothesis* (1877).

- Reif, F., *Fundamentals of Statistical and Thermal Physics*, McGraw-Hill, 1965, 651 pp.
- Huang, K., *Statistical Mechanics*, John Wiley & Sons: New York, 1963, 470 pp.
- Jackson, E.A., *Equilibrium Statistical Mechanics*, Dover, 2000, 241 pp.
- Schrödinger, E., *Statistical Thermodynamics*, Cambridge University Press: Cambridge, 1962, 95 pp.
- Sommerfeld, A., *Thermodynamics and Statistical Mechanics* (Lectures on Theoretical Physics), Vol. 5, Academic Press: New York, 1964, 401 pp.

405 Arises in the determination of the phase-space distribution of particles of an ideal gas in an enclosure on which there act external forces.
Another equation, which also bears Boltzmann's name, is an equation for the evolution of a probability density $\phi(x,t)$ over time (furnished with an initial condition),

$$\frac{\partial \phi(x,t)}{\partial t} = -\lambda \phi(x,t) + \lambda \int_0^\infty K(x,s)\phi(s,t)\, ds$$

406 A generalization of *Hooke's solid*, *Newtonian fluid* and the *viscoelastic models* of **Maxwell**, **Kelvin** and **Voigt** (in retrospect). It has the form of a convolution integral

$$\mathfrak{T}(\boldsymbol{r},t) = \int_{-\infty}^{t} \overset{4}{\Psi}(\boldsymbol{r},t-\tau) : \frac{\partial \mathfrak{E}(\boldsymbol{r},\tau)}{\partial \tau} d\tau,$$

where $\overset{4}{\Psi}$ is a fourth-order *relaxation tensor, $\mathfrak{T}(\boldsymbol{r},\tau)$ is a stress tensor and $\mathfrak{E}$ is a strain tensor*.

- *Stefan-Boltzmann law* for blackbody radiation (1883–1884), through which he connected Maxwell's electrodynamics with thermodynamics.

Boltzmann established the statistical nature of the second law of thermodynamics, which stipulates that heat passes, *in a closed system*, spontaneously from a hot (higher temperature) body to a cold (lower temperature) one and never in the reverse direction. Boltzmann's statistical proof of the second law of thermodynamics addresses only the *average* variation of entropy of an isolated system and *does not rule out* the possibility of an occasional decrease in its value. Fluctuations from the average *must*, in fact, occur, and their frequency and typical magnitudes depend on the *size* of the system. Indeed, **Smoluchowski** (1872–1917) has shown that microscopic phenomena can have no intrinsic arrow of time, as far as internal entropy changes are concerned.

When Boltzmann (1872) first presented a statistical theory purporting to prove that an improbable distribution will *always* proceed, when left alone, to a more probable distribution with higher entropy, his colleague Joseph Loschmidt is said to have questioned the general validity of Boltzmann's theorem by pointing out that, if a skilled experimentalist would at some instant reverse all motions in an equilibrium state that had evolved in the manner envisaged by Boltzmann from a non-equilibrium state, then this reversed equilibrium state would return (without further interference from the outside) to a non-equilibrium state, and the entropy would decrease during that return. This counterexample to Boltzmann's supposed proof, nowadays known under the name "*Loschmidt's paradox*", is clearly based on the validity of symmetry with respect to reversal of motion, as it invokes the existence of the reversed process to every process that may take place.

Boltzmann is reported to have silenced Loschmidt at the time by pointing a finger at him and saying, "*You reverse the momenta*". Ever since, there has existed a body of opinion that dismisses Loschmidt's paradox on the grounds that simultaneously reversing the motions of a huge number of molecules is a task in principle beyond the capability of the most skilled experimenters. If indeed this is what Boltzmann meant by his cryptic remark, then it did not dispose of Loschmidt's objection, because there *are* experiments that precisely realize the kind of reversal of motion Loschmidt had in mind. A Loschmidt type of reversal *is* realized in all so-called *spin echo experiments*, which are conducted daily in many laboratories since their first performance by the American physicist **Erwin Hahn** (1950). Indeed, if one were considering a gas of particles of the degree of dynamic simplicity as that dealt with in spin echo experiments, even Loschmidt might have been able to reverse the momenta.

To deal with Loschmidt's objection satisfactorily one must keep in mind that the precise initial positions and velocities of the molecules remain encoded, through the laws of motion, in the future positions and velocities of the molecules *only if the gas remains completely isolated*.

Definition and measurement of temperature T and/or entropy S, on the other hand, require that the gas be put into thermal contact with a heat bath, thus breaking the condition of isolation. Whenever the gas is in thermal contact with a heat bath, its molecules will be subject, as a matter of principle, to random changes in their positions and velocities *which destroy the encoded memory of the initial state*. In other words, the thermal contact couples the *thermodynamic arrow of time* of the measured sample, to that of its environment.

Thus, if one reverses all motions *after* the equilibrium temperature T and/or entropy S have been measured, the gas will *not* return to a non-equilibrium state of lesser entropy, but rather evolve into other states whose entropy is either equal to or larger than S. Furthermore, as shown by **L. Szilard** (1921) in connection with the *Maxwell's Demon* paradox, any micro-management by a macroscopic (or even microscopic) "Demon" of molecular degrees of freedom, entails *its own* entropy increase due to the attendant data processing. We conclude that sequences of actual measurements of entropy on a real-life, macroscopic sample of matter will never show a decrease, even though reversal experiments of the Loschmidt type are possible in certain special circumstances.

Boltzmann's principle represents the entropy S in terms of the probability W of macroscopic states and expresses it in the formula

$$S = k \log W$$

(carved out on Boltzmann's tombstone in the Central Cemetery in Vienna].

Boltzmann never wrote down the equation in this form. This was done by **Planck** (1906), who also introduced the constant k. Boltzmann only referred to the proportionality between S and the logarithm of the probability of a state. The designation of *Boltzmann's principle* was advocated by **Einstein** for the reverse of this relation, namely:

$$W = \exp\{S/k\}$$

in which S is considered to be known empirically.

The insight that the second law of thermodynamics can be understood only in terms of a connection between entropy and probability, was one of the seminal achievements of 19^{th} century scientific thought.

Boltzmann was born in Vienna and received his doctorate there in 1866. After a few years as assistant to his teacher, **Joseph Stefan**, he taught at Graz and then moved on to Heidelberg and Berlin for further studies with such notables as **Gustav Kirchhoff** and **Hermann von Helmholtz**. He returned to Vienna in 1873 as professor of mathematics, but soon left for Graz again, where he served this time as a professor of experimental physics from 1876 to 1889. From Graz Boltzmann went to Münich as professor of theoretical physics, after refusing an invitation to succeed Kirchhoff in Berlin.

He returned to his native Vienna once more in 1894, this time as professor in his own field, but his wanderings were not over yet. He was to leave Vienna for Leipzig in 1900, and then return to his still vacant chair in Vienna in 1902 for the remaining few years of his life. In beginning his inaugural speech in Vienna in 1902, Boltzmann remarked that he could spare his audience the conventional hymn of praise for his predecessor since he and the speaker were identical!

Boltzmann admired the republic of the United States of America and visited it several times. In 1905 he was invited to give a course of lectures (in English) in the summer session at the University of California in Berkeley, where he arrived on 26 June. During his stay he visited Stanford University and Lick Observatory[407].

By 1906, at age 62, Boltzmann had suffered for years from periods of serious depression, and from the perhaps not unrelated burden of serious asthma. During his later years he was plagued by an anxiety that his own wit and memory would suddenly leave him in the midst of a lecture. The combination of his recurrent depressions and fears became too much for him to bear, and he took his own life on 5 September 1906, while on a summer vacation at Duino, near Trieste.

One of the causes of this tragic event was the intense philosophical opposition to his work, which now forms an integral part of physics. Ironically,

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[407] His recollections of that summer survive in his well-known popular essay: "*Reise eines deutschen Professor ins Eldorado*". Of James Lick, the founder of the observatory, he said: "*I have often asked myself which is a more remarkable fact about America: that millionaires are idealistics, or that idealistics become millionaires. What a fortunate land!...*"

Boltzmann was generally having a wonderful time in California; he smuggled wine into Berkeley and was a weekend house guest at the Hearst estate near Livermore, where he played a Schubert sonata on a Grand Steinway before an audience after dinner. He was astonished to find in the Berkeley bulletin an announcement of a course of lectures, by a female colleague, on the preparation of salads and desserts, alongside the syllabus of his own lectures.

Boltzmann died just a year after the publication of Einstein's first paper on Brownian motion, the harbinger of Boltzmann's ultimate triumph.

1867 CE **Charles-Joseph-Étienne Wolf** (1827–1918, France) and **George-Antoine-Pons Rayet** (1839–1906, France). Astronomers. First to observe visually very broad emission lines in several 8^{th} magnitude stars in Cygnus: the spectra of V1042, MR103 and MR100 was observed by them (1867), before systematic use of photographic plates. It was the first known instance of a *laser* being observed about 100 years before the first artificial one was built. The 'bands' were originally thought to be due to hydrogen molecules.

Stars of this class are called today '*Wolf-Rayet Stars*' (WR). They are very rare (only about 150 in our galaxy of 10^{11} stars) and represent an important phase of stellar evolution.

Wolf-Rayet stars have masses in the range 30–50 solar masses (a solar mass $\approx 2 \times 10^{33}$ gram) and lie near the main sequence of the H-R diagram. A very large percentage of these rare and beautiful stars have been confirmed to be members of close binary systems. Although such stars are few in number, they are important in the generation of the chemical elements, and they play a key role in the life-cycle of stars.

It is believed today that WR stars are at the end of their stellar lives ($\leq 4 \times 10^6$ yr). As these stars age, material which the stars have cooked up in their central nuclear furnaces (like carbon and oxygen) gradually reach the surface of the star. When enough material reaches the surface, it absorbs so much of the intense light from the star that an enormously strong *wind* starts to flow from the star's surface. This wind (which, essentially is an ejected hot gas at a typical velocity of 100 km/sec) becomes so thick that it totally obscures the star. The amount of material which the wind carries away is very large. Typically, WR stars lose mass at a rate of about $10^{-6} - 10^{-5}$ solar masses per year. By comparison, our sun loses about 10^{-14} solar masses per year in its solar wind.

This mass loss is so large that it significantly shortens the stars' life.

Astronomers believe that very massive stars become Wolf-Rayet stars just before they explode as *supernovae*.

WR stars became an important object of research for astronomers and continue to challenge our understanding of massive star evolution and the physics of radiative processes in very dense hot (50,000 to 100,000 degree K) star winds. It has rather broad astrophysical implications; e.g. vigorous stellar winds near the very hot WR star *HD* 56925 (WN4) produces *nebulousity with visible shock fronts*.

Because of their intrinsic brightness and their remarkable spectra, some WR can be observed in distant galaxies up to 60 Mps away (1 parsec=3.2 LY). It makes these stars excellent candidates for measuring distances significantly beyond the Cepheid limit. Because of their remarkable optical spectra, dominated by strong emission lines, WR can be distinguished easily with low-resolution spectroscopy, or with narrow-band photometry. Their spectra is thus useful for determining their atmosphere constituents, radial velocities and their distance from us.

Recently (1995–2003), WR stars were linked to *hypernovae*, which in turn are associated with *gamma-ray bursters*.

Wolf worked in the Paris observatory from 1862 and was a professor of astronomy in Paris during 1875–1901.

1867 CE **Alfred Bernhard Nobel** (1833–1896, Sweden). Invented dynamite (a combination of nitroglycerin with an absorbent substance). Within a few years, he became one of the world's richest men. Nobel set up a fund of about 9 million dollars, the interest from which was to be used for annual award prizes in six different fields [physics, chemistry, physiology (or medicine), literature, peace and economics]. Prizes for the first five categories were first presented in 1901.

Dynamite and Peace

The era of modern explosives began in 1739 with the discovery of glycerin by ***Carl Wilhelm Scheele*** *(1742–1786, Sweden), a struggling apothecary who made many first-class contributions to experimental chemistry. Glycerin — a sweet, syrupy liquid — could be obtained by heating various oils of plant or animal origin. This organic substance, frequently used as a humectant in candy, cosmetics, skin lotion, ink and tobacco, was destined to become a substance of first importance in the manufacturing of modern explosives.*

However, organic chemistry *began to take shape as a definite branch of science only about* 1830, *and it was not until* 1858 *that its fundamental theory of molecular structure was put forward by* ***Friedrich August Kekulé***

*(1829–1896, Germany) and **Archibald Scott Couper** (1831–1892, Scotland). [These dates are significant, because a command of organic chemistry was essential before an organic explosive could be prepared and applied.]*

*Thus, in 1846, an Italian chemist, **Sorbero**, first prepared nitroglycerin (glycerol trinitrate) by treating glycerin with a mixture of sulphuric and nitric acids at low temperatures. This oily liquid with its sweet burning taste, detonates violently on the slightest touch: there is sufficient oxygen in the molecule to convert all the carbon and hydrogen present into carbon dioxide and water, liberating molecular nitrogen. The reaction instantaneously releases a large amount of gas ($7\frac{1}{2}$ moles) into small volume, (initially occupied by liquid) at a relatively high temperature.*

*This physical process, in turn, results in an explosion and shock-wave of enormous proportions. [In marked contrast, nitroglycerin has had an interesting pharmaceutical history in the treatment of angina pectoris, as a coronary vascodilator, when taken in tablet form.] Meanwhile, other developments in organic chemistry were taking place: in 1838 the French chemists **Théophile Jules Pelouze** and **Henri Braconnot** obtained highly inflammable material by treating cotton with strong nitric acid, and thereby opened the way to the study of materials which became known as nitro-celluloses.*

*This process was improved in 1845 by **Christian Friedrich Schönbein** (1799–1868, Switzerland), by using a mixture of nitric and sulphuric acids for the nitration of cotton, and resulted (1846) in a new explosive far exceeding gun powder in its power. It was left for Nobel to unite the two lines of research starting from glycerin and cotton, and to show that the explosive properties of nitrated cotton could be tamed for propellant purposes by gelatinizing the fibrous material with nitroglycerin. This discovery, that the two most powerful explosives then known could be blended to furnish a slow burning propellant was so startling that it was received with incredulity, which soon gave place to astonishment. Table 4.10 summarizes the history of explosives.*

1867–1876 CE **Ludwig Schläfli** *(1814–1895, Switzerland). Mathematician. Pioneered in higher-dimensional point geometry and elliptic modular functions (1870). Also made significant contributions to the theory of Bessel functions. The bulk of his work was not published until several years after his death.*

*Schläfli was born in Grasswil, Bern. He enrolled in the theological faculty at Bern but, not wishing to pursue an ecclesiastical career, accepted a post as a teacher of mathematics and science at the Burgerschule in Thun. He taught there for ten years, using his few free hours to study higher mathematics. In the autumn of 1843 he accompanied **Steiner**, **Jacobi** and **Dirichlet** on their travels in Italy, as an interpreter, and had thus the opportunity to learn*

from the leading mathematicians of his time. It was not until 1868, when he became a full professor at Bern University, that he was free from financial concerns.

An examination of his posthumous manuscripts reveal that in 1867, ten years ahead of **Dedekind**, *Schläfli discovered the domain of discontinuity of the modular group and used it to make a careful analysis of the Hermite modular function. We have today the Schläfli modular equation, as well as Schläfli polynomial, function and hypergeometric series in the theory of Bessel functions.*

Besides his mathematical achievements, Schläfli was an expert on the flora of the canton of Bern and an accomplished student of languages. He possessed a profound knowledge of the Veda, and his posthumous manuscripts include ninety notebooks of Sanskrit and commentary on the Rig-Veda.

1868 CE **Felice Casorati** *(1835–1890, Italy). Mathematician. Proved the important Casorati-Weierstrass theorem which claims that in any neighborhood of an essential singularity of a function, it comes arbitrarily close to any given value.*

Casorati was a student in Pavia and later taught at Pavia and Milan.

Table 4.10 MAJOR EVENTS IN THE HISTORY OF EXPLOSIVE MATERIALS[408]

900–1000	Gunpowder developed in China.
1242	English monk **Roger Bacon** (1220–92) described the preparation of gunpowder (using an anagram).
c.1250	German alchemist **Berthold Schwarz** claimed to have reinvented gunpowder.
1771	French chemist **Pierre Woulfe** discovered picric acid (originally used as a yellow dye).
1807	Scottish cleric **Alexander Forsyth** (1767–1843) discovered mercury fulminate.
1833	French chemist **Henri Braconnot** (1781–1855) nitrated starch, making a highly flammable compound (crude nitrocellulose).
1838	French chemist **Théophile Pelouze** (1807–67) nitrated paper, making crude nitrocellulose.
1845	German chemist **Christian Schönbein** (1799–1868) nitrated cotton, making nitrocellulose.
1846	Italian chemist **Ascania Sobrero** (1812–88) discovered nitroglycerin.
1863	Swedish chemist **J. Wilbrand** discovered trinitrotoluene (TNT).
	Swedish chemist **Alfred Nobel** (1833–96) invented a detonating cap based on mercury fulminate.
1867	Alfred Nobel invented dynamite by mixing nitroglycerin and kieselguhr.
1871	German chemist **Hermann Sprengel** showed that picric acid can be used as an explosive.

[408] For further reading, see:

- Read, John, *Explosives*, Pelican Books, 1942, 159 pp.

1875 Alfred Nobel invented blasting gelatin (nitroglycerin mixed with nitrocellulose).

1885 French chemist **Eugene Turpin** discovered ammonium picrate (Mélinite).

1888 Alfred Nobel invented a propellant from nitroglycerin and nitrocellulose (Ballistite).

1889 British scientists **Frederick Abel** (1826–1902) and **James Dewar** invented a propellant (Cordite) similar to Ballistite.

1891 German chemist **Bernhard Tollens** (1841–1918) discovered pentaerythritol tetranitrate (PETN).

1899 **Henning** discovered cyclotrimethylenetrinitramine (RDX or cyclonite).

1905 US army officer **B.W. Dunn** (1860–1936) invented ammonium picrate explosive (Dunnite).

1915 British scientists invented amatol (TNT + ammonium nitrate).

1955 US scientists developed ammonium nitrate-fuel oil mixtures (ANFO) as industrial explosives.

1868 CE, Jan 30 A bright fireball streaked through the sky over the Polish town of Pultusk (52.42°N; 21.02°E; 30 km north of Warsaw): A small asteroid, with an estimated mass of the order of ten tons ripped through the earth's atmosphere at about 20 km/sec and exploded over the town. It pelted the countryside with a shower of rocks, the fragment of which ranged from the size of peas to chunks weighing about 10 kg. The bombardment occurred when the men of Pultusk were at home instead of working in the fields, so no one was killed or injured. In recent years, scientists have analyzed some of the fragments. It was found that the original asteroid was a piece of the primordial material from which the planets formed $4\frac{1}{2} \times 10^9$ years ago.

1868 CE **Joseph Norman Lockyer** (1836–1920, England). Astronomer. Suggested the existence of a new element, not yet discovered on earth. It was named *Helium* (from the Greek word for the sun), because its characteristic lines were found in the spectrum of solar radiation. In 1895, this sun-element was discovered on earth.

1868–1870 CE **Paul Albert Gordan** (1837–1912, Germany). Mathematician. Contributed to invariant theory and algebraic geometry. Proved that every binary form has an associated finite complete system of invariants and covariants. He also showed that any finite system of binary forms has associated with it such a system of invariants and covariants. Gordan was born in Breslau of Jewish parents. He studied under **Kummer** and **Jacobi** and worked with **Clebsch**. **Emmy Noether** was his only doctoral student.

1868–1893 CE **Karl Hermann Amandus Schwarz** (1843–1921, Germany). Mathematician. A pupil of Weierstrass and his successor as professor of mathematics at Berlin (1897). One of the most distinguished researchers on the calculus of variations in the 19^{th} century. Contributed significantly to many branches of mathematics, including the theory of minimal surfaces, the theory of functions and its applications to potential theory (Dirichlet problem), set theory and conformal mappings.

Schwarz showed that smooth parts of a soap film will intersect a smooth supporting surface perpendicularly: he proved that if a *minimal surface* has a free boundary Σ on a support surface S, then it meets S along the curve Σ at a right angle. **Gergonne** (1816) posed the problem: Divide a cube into two parts by a surface M in such a way that M is attached at two inverse diagonals that lie on opposite faces of the cube, and M is of minimal surface (*Gergonne's surface*). A solution was found by Schwarz in 1872.

H.A. Schwarz was the first to solve the *Plateau problem* for the simplest contour which does not lie in a plane (1865). He also discovered two important *reflection principles* for minimal surfaces and periodic minimal surfaces known as *Schwarz chain*.

Named after him are: *Schwarz' inequality*[409], *Schwarz' theorem*, *Schwarz' lemma*, *Schwarz-Christoffel transformation*, *Schwarzian derivative or differential invariant*[410] and the *Schwarz problem*[411].

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409 $\int_a^b \phi^2(x)dx \int_a^b \psi^2(x)dx \geq \left\{\int_a^b \phi(x)\psi(x)dx\right\}^2$.

410 $\{z, u\} = \frac{z'''(u)}{z'(u)} - \frac{3}{2}\left\{\frac{z''(u)}{z'(u)}\right\}^2$.

411 Schwarz's problem: Given an acute triangle, find an inscribed triangle with the *smallest possible perimeter*. Schwarz discovered that the solution is given by

Schwarz was the son of a Jewish architect. He held the chair of mathematics successively at Halle (1867), Zürich (1869), Göttingen (1875) and Berlin (1892). His research was marked by originality of thought, vivid imagination and a love of detail. His problems were concretely and clearly outlined and usually derived from geometrical or mathematical-physics sources. He exercised a decisive influence on the development of mathematics in the second half of the 19^{th} century.

He was married to Kummer's daughter.

1869 CE, Nov. 17 The *Suez Canal* opened (construction began April 25, 1859). Constructed by a French company under the direction of **Ferdinand Marie de Lesseps** (1805–1894, France). It is a narrow waterway, 160 km long, that connects the Mediterranean and the Red Sea. Lesseps, canal builder and diplomat, was born in Versailles, and during 1825–1849 worked in the French diplomatic service.

1869 CE **John Wesley Hyatt** (1837–1920, U.S.A.). Inventor. A printer in Albany, N.Y., who invented *Celluloid*, the first synthetic plastics material to receive wide commercial use. Hyatt was seeking a substitute for ivory to make billiard balls. Celluloid could be sawed,carved, and made into sheets. As a result, new plastics products appeared on the market, including the first photographic roll film. But celluloid was hard to mold and it caught fire easily.

1869 CE First systematic *color photography*[412] (subtractive method) done simultaneously by the Frenchmen **Emile Hortensius Charles Cros** (1842–1888) and **Louis Ducos de Hauron** (1837–1920). Both based their system

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the *altitude triangle*, whose vertices are obtained by dropping the perpendicular from each vertex of the original triangle to its opposite side. It is easy to see that the altitude triangle must be a *light-ray triangle*; if we think of a triangular room with mirror walls, the inscribed triangle represents a closed path of travel for a ray of light in the room. It can be shown that the perimeter of the inscribed light-triangle is equal to $a\cos A + b\cos B + c\cos C$, where ABC is the original triangle.

[412] The first color photograph was demonstrated by **James Clerk Maxwell**, already in 1861.
The physicist, **Gabriel Jonas Lippman** (1845–1921, France) invented in 1891 the first *color photographic process* (based on the phenomenon of light interference). This earned him the physics Nobel prize for 1908. The relative long time required for a film to develop by this method precluded its commercial success, and it was therefore superseded by the Maxwell 3-color procedure. However, *Lippman's plate* found new applications in *holography* (1962).

on the *Young-Maxwell* color separation and mixing theory, by superposing three color positive pictures on one another.

1869–1871 CE **Dimitri Ivanovich Mendeleev** (1834–1907, Russia). Chemist. Introduced order into inorganic chemistry by devising the Periodic Table, that systematized the properties of the elements known at his time and permitted prediction of the existence of new ones. The later synthesis of new elements has been based on his work. Various chemists[413] had traced numerical sequences among the atomic weights of some of the elements and noted connections between them and the properties of the different substances, but it fell to him to give a full expression to the generalization, and to treat it not merely as a system of classifying the elements according to certain observed facts, but as a *law of nature* — which could be relied upon to predict new facts and to disclose errors in what were supposed to be old facts. Thus in 1871 he was led, by certain gaps in his tables, to assert the existence of 3 new elements, hitherto unknown to chemistry, and to assign them definite properties [*gallium*, discovered 1871; *scandium*, 1879; *germanium*, 1886].

The youngest of a family of 17, Mendeleev was born at Tobolsk, Siberia. After attending the gymnasium of his native town, he went to study natural science at St. Petersburg where he graduated in chemistry (1856), subsequently becoming *privatdocent*. In 1860 he went to Heidelberg where he started a laboratory of his own, but returned to St. Petersburg in 1861. He became professor of chemistry in the technological institute there in 1863, and three years later succeeded to the same chair at the University. In 1890

[413] The periodic law was proposed in 1869 independently by **Julius Lothar Meyer** (1830–1895, Germany), who plotted *atomic volumes* (atomic weight/density) of the elements against their atomic weight. The resulting curve exhibits periodicity in the case of other properties, such as expansion by heat, thermal and electrical conductivities, magnetic susceptibility, melting point, refractive index, boiling point, crystalline form, compressibility, atomic heat at low temperatures, heats of formation of oxides and chlorides, hardness, malleability, volatility, volume change on fusion, viscosity and color of salts in aqueous solution, mobility of ions, electrode potentials of metals, frequency of atomic vibrations in solids, distribution of elements in nature, distribution of spectral lines, and valence. As Mendeleev said, "*these regularities can hardly be the result of chance*".

Lothar Meyer pointed out that gaseous elements, occur at the maxima and on ascending portions of the atomic volume curve.

Lewis Reeve Gibbes (1810–1894, U.S.A.) constructed in 1870 a table of the chemical elements which arranged the elements into families according to valences. This work was not published until 1886, by which time the tables of Mendeleev and Meyer were well established.

he resigned the professorship, and in 1893 he was appointed director of the Bureau of Weights and Measures, a post which he occupied till his death.

Boyle, who laid the foundations of modern chemistry in the 17^{th} century, was familiar with the concept of atoms, which may have assisted him in his attempts to classify all substances into elements, compounds and mixtures. However, the full importance of the atomic theory in chemistry was not realized until **Dalton** used it to expound the laws of chemical combinations in 1803. Since the chemical elements react with each other in *fixed proportions by weight*, it appeared that atoms of different elements are combining to form compound atoms or *molecules*. The success of this idea led to the introduction of chemical formulae for the simpler compounds, each formula indicating the atoms present in a single molecule of the compound.

An important development was the introduction of the concept of the *valency* concept in 1852 by **Edward Frankland** (1825–1899, England), which is, ideally, the number of hydrogen atoms combining with one atom of the element considered. Through the introduction of single, double, and triple bonds of the 'covalent' type, the greater part of organic chemistry could be brought into one scheme. It was found, however, that a single valency could not be assigned for many atoms [e.g. nitrogen and sulphur, which do not form ions and thus exhibit variable valency].

Mendeleev's periodic classification was a major development amid the confusion of chemical ideas which prevailed in the middle years of the 19^{th} century. This table grouped elements with valence properties in vertical columns, numbered from I to VIII. The chemical resemblance of the first two rows (*lithium* to *fluorine* and *sodium* to *chlorine*) had been recognized for some time. Although the scheme was received with skepticism, its essential correctness was demonstrated eventually by the discovery of the inert gases (*helium*, *neon*, *argon*, *krypton*, and *xenon*), which provided a completely new column in the table.

In its modern form (Table 4.11), the periodic table is based on the *atomic number* (Z) of the elements, where Z is the number of electrons accommodated outside the nucleus of the atom. The periodic classification shows immediately that the electrons in an atom possesses some kind of *shell structure*. It was only since 1925, the year of the *Pauli exclusion principle*, that the '7^{th} veil' was finally lifted, and the true infrastructure of the periodic table was understood.

Table 4.11: Periodic Table of the Elements

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1A	2A	3B	4B	5B	6B	7B	8B			1B	2B	3A	4A	5A	6A	7A	8A
1 **H** 1.00794																	2 **He** 4.00260
3 **Li** 6.941	4 **Be** 9.01218											5 **B** 10.811	6 **C** 12.011	7 **N** 14.0067	8 **O** 15.9994	9 **F** 18.9984	10 **Ne** 20.1797
11 **Na** 22.9898	12 **Mg** 24.3050											13 **Al** 26.9815	14 **Si** 28.0855	15 **P** 30.9738	16 **S** 32.066	17 **Cl** 35.4527	18 **Ar** 39.948
19 **K** 39.0983	20 **Ca** 40.078	21 **Sc** 44.9559	22 **Ti** 47.88	23 **V** 50.9415	24 **Cr** 51.9961	25 **Mn** 54.9381	26 **Fe** 55.847	27 **Co** 58.9332	28 **Ni** 58.693	29 **Cu** 63.546	30 **Zn** 65.39	31 **Ga** 69.723	32 **Ge** 72.61	33 **As** 74.9216	34 **Se** 78.96	35 **Br** 79.904	36 **Kr** 83.80
37 **Rb** 85.4678	38 **Sr** 87.62	39 **Y** 88.9059	40 **Zr** 91.224	41 **Nb** 92.9064	42 **Mo** 95.94	43 **Tc** (98)	44 **Ru** 101.07	45 **Rh** 102.906	46 **Pd** 106.42	47 **Ag** 107.868	48 **Cd** 112.411	49 **In** 114.818	50 **Sn** 118.710	51 **Sb** 121.757	52 **Te** 127.60	53 **I** 126.904	54 **Xe** 131.29
55 **Cs** 132.905	56 **Ba** 137.327	57 ***La** 138.906	72 **Hf** 178.49	73 **Ta** 180.948	74 **W** 183.84	75 **Re** 186.207	76 **Os** 190.23	77 **Ir** 192.22	78 **Pt** 195.08	79 **Au** 196.967	80 **Hg** 200.59	81 **Tl** 204.383	82 **Pb** 207.2	83 **Bi** 208.980	84 **Po** (209)	85 **At** (210)	86 **Rn** (222)
87 **Fr** (223)	88 **Ra** 226.025	89 **Ac** 227.028	104 **Rf** (261)	105 **Db** (262)	106 **Sg** (263)	107 **Bh** (262)	108 **Hs** (265)	109 **Mt** (266)	110 (269)	111 (272)	112 (272)		114 (287)		116 (289)		118 (293)

*Lanthanide series	58 **Ce** 140.115	59 **Pr** 140.908	60 **Nd** 144.24	61 **Pm** (145)	62 **Sm** 150.36	63 **Eu** 151.965	64 **Gd** 157.25	65 **Tb** 158.925	66 **Dy** 162.50	67 **Ho** 164.930	68 **Er** 167.26	69 **Tm** 168.934	70 **Yb** 173.04	71 **Lu** 174.967
Actinide series	90 **Th** 232.038	91 **Pa** 231.036	92 **U** 238.029	93 **Np** 237.048	94 **Pu** (244)	95 **Am** (243)	96 **Cm** (247)	97 **Bk** (247)	98 **Cf** (251)	99 **Es** (252)	100 **Fm** (257)	101 **Md** (258)	102 **No** (259)	103 **Lr** (260)

Atomic masses are relative to carbon-12. For certain radioactive elements, the numbers listed (in parentheses) are the mass numbers of the most stable isotopes.

The Elements

Name	Symbol	Atomic Number	Relative Atomic Weight	Name	Symbol	Atomic Number	Relative Atomic Weight
Actinium	Ac	89	227.028	Mendelevium	Md	101	(258)
Aluminum	Al	13	26.9815	Mercury	Hg	80	200.59
Americium	Am	95	(243)	Molybdenum	Mo	42	95.94
Antimony	Sb	51	121.757	Neodymium	Nd	60	144.24
Argon	Ar	18	39.948	Neon	Ne	10	20.1797
Arsenic	As	33	74.9216	Neptunium	Np	93	237.048
Astatine	At	85	(210)	Nickel	Ni	28	58.693
Barium	Ba	56	137.327	Niobium	Nb	41	92.9064
Berkelium	Bk	97	(247)	Nitrogen	N	7	14.0067
Beryllium	Be	4	9.01218	Nobelium	No	102	(259)
Bismuth	Bi	83	208.980	Osmium	Os	76	190.23
Bohrium	Bh	107	(262)	Oxygen	O	8	15.9994
Boron	B	5	10.811	Palladium	Pd	46	106.42
Bromine	Br	35	79.904	Phosphorus	P	15	30.9738
Cadmium	Cd	48	112.411	Platinum	Pt	78	195.08
Calcium	Ca	20	40.078	Plutonium	Pu	94	(244)
Californium	Cf	98	(251)	Polonium	Po	84	(209)
Carbon	C	6	12.011	Potassium	K	19	39.0983
Cerium	Ce	58	140.115	Praseodymium	Pr	59	140.908
Cesium	Cs	55	132.905	Promethium	Pm	61	(145)
Chlorine	Cl	17	35.4527	Protactinium	Pa	91	231.036
Chromium	Cr	24	51.9961	Radium	Ra	88	226.025
Cobalt	Co	27	58.9332	Radon	Rn	86	(222)
Copper	Cu	29	63.546	Rhenium	Re	75	186.207
Curium	Cm	96	(247)	Rhodium	Rh	45	102.906
Dubnium	Db	105	(262)	Rubidium	Rb	37	85.4678
Dysprosium	Dy	66	162.50	Ruthenium	Ru	44	101.07
Einsteinium	Es	99	(252)	Rutherfordium	Rf	104	(261)
Erbium	Er	68	167.26	Samarium	Sm	62	150.36
Europium	Eu	63	151.965	Scandium	Sc	21	44.9559
Fermium	Fm	100	(257)	Seaborgium	Sg	106	(263)
Fluorine	F	9	18.9984	Selenium	Se	34	78.96
Francium	Fr	87	(223)	Silicon	Si	14	28.0855
Gadolinium	Gd	64	157.25	Silver	Ag	47	107.868
Gallium	Ga	31	69.723	Sodium	Na	11	22.9898
Germanium	Ge	32	72.61	Strontium	Sr	38	87.62
Gold	Au	79	196.967	Sulfur	S	16	32.066
Hafnium	Hf	72	178.49	Tantalum	Ta	73	180.948
Hassium	Hs	108	(265)	Technetium	Tc	43	(98)
Helium	He	2	4.00260	Tellurium	Te	52	127.60
Holmium	Ho	67	164.930	Terbium	Tb	65	158.925
Hydrogen	H	1	1.00794	Thallium	Tl	81	204.383
Indium	In	49	114.818	Thorium	Th	90	232.038
Iodine	I	53	126.904	Thulium	Tm	69	168.934
Iridium	Ir	77	192.22	Tin	Sn	50	118.710
Iron	Fe	26	55.847	Titanium	Ti	22	47.88
Krypton	Kr	36	83.80	Tungsten	W	74	183.84
Lanthanum	La	57	138.906	Uranium	U	92	238.029
Lawrencium	Lr	103	(260)	Vanadium	V	23	50.9415
Lead	Pb	82	207.2	Xenon	Xe	54	131.29
Lithium	Li	3	6.941	Ytterbium	Yb	70	173.04
Lutetium	Lu	71	174.967	Yttrium	Y	39	88.9059
Magnesium	Mg	12	24.3050	Zinc	Zn	30	65.39
Manganese	Mn	25	54.9381	Zirconium	Zr	40	91.224
Meitnerium	Mt	109	(266)				

Atomic masses in this table are relative to carbon-12 and limited to six significant figures, although some atomic masses are known more precisely. For certain radioactive elements the numbers listed (in parentheses) are the mass numbers of the most stable isotopes.

The Elements[414]

Chemistry started to emerge from its alchemical roots in the 18th century, partly with the discovery of new elements: between 1735 and 1826, no fewer than 40 were added to the 9 known to the ancients (copper, silver, gold, iron, mercury, lead, tin, sulphur and carbon) and the few discovered in the Middle Ages (arsenic, antimony and bismuth). The discovery of these new elements forced certain questions on every chemist: How many elements were there? Was there any limit to their number? Were they all related somehow? And if so, how could they be classified?

Kinships were recognized among some. Chlorine, bromine and iodine — all colored, volatile, hungrily reactive — seemed a natural family, the halogens. Calcium, strontium and barium, the alkaline earth metals, were another family, for they were all light, soft, readily set alight and strongly reactive with water.

In 1817, **Johann Döbereiner** *observed that the atomic weights of the alkaline earth metals formed a series, the atomic weight of strontium being just midway between those of calcium and barium. He later discovered other such triads, as well as triads in which the elements had similar properties but almost identical atomic weights.*

Döbereiner's triads convinced many chemists that atomic weight must represent a fundamental characteristic of all elements. But confusion about the basics remained — about the difference between atoms and molecules and about the combining power, or valency, of atoms. As a consequence, many accepted atomic weights were wrong. Dalton himself — the originator of the atomic hypothesis — assumed, for instance, that the formula of water was HO *and not* H_2O, *giving him an atomic weight for oxygen that was only half the correct number.*

In 1860, the first international gathering of chemists was convened at Karlsruhe, Germany, for the expressed purpose of clearing up this confusion. Here, **Stanislao Cannizzaro** *proposed a reliable way of calculating atomic weights from vapor density, and his beautifully argued presentation carried the day, leading to a consensus: now, at last, with corrected atomic weights and a*

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414 For further reading, see:

- Emsley, John, *Nature's Building Blocks*, Oxford University Press, 2003, 539 pp.

clear idea of valency, the way was open for a comprehensive classification of the elements.

It is a remarkable example of synchronicity that no fewer than six such classifications, all pointing toward the discovery of periodicity, were independently devised in the next decade. Of these, ***Dmitri Ivanovich Mendeleev's*** *system was the most comprehensive, and also the most audacious, for it ventured to make detailed predictions of elements as yet unknown.*

Mendeleev was the author of a chemistry text "The Principles of Chemistry", and he had brooded since 1854 on how the chemical elements might be classified.

With the old, pre-Karlsruhe atomic weights, one could get, as Döbereiner did, a sense of local triads, or groups. But one could not easily see that there was a numerical relationship between the groups themselves, Only when Cannizzaro showed that the proper atomic weights for the alkaline earth metals, calcium, strontium and barium, were 40, 88 and 137 did it become clear how close these were to those of the alkali metals, potassium (39), rubidium (85) and cesium (133). It was this closeness, and the closeness of the atomic weights of the halogens — chlorine, bromine and iodine — that incited Mendeleev in 1868 to make a small, two-dimensional grid juxtaposing the three groups:

Cl	35.5	K	39	Ca	40
Br	80	Rb	85	Sr	88
I	127	Cs	133	Ba	137

And it was at this point, seeing that arranging the three groups of elements in order of atomic weight produced a repetitive pattern — a halogen followed by an alkali metal followed by an alkaline earth metal — that Mendeleev felt this must be a fragment of a larger pattern and leaped to the idea of a periodicity governing all the elements, a periodic law.

Mendeleev's first small table had to be filled in and then extended in all directions, as if filling up a crossword puzzle. Alternating between conscious calculation and hunch, between intuition and analysis, Mendeleev arrived within a few weeks at a tabulation of 30-odd elements in order of ascending atomic weight, a tabulation that suggested that there was a recapitulation of properties with every eighth element.

On the night of Feb. 16, 1869, it is said, Mendeleev had a dream in which he saw almost all of the 65 known elements arrayed in a grand table. The following morning, he committed this to paper.

This first table was to undergo considerable revision over the next few years, but by 1871 it had taken its now familiar form of a chunky rectangle with intersecting groups and periods.

It was this table that was to be found in every textbook, lecture room and museum for a century. One could read the table up and down, going from one group to another (each vertical group was a family of elements with similar reactivity and valency) — this was what Döbereiner and the pre-1860 chemists would have done.

But one could also read it horizontally, getting a feel for each period as it moved through the eight groups. One could see the way in which the properties of the elements changed with each increment of atomic weight, until suddenly the period came to an end and one found oneself on the next row and period, where all the elements echoed the properties of those above. It was this, above all, that gave one a feel for the mysterious periodicity of the table, the reality of the great law it enshrined.

The periodic table did not actually tell one the properties of the elements, but like a family tree, it assigned them places. One could plot the physical and chemical properties of all the elements against their atomic weights and obtain the most tantalizing graphs. If one plotted atomic volume against atomic weight, for example, one would get a many-peaked curve, with summits for the light Group I metals, valleys for the dense Group VIII metals. Every property, it seemed, varied periodically and was somehow linked with atomic weight. But why any of the elements should have the properties they had, and why such properties should recur in periodicity with atomic weight, were complete mysteries.

From 1869 to 1871, Mendeleev expanded the table, going so far as to reposition elements that did not fit, revising their accepted atomic weights to make them fit, a practice that shocked some of his contemporaries. Further challenges were presented by two groups of elements, the transition elements (these included rare metals like vanadium and platinum, as well as common ones like iron and nickel) and the rare-earth elements. Neither of these seemed to fit in the neat "octaves" of the earlier periods. To accommodate them, Mendeleev and others experimented with new forms of the table — helical forms, pyramidal forms, etc. — that, in a sense, gave it extra dimensions.

In an act of supreme confidence, Mendeleev reserved several empty spaces in his table for elements "as yet unknown". He asserted that by extrapolating from the properties of the elements above and below (and also, to some extent, from those to either side), one might make a confident prediction as to what these unknown elements would be like. He did exactly this, predicting in great detail a new element that would follow aluminum in Group III: it would be a silvery metal, he thought, with a density of 6.0 and an atomic weight of 68. Four years later, in 1875, just such an element was found: GALLIUM.

He also predicted with equal precision the existence of SCANDIUM *and* GERMANIUM, *and these too were soon discovered. It was this ability to predict*

elements in such detail that stunned his fellow chemists and convinced many of them that Mendeleev's system was not just an arbitrary ordering of the elements but a profound expression of reality.

But Mendeleev was astonished, as everyone was, by the discovery in the 1890's of an entire new family of elements, the inert gases. He was at first skeptical of their existence. (He initially thought that ARGON, *the first found, was just a heavier form of nitrogen.) But with the discovery of* HELIUM, NEON, KRYPTON, XENON *and finally* RADON, *it was clear that they formed a perfect periodic group. They were identical in their inability to form compounds; they had a valency, it seemed, of zero. So to the eight groups of the table, Mendeleev now added a final Group 0.*

With the inert gases in place, the number of elements in each period stood out: 2 (hydrogen and helium) in the first period; 8 each in the second and third; 8 typical plus 10 transition elements, or 18 each, in the fourth and fifth periods; 8 plus 10 plus 14 rare-earth elements, or 32, in the sixth period. These were the magical numbers — 2, 8, 8, 18, 18, 32. But what did they mean? And what, in broader terms, was the basis of chemical properties?

Mendeleev constantly returned to these questions. He yearned for a new "chemical mechanics", comparable to the classical mechanics of Newton. And yet one wonders what he might have thought of the actual form of the revolution that took place after his death, a revolution wholly unimaginable in terms of classical mechanics.

The new insight into the internal constitution of atoms came in 1911, four years after Mendeleev's death, when ***Ernest Rutherford*** *(bombarding gold foil with alpha particles and finding that, very occasionally, one was deflected back) inferred that the atom must have a structure like a miniature solar system, with almost all of its mass concentrated in a minute, very dense, positively charged nucleus surrounded at relatively great distances by relatively light electrons.*

But the very essence of atoms was their absolute stability. And such an atom as Rutherford's, if ruled by the laws of classical mechanics, would not be stable; its electrons would lose energy (by EM radiation) as they orbited, eventually diving into the nucleus.

Niels Bohr*, working with Rutherford in 1912, was intensely aware of this, and of the need for a radically new approach. This he found in the quantum theory, which postulated that electromagnetic energy — light, radiation — was not emitted or absorbed continuously, but rather in discrete packets, or "quanta". Bohr, by an astounding leap, connected these concepts with the Rutherford model and with the well-known but previously inexplicable nature of optical spectra — that these were not only characteristic for each element*

but consisted of a multitude of discrete lines or frequencies which obeyed Ritz's combination law.

All of these considerations came together in the Bohr atom, where electrons were conceived to occupy a series of orbits, or "shells", about the nucleus – of differing radii and energies. Unlike classical orbits, which would decay, these quantum orbits had a stability that allowed them to maintain themselves, potentially, forever. (But if the atom was excited, some of its electrons might leap to higher energy orbits for a while and in returning to their ground state emit a quantum of light energy of a certain frequency; it was this that caused the characteristic absorption and emission lines in their spectra.)

Bohr presented his model of the atom in the spring of 1913. A few months later, ***Henry Moseley*** *found a most intimate relationship between the order of the elements and their X-ray spectra. These spectra could be correlated, Moseley thought, with the number of positive charge units carried by the nucleus, and for this the term "atomic number" was used. With atomic numbers, there were no gaps or fractions or irregularities, as with atomic weights. It was atomic number, not atomic weight, that determined the order of the elements. And Moseley could now say with absolute confidence that there were only 92 elements between hydrogen and uranium, including half a dozen as yet undiscovered. (Three of these had been predicted, though vaguely, by Mendeleev.)*

Bohr's model suggested that every element's chemical properties, its position in the periodic table, depended on the number of its electrons and how these were organized in successive shells. Valency and chemical reactivity, the definers of Mendeleev's groups, were correlated with the number of valence electrons in the outer shell: with the maximum of eight electrons, an atom was chemically inert; with more, or less, than the maximum, it would tend to be more reactive. Thus the halogens, only one electron short in their outermost shells, were avid to pick up an eighth electron, whereas the alkali metals, with only a single electron in their outer shells, were avid to get rid of it, to become stable in their own way.

To this basic "eightness", extra sub-shells were added in the later periods: two 10-electron shells for the transition elements and two 14-electron shells for the rare-earth elements.

Bohr *and* ***Moseley*** *thus provided a spectacular confirmation of the periodic table, grounding it, as* ***Mendeleev*** *had hoped, in "the invisible world of chemical atoms". The periodicity of the elements, it was now clear, emerged from their electronic structure. And the mysterious numbers that governed the periodic table — 2, 8, 8, 18, 18, 32 (and eventually, another "32") — could now be understood as the number of electrons added in each period.*

Such an electronic periodic table is basically identical with Mendeleev's table, posited nearly half a century earlier on purely chemical grounds. Moseley and Bohr worked from the inside, with the invisible world of chemical atoms, and Mendeleev and his contemporaries worked from the outside, with the visible macroscopic and manifest physico-chemical properties of the elements — and yet they arrived at the same point. This is the beauty of the periodic table, indeed, that it looks both ways, uniting classical material science and chemistry and quantum physics in a magical synthesis.

Given Bohr's orbits[415] *of different energy levels, together with the property of* electron spin *and the* Pauli exclusion principle *(both discovered in the 1920's, close on the heels of Quantum Mechanics), one can, in principle, build up the whole periodic table by adding electrons to the atom (and protons to its nucleus) one at a time, climbing the rungs of an atomic ladder from hydrogen to uranium. And it is by such a building-up that we have been able to create new elements absent in nature, like the 20 elements (93–112) that now follow uranium in the periodic table, heavier atoms that do not depart from the regularities of the periodic law.*

In principle, one can work out the periodic table to element 200 and beyond and predict some of the properties of such elements. (These predictions are largely theoretical, because the highly radioactive transuranic elements tend to get more and more unstable. One may only be able to produce an atom at a time, and this may be gone in a few millionths of a second. And there are theoretical reasons for believing that for atomic numbers above about 137, the intense electric fields near the nucleus might locally destabilize the vacuum*, producing pair-created electrons and positrons and destroying such heavy nuclei before they have had a chance to form atoms.)*

The periodic table is still the icon of chemistry, as it has been since 1869. It continues to guide chemical research, to suggest new syntheses, to allow predictions of the properties of never-before-seen materials. It is a marvelous map to the whole geography of the elements and their compounds and alloys.

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[415] (later renamed *orbitals* when quantum mechanics – which supplanted the old quantum theory of Planck, Einstein and Bohr – showed them to be fuzzy, variously-shaped, continuous probability distributions rather than Keplerian orbits.)

1869–1874 CE **Johann Friedrich Miescher** (1844–1895, Switzerland). Chemist and physiologist. Professor at Basel (1871–1895). Discovered (1869) *nucleic acids* in cell nuclei. Miescher, isolated (1874) a substance from tissue, that turned out to be neither carbohydrate, lipid, nor protein. Since he had obtained it from cell nuclei, he named it *nuclein.* In time the substance turned out to have acid properties, so it was renamed *nucleic acid*, but it was not connected either to heredity or to chromosomes. This substance was eventually (1944) found to be joined to the protein of chromosomes, and given the name *nucleoprotein.* In Miescher's time, however, no one understood its significance. [Miescher later discovered that salmon sperm are almost entirely nucleic acid plus a simple protein; but he failed to connect this fact with heredity.]

1869–1877 CE **Elwin Bruno Christoffel** (1829–1900, Germany). Mathematician. Discovered the procedure now known as '*covariant differentiation*' (1869) and introduced two symbols, now named after him. Independently of Riemann, he discovered the concepts of space curvature and metric. In 1877 Christoffel derived the cubic equation for the three plane-wave phase velocities in general anisotropic elastic media. He was also one of the first contributors to the theory of shock waves.

Christoffel studied at the University of Berlin, where he was taught by Dirichlet. Obtained his doctorate in 1856 and became eventually a professor of mathematics at the University of Strasbourg (1872–1892).

1869–1882 CE **Rudolf Otto Sigismund Lipschitz** (1832–1903, Germany). Mathematician. Invented independently the process of *covariant differentiation* (1869). Discovered new theorems concerning subspaces of Riemannian and Euclidean manifolds, the mean curvature vector and minimal subspaces.

Lipschitz was born to Jewish parents, on his father's estate near Königsberg. At the age of 15 he began the study of mathematics at the Königsberg University. He received his doctorate from the University of Berlin in 1853. From 1864 onwards he was a full professor at Bonn. Lipschitz was a corresponding member of the academies of Paris, Berlin, Göttingen and Rome.

With Christoffel, Aronhold and Clebsch he laid the foundations to Ricci's absolute differential calculus[416].

[416] If $f(x,y)$ is defined in a region S such that, for *any* two points (x,y) and $(x,\overline{y})$ in S, $|f(x,y)-f(x,\overline{y})| \leq N|y-\overline{y}|^{\alpha}$, where N, α are positive constants, then $f(x,y)$ is said to satisfy the *Lipschitz condition* in S.

1870 CE[417] **Eugène Rouché** (1832–1910, France). Mathematician. Worked on complex functions, descriptive geometry, algebra and probability theory. Was born in Sommiéres, and died in Lunel, France. Known mainly for *Rouché's Theorem*[418].

Science Progress Report No. 10
Darwinism ad Absurdum

A scientific theory that had the most revolutionary impact on almost every facet of Western thought and society in the second half of the 19^{th} *century was Darwin's theory of evolution. To an age that worshiped science, the thought that man was just as much subject to the logic of science as was everything else in nature, also held great fascination.*

Underlying much of Darwin's work was the idea of progress, an idea dear to the 19^{th} *century. History, the study of man's past, suddenly appeared in a new light — as a march toward some far-off, lofty goal. The concept of life as a struggle for existence in which the fittest would survive had particular appeal to his contemporaries. The philosophy of laissez faire, with its emphasis on competition, had long been hailed as the root of economic success. With the advent of Darwinism, this belief seemed to have been given scientific sanction.*

If $f(x, y)$ has a continuous partial derivative w.r.t. y in S, and if this partial derivative is bounded in S, then $f(x, y)$ satisfies the Lipschitz condition with $\alpha = 1$.

If $f(x, y)$ is continuous and satisfies the Lipschitz condition with $\alpha = 1$ in $S\{0 \le x \le b, |y - y_0| < c\}$, then the initial-value problem $\frac{dy}{dx} = f(x, y)$, $y(0) = y_0$, $M = \max|f(x, y)|$ in S has a unique solution for $0 \le x \le \min\left[b, \frac{c}{M}\right]$.

417 In the absence of additional biographical material, this year was arbitrarily chosen for the inception of the *Rouché theorem*.

418 Let $f(z)$ and $g(z)$ be analytic within and on a simple closed contour C and satisfy the inequality $|g(z)| < |f(z)|$ on C, where $f(z)$ does not vanish. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.
This theorem provides a proof that an algebraic equation of degree n has n roots (cannot be proved by *purely* algebraic methods!).

Big business, according to John D. Rockefeller, was "merely a survival of the fittest... the working out of a law of nature and a law of God". But not only the Capitalists derived great comfort from Darwin. His emphasis on the importance of environment for the improvement of man also gave hope to the socialists in their demands for social and economic reform. More than ten years before Darwin published his Origin of the Species, ***Karl Marx**, the "Darwin of the Social Sciences", had already sketched the evolution of society through a series of struggles among social classes.*

The application of Darwin's theory to groups and states, rather then individuals, was promoted by ***Herbert Spencer*** *and* ***Walter Bagehot*** *(1872), and is known as 'Social Darwinism'. It blends evolutionary and nationalist elements, and argues that the majority of* groups *which win and conquer are better than the majority of those which fail and perish.*

To a generation that had recently experienced several major wars and that was actively engaged in numerous colonial expeditions against native peoples overseas, Social Darwinism with its glorification of war came as welcome rationalization[419] *[e.g. President Theodore Roosevelt held that war alone enabled man to "acquire those virile qualities necessary to win in the stern strife of actual life".]*

While Darwin's ideas (1844) were accepted almost everywhere around the world, they were somehow slow to reach the "Bible belt" in the deep south of the U.S. It thus came to pass that 81 years later, in the town of Dayton, Tennessee, a public school science teacher by the name of John T. Scopes was arrested for violating the Anti-Evolution Law *that prohibited teaching Darwin's theory of evolution in public schools of that state.*

The ensuing trial (known as the "Monkey Trial") took place in July 1925. Assisting the state prosecution was ***William Jennings Bryan*** *(1860–1925), a famous orator and statesman who strongly advocated literal interpretation of the Bible and who believed in religious fundamentalism.*

Opposite him stood ***Clarence Seward Darrow*** *(1857–1938), the renowned criminal lawyer who defended Scopes and the right to teach evolution.*

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[419] The maxim that 'might makes right' had little to do with Darwin's original theory. In fact, the emphasis on struggle as a *necessary condition* for progress was a narrow and one-sided interpretation of Darwinism, not shared by its author. In his *Descent of Man*, Darwin had emphasized that a feeling of sympathy and coherence, social and moral activities, were needed for the advancement of society.

When the legal aspects of the case had been fought to conclusion[420]*, when both sides had belabored the right of sovereign people to pass any legislation they saw fit, and when the question of whether the Anti-Evolution Law violated the Constitution of the United States had been obscured, it was the fifty questions that Darrow had put to Bryan, which suddenly flashed the trial into focus an discredited the Anti-Evolution Bill.*

However, the anti-evolution laws remained on the books in half a dozen states for another forty years.

Social Darwinism also led to the clothing of racism in the disguise of a scientific doctrine. The first 'theorist' of the superiority of the Germanic 'Aryans' over the inferior Slavs and Jews was the French ***Joseph Arthur de Gobineau*** *in his Inequality of the Human Races (1852). The idea of white, specifically Anglo-Saxon, superiority found its main echo in Germany but was also popular in England and America. Ever since, it remained one of the underlying ideological sources and justifications of modern antisemitism.*

Recently, a number of authors[421] *have linked Darwin's theory of evolution to the major mass murders and genocides in the 20*th *century. It seems that Darwinism has infected the whole culture. Indeed, the world would witness Nazi Germany, Stalinist gulags and the slaughter of 70 million Chinese at the hands of their exalted chairman.*

Scientists such as ***Francis Galton*** *and* ***Ernst Haeckel*** *extended Darwinism to advance their ideas for selective breeding of humans and forced sterilization of "unfit", calling politics "applied biology" — a phrase later appropriated by the Nazis.*

It is impossible to understand Hitler's monstrous views apart from his belief in natural selection applied to races. He believed Darwin's theory of natural selection showed that "science" justified the extermination of the Jews.

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420 The real story of the Scopes trial is told in the book *Summer of the Gods* (by E.J. Larson, Basic Books: New York, 1997): the trial was nothing but a publicity stunt. The idea for a trial on evolution was hatched by the ACLU in New York and seized upon by civic leaders in Dayton, Tennessee as a way to drum up publicity for their town.

421 In his book "*From Darwin to Hitler*" (Palgrave MacMillan, 2004), **Richard Weikart** documents the proliferation of eugenics organization in Germany around 1900. Darwin's theory was quickly and widely accepted among German biologists and Darwinism provided the lingo for "scientific" racism. Not only were all eugenicists Darwinists, but nearly all Darwinists were scientific racist. See also the last chapter of the Ann Coulter's book *Godless* (Crown Forum Publ.: New York, 2006).

Hitler embraced an evolutionary ethic that made Darwinian struggle for existence between races, become the sole arbiter for morality.

Indeed, within one century of the appearance of this book, the 'theory' has been applied by the Germans in a most efficient way, for the "final solution" of the "Jewish problem" in Europe.

1870–1871 CE The *Franco-Prussian War* resulted in the foundation of the German Empire. To it, France ceded Alsace-Lorraine and paid one billion dollars in reparations. The war led France to withdraw the French troops that were protecting Rome for the Pope. The Italian army moved into Rome, and Italy at last included the entire peninsula. The Pope's territory was reduced to Vatican City, and Rome became the capital of Italy in 1871.

1870–1893 CE **Marius Sophus Lie** (1842–1899, Norway). A path-breaking mathematician whose work has found wide applications in 20^{th} century analysis and physics.

Developed his notions of continuous transformation groups and their role in the theory of differential equations. Today the theory of continuous groups is a fundamental tool in such diverse areas as analysis, differential geometry, number theory, differential equations, mechanics, atomic structure and high energy physics. *Lie groups* and *Lie algebras* are named after him.

Lie groups are smooth Riemannian manifolds in which each point is an element in a continuous group of matrices. A tensor, which in ordinary manifolds enters via the study of tangent curves, enters in Lie groups in a two-fold manner: (1) In the ordinary way on manifolds. (2) Each point on the manifold is itself a matrix. Lie invented the so-called 'Lie-derivative'[422].

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[422] *The Lie derivative* is a covariant process of *directional* differentiation which is distinct from "covariant differentiation" (absolute differentiation, based on the affine connection). The Lie derivative depends only on the *tensor field* it acts on and on the *vector field* defining the local direction, and in this sense is more natural.

Both derivatives obey the basic laws of differentiation: they are linear, obey Leibniz' rule, and reduce to ordinary directional derivative when acting on a scalar field. Thus, for a scalar field $\phi(x)$ and a contravariant vector field $v^i(x)$, the Lie derivative of ϕ along v is by definition $\mathcal{L}_{\boldsymbol{V}}\phi \equiv v^i \frac{\partial \phi}{\partial x^i} = \boldsymbol{V} \cdot \boldsymbol{\nabla}\phi$. When acting on another contravariant vector field A^i, or on a covariant vector field

Lie was born at Nordfjordeif, near Bergen, and was educated at the University of Christiania, where he took his doctorate degree in 1868 and became an associate professor of mathematics four years later[423]. In 1886 he was cho-

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B_i, the Lie derivative is defined as follows:

$$\mathcal{L}_{\boldsymbol{V}} A^i = v^k \frac{\partial A^i}{\partial x^k} - A^k \frac{\partial v^i}{\partial x^k}; \qquad \mathcal{L}_{\boldsymbol{V}} B_i = v^k \frac{\partial B_i}{\partial x^k} + B_k \frac{\partial v^k}{\partial x^i}.$$

In index-free notation, the above formulae are written as

$$\mathcal{L}_{\boldsymbol{V}} \boldsymbol{A} = \boldsymbol{V} \cdot \boldsymbol{\nabla} \boldsymbol{A} - \boldsymbol{A} \cdot \boldsymbol{\nabla} \boldsymbol{V}; \qquad \mathcal{L}_{\boldsymbol{V}} \boldsymbol{B} = \boldsymbol{V} \cdot \boldsymbol{\nabla} \boldsymbol{B} + \boldsymbol{\nabla} \boldsymbol{V} \cdot \boldsymbol{B}.$$

It is straightforward to verify that $\mathcal{L}_{\boldsymbol{V}} \boldsymbol{A}$, $\mathcal{L}_{\boldsymbol{V}} \boldsymbol{B}$ again transform as contravariant and covariant vector fields respectively, and that $\mathcal{L}_{\boldsymbol{V}}$ acts on the scalar field $A^i B_i$ in accordance with the Leibniz rule. By using this rule on arbitrary *dyadic products* of $\boldsymbol{A}$ and $\boldsymbol{B}$, we easily discover the law of action of $\mathcal{L}_{\boldsymbol{V}}$ on any mixed tensor field; thus

$$\mathcal{L}_{\boldsymbol{V}} T^i_{\cdot j} = v^k \frac{\partial T^i \cdot j}{\partial x^k} - T^k_{\cdot j} \frac{\partial v^i}{\partial x^k} + T^i_{\cdot k} \frac{\partial v^k}{\partial x^j},$$

which in index-free notation reads

$$\mathcal{L}_{\boldsymbol{V}} \mathfrak{T} = \boldsymbol{V} \cdot \boldsymbol{\nabla} \mathfrak{T} - \widetilde{\mathfrak{T}} \cdot \boldsymbol{\nabla} \boldsymbol{V} + \mathfrak{T} \cdot (\widetilde{\boldsymbol{\nabla} \boldsymbol{V}}),$$

where twiddle ($\sim$) denotes dyadic transposition. Note that the price we pay for the freedom of the Lie derivative from the concept of connection, is that $\mathcal{L}_{\boldsymbol{V}}$ depends both on the field $\boldsymbol{V}$ *and* its gradient.

It is possible to introduce the Lie derivative in an alternative way; we associate with the field $\boldsymbol{V}$ the infinitesimal coordinate transformation $\overline{x}^i = x^i + \lambda v^i(x)$. It can then be shown that $\mathcal{L}_{\boldsymbol{V}} T = \lim_{\lambda \to 0} \frac{T(x) - \overline{T}(x)}{\lambda}$.

When $\boldsymbol{V}$ is a *constant vector*, $\mathcal{L}_{\boldsymbol{V}}$ reduces to the ordinary directional derivative

$$\mathcal{L}_{\boldsymbol{V}}(\mathfrak{T}) = \frac{d\mathfrak{T}}{d\boldsymbol{V}} = \boldsymbol{V} \cdot \boldsymbol{\nabla} \mathfrak{T}.$$

If on the other hand the transformation $x \to \overline{x}$ is an infinitesimal rotation $\boldsymbol{V} = \boldsymbol{\omega} \times \boldsymbol{r}$, with $\boldsymbol{\omega}$ as a constant vector, we find for a scalar field

$$\mathcal{L}_{\boldsymbol{V}} \phi = (\boldsymbol{\omega} \times \boldsymbol{r}) \cdot \nabla \phi = \boldsymbol{\omega} \cdot (\boldsymbol{r} \times \nabla \phi).$$

For a contravariant vector field, the result is

$$\mathcal{L}_{\boldsymbol{V}} \boldsymbol{A} = \boldsymbol{\omega} \cdot \{\boldsymbol{r} \times \boldsymbol{\nabla} \boldsymbol{A} - \mathfrak{I} \times \boldsymbol{A}\}.$$

[423] Because of the French-German war in 1870, Lie left France and decided to go to

sen to succeed Felix Klein to the chair of geometry at Leipzig. As his fame grew, a special post was arranged for him in Christiania. But his health had deteriorated by a life of assiduous study, and he died in Christiania six months after his return.

Italy. On the way he was arrested as a German spy and his mathematics notes were assumed to be coded messages. Only after the intervention of Darboux was Lie released.

Lie Algebras and Lie Groups[424]

Consider the set of all three-dimensional vectors $\boldsymbol{A}$ *as oriented line segments. On defining the sum* $\boldsymbol{A}+\boldsymbol{B}$*, the negative* $-\boldsymbol{A}$ *and the scalar multiple* $\lambda\boldsymbol{A}$ *such that*

$$\lambda(\mu\boldsymbol{A}) = (\lambda\mu)\boldsymbol{A}\ , \quad \lambda(\boldsymbol{A}+\boldsymbol{B}) = \lambda\boldsymbol{A}+\lambda\boldsymbol{B}$$

and

$$(\lambda+\mu)\boldsymbol{A} = \lambda\boldsymbol{A}+\mu\boldsymbol{A},$$

we create a linear vector space over the field of real numbers. If to this structure we now add the product *of two vectors defined via the* vector product $\boldsymbol{A}\times\boldsymbol{B}$*, then every three vectors satisfy the relations*

$$\boldsymbol{A}\times\boldsymbol{B} = -\boldsymbol{B}\times\boldsymbol{A} \quad \text{(anticommutativity)} \tag{1}$$

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})+\boldsymbol{B}\times(\boldsymbol{C}\times\boldsymbol{A})+\boldsymbol{C}\times(\boldsymbol{A}\times\boldsymbol{B}) = 0 \tag{2}$$

We also have

$$(\boldsymbol{A}+\boldsymbol{B})\times\boldsymbol{C} = (\boldsymbol{A}\times\boldsymbol{C})+(\boldsymbol{B}\times\boldsymbol{C}) \quad \text{(distributivity)} \tag{3}$$

$$(\lambda\boldsymbol{A})\times\boldsymbol{B} = \lambda(\boldsymbol{A}\times\boldsymbol{B}) \quad \text{(associativity)} \tag{4}$$

Consider next all square matrices A *of order* n *under the usual vector-space laws of addition and multiplication by a scalar. Define a new 'product'*

$$A\odot B = AB-BA \equiv [A,B]$$

where AB *is the ordinary matrix product. The symbol* $[A,B]$ *is called the* commutator *of* A *and* B*. It satisfies the two conditions*

$$[A,B] = -[B,A]$$

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[424] To dig deeper, see:

- Sattinger, D.H. and O.L. Weaver, *Lie Groups and Algebras with Applications to Physics, Geometry and Mechanics*, Springer-Verlag: New York, 1986, 215 pp.
- Altmann, S.L., *Rotations, Quaternions, and Double Groups*, Dover, 1986, 317 pp.
- Srinivasa Rao, K.N., *The Rotation and Lorentz Groups*, Wiley, 1988, 351 pp.

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0$$

As a third case we specialize to skew-symmetric matrices of order n. They have the general form

$$S=(I\times\boldsymbol{V})=\begin{bmatrix}0 & -c & b\\ c & 0 & -a\\ -b & a & 0\end{bmatrix},$$

where I is the unit matrix and $\boldsymbol{V}(a,b,c)$ is a vector in $\mathbb{R}^3$.

The commutator of any two such matrices

$$[S_1,S_2]=(I\times\boldsymbol{V}_1)\cdot(I\times\boldsymbol{V}_2)-(I\times\boldsymbol{V}_2)\cdot(I\times\boldsymbol{V}_1)\equiv I\times(\boldsymbol{V}_1\times\boldsymbol{V}_2)=-[S_2,S_1]$$

It is not difficult to verify that in this case too

$$[S_1,[S_2,S_3]]+[S_2,[S_3,S_1]]+[S_3,[S_1,S_2]]=0.$$

Note that

$$(I\times\boldsymbol{V}_1)\cdot(I\times\boldsymbol{V}_2)=\boldsymbol{V}_2\boldsymbol{V}_1-I(\boldsymbol{V}_1\cdot\boldsymbol{V}_2)$$

and

$$I\times(\boldsymbol{V}_1\times\boldsymbol{V}_2)=\boldsymbol{V}_2\boldsymbol{V}_1-\boldsymbol{V}_1\boldsymbol{V}_2=\boldsymbol{V}_2\wedge\boldsymbol{V}_1\quad\text{('wedge product')}.$$

A fourth case concerns the linear differential operator

$$X=\boldsymbol{a}\cdot\boldsymbol{\nabla}=a_1\frac{\partial}{\partial x_1}+a_2\frac{\partial}{\partial x_2}+a_3\frac{\partial}{\partial x_3}$$

where $a_i(x_1,x_2,x_3)$ are three differentiable functions. When this operator is applied to a smooth function $f(x_1,x_2,x_3)$, it assigns to it a real number known as the directional derivative *along the vector field $\boldsymbol{a}$:*

$$X(f)=a_1\frac{\partial f}{\partial x_1}+a_2\frac{\partial f}{\partial x_2}+a_3\frac{\partial f}{\partial x_3},$$

where $(\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\frac{\partial f}{\partial x_3})$ are the components of a vector normal to the surface $f=\text{const}$, at a given point. To each pair of operators $X=\boldsymbol{a}\cdot\boldsymbol{\nabla}$, $Y=\boldsymbol{b}\cdot\boldsymbol{\nabla}$, we can associate a Lie bracket

$$\begin{aligned}[X,Y] &= XY - YX = (\boldsymbol{a}\cdot\boldsymbol{\nabla})(\boldsymbol{b}\cdot\boldsymbol{\nabla}) - (\boldsymbol{b}\cdot\boldsymbol{\nabla})(\boldsymbol{a}\cdot\boldsymbol{\nabla}) \\ &= (\boldsymbol{a}\cdot\nabla\boldsymbol{b} - \boldsymbol{b}\cdot\nabla\boldsymbol{a})\cdot\boldsymbol{\nabla} = L_1\frac{\partial}{\partial x_1} + L_2\frac{\partial}{\partial x_2} + L_3\frac{\partial}{\partial x_3}\end{aligned}$$

where

$$L_i = (\boldsymbol{a}\cdot\nabla\boldsymbol{b}_i - \boldsymbol{b}\cdot\nabla\boldsymbol{a}_i) \qquad i = 1,2,3.$$

Here also

$$[X_1 + X_2, Y] = [X_1, Y] + [X_2, Y],$$

$$[\lambda X, Y] = \lambda[X,Y] \qquad (\lambda \text{ constant})$$

and

$$[Y,X] = -[X,Y],$$

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

Since systems such as the four described above hold importance in mathematics and theoretical physics, it is advantageous to bring them under the umbrella of a common new algebra, the Lie algebra, abstractly defined as follows:

A Lie algebra is a vector space L over some field F (typically the real or complex numbers) together with a binary operation $[X,Y] \in L$ called the Lie bracket, which satisfies the conditions:

- *It is bilinear, i.e.* $[aX + bY, Z] = a[X,Z] + b[Y,Z]$; *also* $[Z, aX + bY] = a[Z,X] + b[Z,Y]$ *for all* a, b *in* F *and* X, Y, Z *in* L.
- *It is antisymmetric, i.e.* $[X,Y] = -[Y,X]$ *for all* X, Y *in* L.
- *It satisfies the Jacobi identity,*

$$J_{XYZ} \equiv [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$$

for all X, Y, Z in L.

This mathematical definition requires some clarifications:

(1) Whenever the binary operation on the vector space is defined as the commutator of ordinary matrix multiplication, then the above three conditions are met automatically. However, one must still impose the closure condition, i.e. that $xy - yx$, for every x and y in L, belong to the same vector space.

Since every vector space has a basis $x_1, \dots, x_n$, then it suffices to demand that for any i, j ($i, j = 1, \dots, n$) the entity $x_i x_j - x_j x_i$ is a linear combination of x_k, where the coefficients belong to the field. We shall soon see how this condition works, for example, in the case of the generators of the rotation group and the Lorentz transformation group.

(2) *Some Lie algebras of importance in physics and applied mathematics are of infinite dimensionality. An example is the original Lie group considered by Lie himself, which is the group of coordinate transformations on a surface. In the case of infinite dimensional Lie groups, the Jacobi identity is non-trivial and there are examples where it exhibits "anomalies".*

(3) *Lie algebras can be represented or realized: representation is usually restricted to a set of matrices or operators. Realization is a generalization of the concept of representation to also include sets of functions or other entities with appropriate group or algebraic properties.*

Note that the Lie bracket is not a multiplication in the usual sense because it is not associative. If an associative algebra with multiplication $(*)$ is given, it can be turned into a Lie algebra by defining the commutator $[X, Y] = X^*Y - Y^*X$. Conversely, every Lie algebra is embedded into one that arises from an associative algebra in this fashion. An example is linear associative algebra w.r.t. ordinary matrix product.

The proof of the Jacobi identity is elementary:

Deleting the star $(*)$ and assuming the associative law under multiplication, we have

$$\begin{aligned} J_{XYZ} = {} & (XY - YX)Z - Z(XY - YX) + \\ & (YZ - ZY)X - X(YZ - ZY) + (ZX - XZ)Y - Y(ZX - XZ) \equiv 0 \end{aligned}$$

While the entity $[X, Y]$ is known as the commutators of X and Y, the entity J_{XYZ} is known as the associator.

The algebra of commutators leads to expressions for commutators of functions of the elements. Thus if

$$[A, B] = AB - BA = \gamma$$

then

$$[A^2, B] = A^2B - BA^2 = AAB - BAA = \\ A(BA + \gamma) - BAA = ABA + A\gamma - BAA = A\gamma + \gamma A;$$

$$[A^3, B] = A(A^2B) - (BA^2)A = A(A^2B) - (A^2B - A\gamma - \gamma A)A \\ = A\gamma A + \gamma A^2 + A^2(AB - BA) = A\gamma A + \gamma A^2 + A^2\gamma$$

Since $e^A = \sum_{m=0}^{\infty} \frac{A^m}{m!}$, *it is not difficult to show that*

$$e^{-A}Be^A = B + \frac{1}{1!}[B, A] + \frac{1}{2!}[[B, A], A] + \dots, \quad \textit{and}$$

$$[e^A, B] = \sum_{m=0}^{\infty} \frac{1}{m!}[A^m, B]$$

If $A\gamma = \gamma A$ *(see an example in the footnote*[425]*) we have* $[A^m, B] = m\gamma A^{m-1}$ *and therefore*

$$[e^A, B] = \gamma e^A, \qquad [e^A, e^B] = (e^\gamma - 1)e^Be^A$$

The ***Campbell-Baker-Hausdorff*** *theorem states that if A and B are matrices which do not necessarily commute, then*

$$e^Ae^B = e^C$$

where

$$C = A + B + \frac{1}{2}[A, B] + \frac{1}{12}([A, [A, B]] - [B, [B, A]]) + \dots$$

and the coefficients are the Bernoulli numbers.

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[425] valid, for example for $A = x$, $B = \frac{d}{dx}$. In this case $[A, B] = -1$, so

$$[A, [A, B]] = 0.$$

Thus

$$AAB + BAA = 2ABA,$$

which is indeed satisfied for any function f since

$$x^2\frac{df}{dx} + \frac{d}{dx}(x^2f) = 2x\frac{d}{dx}(xf).$$

To prove this theorem we consider $e^{C(t)} = e^{tA}e^{B}$, *where* t *is a parameter. Evaluating via a Taylor's expansion about* $t = 0$, *i.e.*

$$C(t) = C(0) + \frac{1}{1!}C'(0) + \frac{1}{2!}C''(0) + \ldots, \text{ and setting } t = 1,$$

the desired result is obtained by successively evaluating $C^{(n)}(0),\ n = 1, 2, \ldots,$ *using* $C(0) = B$ *and the differential equation*

$$e^{-B}Ae^{B} = e^{-C(t)}\frac{d}{dt}\left(e^{C(t)}\right)$$

for $C(t)$.

This theorem is applicable to Wigner's rotation in the Lorentz group (special relativity), the Gibbs formula for addition of finite rotations (geometry and rotational dynamics), and various problems in quantum mechanics.

We have shown that all 3-dimensional skew-symmetric matrices form a Lie algebra. We know that proper active rotations in 3-dimensional space are represented by 3×3 *orthogonal matrices* R *with determinant* $+1$. *The* Lie group *formed under these operations is denoted by the symbol* SO(3). *We also know that*

$$R = e^{(I \times \boldsymbol{V})}$$

where

$$(I \times \boldsymbol{V}) = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} = v_1 I_1 + v_2 I_2 + v_3 I_3,$$

$$I_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Consequently, there is a close connection between the geometry *of the rotations* R *and their composition law on one hand, and the* algebra *of skew-symmetric matrices on the other. Given* R, *the vector* $\boldsymbol{V}$ *can be explicitly constructed from the matrix elements of* R. *It specifies both the direction of the axis of rotation in space and the finite angle of rotation about that axis, which are* $\frac{\boldsymbol{v}}{|\boldsymbol{v}|}$ *and* $|\boldsymbol{v}|$, *respectively. The three matrices* (I_1, I_2, I_3) *are said to* generate *the Lie group* SO(3), *as well as furnishing a basis for the corresponding Lie Algebra. The parameters* (v_1, v_2, v_3) *are the "coordinates" or "components" of* $(I \times \boldsymbol{V})$ *relative to this basis.*

Since the elements of $(I \times \boldsymbol{V})$ *and* R *determine each other uniquely in small enough neighborhood of* I *through the relation* $R = e^{(I \times \boldsymbol{V})}$, v_i *can be considered as local coordinates of* R *in the group manifold* $\mathrm{SO}(3)$.

The matrices I_i *obey the commutation relations*

$$I_1 I_2 - I_2 I_1 = I_3, \qquad I_2 I_3 - I_3 I_2 = I_1, \qquad I_3 I_1 - I_1 I_3 = I_2;$$

one also has $I_1^2 + I_2^2 + I_3^2 = -2I$.

We can write $\boldsymbol{V} = \boldsymbol{e}\theta$, *where* $\boldsymbol{e}$ *is a unit vector along the axis of rotation and* θ *the angle of rotation. Denoting by* (α, β, γ) *the three Euler-like angles representing* R, *we have* $R = M_\alpha M_\beta M_\gamma$ *where*

$$M_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix},$$

$$M_\beta = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}, \qquad M_\gamma = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For an infinitesimal rotation, we expand $R(\alpha, \beta, \gamma)$ *in Taylor series about* $\alpha = 0$, $\beta = 0$, $\gamma = 0$. *We find*

$$R = I + \frac{\partial M_\alpha}{\partial \alpha}\Big|_{\alpha=0}\delta\alpha + \frac{\partial M_\beta}{\partial \beta}\Big|_{\beta=0}\delta\beta + \frac{\partial M_\gamma}{\partial \gamma}\Big|_{\gamma=0}\delta\gamma,$$

where

$$\frac{\partial M_\alpha}{\partial \alpha}\Big|_{\alpha=0} = I_1, \qquad \frac{\partial M_\beta}{\partial \beta}\Big|_{\beta=0} = I_2, \qquad \frac{\partial M_\gamma}{\partial \gamma}\Big|_{\gamma=0} = I_3, \quad \text{and}$$
$$\delta\alpha = e_1\delta\theta, \quad \delta\beta = e_2\delta\theta, \quad \delta\gamma = e_3\delta\theta.$$

Thus, under an infinitesimal rotation $\delta\theta$ *about the* z*-axis* $(e_1 = e_2 = 0, e_3 = 1)$ *we have*

$$R_z = e^{I_3\delta\theta}.$$

The matrices I_1, I_2, I_3 *are known as* generators *of the Lie group.*

These generators are associated with the local structure *of the Lie group in the neighborhood of the identity element.*

In general, in n-dimensional Euclidean space the number of generators (i.e. the number of Euler angles) is $N = \frac{n(n-1)}{2}$. We can then write symbolically for a general rotation:

$$g(x) = \exp[\sum_{i=1}^{N} x_i T_i],$$

where x_i are local coordinates in the N-dimensional group manifold *(an abstract set of points, each representing one element of the Lie group). The operators T_i are the generators of the Lie group in these coordinates.*

The point $x_i = 0$ $(i = 1, \ldots, n)$ in the manifold is the identity element $g = I$. Each T_i is an operator with a matrix representation. This formalism can be extended to non-vector representations of the rotation groups, and indeed to any Lie group and its corresponding algebra. Examples are:

- *For the vectorial rotation group* SO(3), *we saw that the generators are the 3×3 matrices*

$$(T_j)_{ab} \equiv (I_j)_{ab} = \epsilon_{jba}, \quad 1 \le j, a, b \le 3,$$

where ϵ_{jab} is the Levi-Civita symbol.

- *For the $\mathbb{R}^3$ rotation group of 3D* spinors, SU(2), *the 3 generators in the spinor representation are*

$$(T_i)_{ab} = \frac{1}{2} i(\sigma_j)_{ab}, \quad i = \sqrt{-1}, \quad 1 \le j \le 3, \quad 1 \le a, b \le 2,$$

where σ_j are the Pauli matrices. The SU(2) *and* SO(3) *Lie group manifolds differ in their* global *topology (e.g. the* SU(2) *manifold is topologically equivalent to the 3-sphere S^3), but are* locally *equivalent since they share the same* Lie algebra; *there is also a* two-to-one *mapping from* SU(2) *to* SO(3) *which preserves the rotation-composition law*[426].

How do we express the "addition law" of the group [namely the law by which $g(x)g(y)$ gives a new $g(z)$] in terms of the $\{T_j\}$?

For that we have the above-mentioned Campbell-Baker-Hausdorff theorem

$$e^{A_1} e^{A_2} = e^{A_1 + A_2 + \phi(A_1, A_2)},$$

426 To wit, both SU(2) elements $\pm e^{\frac{i}{2}\boldsymbol{v}\cdot\boldsymbol{\sigma}}$ correspond under this map to the same SO(3) element $e^{\boldsymbol{I}\times\boldsymbol{v}}$.

where

$$\phi(A_1, A_2) = \sum_{n=1}^{\infty} \sum_{\{j\}} C_{\{j\}}[\dots[A_{j_1}, A_{j_2}], \dots A_{j_{n+1}}],$$

$C_{\{j\}}$ *are known constants and the second sum ranges over* $\{j_p \mid 1 \le j_p \le 2,$ $p = 1, \dots, n+1\}$.

Since ϕ *is built by repeatedly commuting* A_1, A_2 *with each other, it follows that if* $g(x) \cdot g(y)$ *is again to be of the form*

$$g(z) = exp\{\sum_j z_j T_j\},$$

we must have

$$[T_i, T_j] = \sum_k \gamma_{ijk} T_k,$$

where the numbers γ_{ijk} *are called the structure constants of the Lie algebra and fully characterize it.*

Note that all the local structure of a Lie group is contained in its Lie algebra, since for any two elements A, B *of the latter,*

$$[A, B] = AB - BA$$

is also an element of the algebra (i.e. the algebra is closed under commutation); in sufficiently small neighborhood of the identity $g = I$*, we have a one-to-one corresponding between the group and the algebra, namely*

$$g(x) \leftrightarrow X = \sum_{j=1}^{N} x_j T_j, \quad g(x) = e^X$$

(the so-called exponential map).

Since the SU(2) *and* SO(3) *generators have the same commutation relations*

$$[T_i, T_j] = \sum_{k=1} \epsilon_{ijk} T_k, \quad 1 \le i, j \le 3,$$

they must have the same Lie algebra and the same local structure as discussed above.

In the foregoing discussion the matrix R *effected the rotation of the coordinate axes or the space coordinates. We may, however, consider the idea of rotation of a function relative to fixed axes (coordinates) under the definition*

$Rf(\boldsymbol{r}) = f(R \cdot \boldsymbol{r})$. Let us, for example, apply it to a rotation of a function about the z-axis with

$$R_z = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This means that

$$f(x, y, z) \to f(x\cos\varphi + y\sin\varphi, -x\sin\varphi + y\cos\varphi, z).$$

For $\varphi = \frac{\pi}{2}$, $\qquad R(\boldsymbol{e}_z, \frac{\pi}{2})f(x, y, z) = f(y, -x, z)$.

Under an infinitesimal rotation $R(\boldsymbol{e}_z, \delta\varphi)$, *a Taylor expansion yields*

$$\begin{aligned} R(\boldsymbol{e}_z, \delta\varphi)f(x, y, z) &= f(x + y\delta\varphi, y - x\delta\varphi, z) \\ &= f(x, y, z) - \delta\varphi[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}]f + O(\delta\varphi)^2 \\ &= \{e^{-i(\delta\varphi)L_z}\}f \end{aligned}$$

where

$$L_z = -i(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}).$$

Similar results are obtained for rotations about the other axes, with

$$L_x = -i(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}), \quad L_y = -i(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}).$$

We note that L_x, L_y, and L_z satisfy exactly the same commutation relation as iT_a: $[L_i, L_j] = i\epsilon_{ijk}L_k$. It can be shown that the Casimir operator $L^2 = L_x^2 + L_y^2 + L_z^2$ has vanishing commutators:

$$[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0.$$

In quantum mechanics, L_a is the a-th component of the orbital angular momentum operator, in units where $\hbar = 1$. The quantum angular-momentum Casimir operator is related to the Laplacian operator as follows:

$$\boldsymbol{\nabla}^2 = \frac{1}{r^2}\boldsymbol{L}^2 + \frac{\partial^2}{\partial r^2},$$

where in the partial differentiation $\frac{\boldsymbol{r}}{r}$ is held fixed and $r = \sqrt{x^2 + y^2 + z^2}$.

In general

$$f(\boldsymbol{r}+\boldsymbol{a}) = \sum_{n=0}^{\infty} \frac{1}{n!}(\boldsymbol{a}\cdot\boldsymbol{\nabla})^n f(\boldsymbol{r}) = e^{\boldsymbol{a}\cdot\boldsymbol{\nabla}} f(\boldsymbol{r}).$$

But quantum-mechanically

$$\boldsymbol{p} \text{ (momentum operator)} = -i\hbar\nabla, \qquad \text{and so} \qquad f(\boldsymbol{r}+\boldsymbol{a}) = e^{i\boldsymbol{a}\cdot\boldsymbol{p}/\hbar} f(\boldsymbol{r}).$$

The special case $(r,\theta,\varphi) \to (r,\theta,\varphi+\alpha)$ *designates a rotation about the* z *axis by an angle* α*; therefore* $e^{\alpha\frac{\partial}{\partial\varphi}} = e^{i\alpha L_z/\hbar}$, *where* $L_z = -i\hbar\frac{\partial}{\partial\varphi}$.

It must be emphasized that the basic idea of the Lie group is that from a generator with elements infinitesimally close to I*, such as* L_z*, which shifts the point* $s=(x,y,z)$ *(or in general, the Lie-group representation or realization) to the point* $s+ds$*, one may generate an operator which shifts the point* s *into a point* s' *at a finite distance along the 'path curve' of a one-parameter group generated by the infinitesimal operator.*

This idea can be applied to obtain addition theorems for special functions of mathematical physics from their respective recursion relations. The infinitesimal differential operators which appear in the recursion-relations of the various special functions, can be used to generate from them finite operators. For example, the Bessel functions obey the recursion equations

$$\mathcal{L}_+\{e^{in\phi}J_n(r)\} = e^{i(n+1)\phi}J_{n+1}(r); \mathcal{L}_-\{e^{in\phi}J_n(r)\} = e^{i(n-1)\phi}J_{n-1}(r)$$

where

$$\mathcal{L}_+ = -\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}, \qquad \mathcal{L}_- = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}; \qquad r = \sqrt{x^2+y^2+z^2}.$$

Since $[\mathcal{L}_+,\mathcal{L}_-]=0$*, the composition law within the Lie group is additive:*

$$e^{\alpha\mathcal{L}_+}\cdot e^{\beta\mathcal{L}_-} = e^{\alpha\mathcal{L}_+ + \beta\mathcal{L}_-}.$$

It is then a trivial matter to derive the addition theorem

$$\frac{J_n[\sqrt{r^2+h^2}]}{(r^2+h^2)^{n/2}} = \sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{-h}{2}\right)^m r^{-n-m}J_{n+m}(r).$$

Lie algebra and classical mechanics

Hamilton *(1835) converted the* ***Lagrange*** *equations (1788) into a set of coupled, 1st-order ODE's representing the solution of a conservative mechanical system in phase space:*

$$\dot{q}_i = \frac{\partial H}{\partial p_i}; \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}; \qquad (p_i = \frac{\partial L}{\partial \dot{q}_i})$$

where $H(q,p,t) = \sum_{i=1}^{n} p_i \dot{q}_i - L$ *is the Hamilton function (Hamiltonian) and* $L(q,\dot{q},t)$ *is the Lagrangian of the system. Let* $f(p,q,t)$ *be some function of the coordinates, momenta and time. Its total time-derivative is*

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_k (\frac{\partial f}{\partial q_k}\dot{q}_k + \frac{\partial f}{\partial p_k}\dot{p}_k) = \frac{\partial f}{\partial t} + [H,f],$$

where

$$[H,f] = \sum_k \left(\frac{\partial f}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial H}{\partial q_k} \right)$$

is known as the Poisson bracket of H *and* f*.*

For any two given functions of the dynamical variables, $u(p,q,t)$ *and* $v(p,q,t)$*, the Poisson bracket is defined analogously as*

$$[u,v] = \sum_k \left(\frac{\partial u}{\partial p_k}\frac{\partial v}{\partial q_k} - \frac{\partial u}{\partial q_k}\frac{\partial v}{\partial p_k} \right).$$

It has the following basic properties:

$$[u,v] = -[v,u]; \quad [u+v,w] = [u,w] + [v,w];$$
$$[uv,w] = u[v,w] + v[u,w];$$
$$\frac{\partial}{\partial t}[u,v] = [\frac{\partial u}{\partial t}, v] + [u, \frac{\partial v}{\partial t}];$$
$$\frac{d}{dt}[u,v] = [\frac{du}{dt}, v] + [u, \frac{dv}{dt}];$$
$$[u,q_k] = \frac{\partial u}{\partial p_k}; \quad [u,p_k] = -\frac{\partial u}{\partial q_k};$$
$$[q_i,q_k] = 0; \quad [p_i,p_k] = 0; \quad [p_i,q_k] = \delta_{ik};$$

$$[u,[v,w]]+[v,[w,u]]+[w,[u,v]]=0 \quad \text{(Jacobi's identity)}$$

If u and v do not depend on time (integrals of the motion) then

$$\frac{d}{dt}[u,v]=0$$

This follows immediately from the Jacobi identity with $w=H$. Note that if $\boldsymbol{M}=\boldsymbol{r}\times\boldsymbol{p}$ (regular momentum), then

$$[M_x,M_y]=-M_z, \qquad [M_y,M_z]=-M_x, \qquad [M_z,M_x]=-M_y$$

Lie algebra of the Lorentz group

We know that every Lorentz transformation L between two frames of reference with arbitrary orientation with respect to each other, is represented by a (4×4) orthogonal matrix acting in Euclideanized Minkowski space (space-time) having metric δ_{ij}:

$$L=e^S,$$

where S is the (4×4) skew-symmetric matrix

$$S=\begin{bmatrix} 0 & -h_3 & h_2 & -ie_1 \\ h_3 & 0 & -h_1 & -ie_2 \\ -h_2 & h_1 & 0 & -ie_3 \\ ie_1 & ie_2 & ie_3 & 0 \end{bmatrix}.$$

In S, (e_1,e_2,e_3) are the direction cosines of a vector *and (h_1,h_2,h_3) are the three components of a* pseudo-vector*. We can write*

$$S=\boldsymbol{h}\cdot\boldsymbol{u}+\boldsymbol{e}\cdot\boldsymbol{w}=h_1u_1+h_2u_2+h_3u_3+e_1w_1+e_2w_2+e_3w_3$$

where

$$u_1=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, u_2=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, u_3=\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$w_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, w_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, w_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

The six skew-symmetric matrices u_i, w_i form a basis in the vector space of our Lie algebra.

Note that e_i, h_i are functions of the components of the velocity vector $\boldsymbol{v}$ and the Euler angles which determine the relative orientation and motion of the two reference frames. Two limiting cases are simple:

- $h_i = 0$ $\quad\boldsymbol{e} \parallel \boldsymbol{v}, \quad |\boldsymbol{e}| = \mathrm{th}^{-1} \frac{|\boldsymbol{v}|}{c}$ *(pure boost)*
- $e_i = 0, \;\; v_i = 0;$ $\quad h_i$ *are functions of the Euler angles (pure rotation)*

The Lie group of Euclideanized matrices L is $\mathrm{SO}(4)$*; in the non-Euclidean (real) Minkowski space of STR, the Lorentz group is called* $\mathrm{SO}(3,1)$.

Lie algebra and the symplectic group (Weyl 1938)

Symplectic transformations preserve skew-symmetric products, which are abundant in physical applications. We know, for example, that the "dot" product of two plane vectors $\boldsymbol{A} = (A_x, A_y)$, $\boldsymbol{B} = (B_x, B_y)$, namely $\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y$ is invariant under rotation in the two-dimensional space. On the other hand, the "cross" product $\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{e}_z(A_x B_y - A_y B_x)$, where $(A_x B_y - A_y B_x)$ is the area of the parallelogram formed by the two vectors, is invariant under a group of transformations which preserves this skew-symmetric product and is known as the symplectic group in 2-dimensional space, or $S_p(2)$. To meet this group, we write

$$\text{area} = A_x B_y - A_y B_x = [A_x, A_y] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} B_x \\ B_y \end{bmatrix} = AJB \tag{1}$$

where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. *Define a* 2×2 *symplectic matrix* M *such that*

$$A \to AM^T, \quad B \to MB. \tag{2}$$

Then, the preservation of the area implies $A'JB' = AJB$, *or*

$$M^T JM = J \tag{3}$$

Even without going into the detailed form of M, *we can deduce that the unit* 2×2 *matrix* $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *is symplectic, that* M *is nonsingular, that if two matrices* M *and* N *are symplectic, so are* MN *and* NM; *and that the inverse of a symplectic matrix is also symplectic. From these observations we can conclude that the* 2×2 *symplectic matrices form a group. It is called* $Sp(2)$, *and includes the planar-rotations group,* SO(2), *as a subgroup.*

Geometrical illustration of the $Sp(2)$ group

Consider a unit circle around the origin of a Cartesian system (x, y), *namely* $x^2 + y^2 = 1$. *Next, consider an ellipse, whose equation in another Cartesian system* (X, Y) *is*

$$e^{-\eta} U^2 + e^{\eta} V^2 = 1$$

with

$$U = X \cos\frac{\theta}{2} + Y \sin\frac{\theta}{2}, \qquad V = X \sin\frac{\theta}{2} - Y \cos\frac{\theta}{2}. \tag{4}$$

If $\eta > 0$, *the major and minor axes of this ellipse are* $e^{\eta/2}$ *and* $e^{-\eta/2}$ *respectively. The major axis is along the* $\frac{\theta}{2}$ *direction. The area of the ellipse is* π *(same as that of the unit circle) and remains invariant as we change the values of* θ *and* η.

The question now arises as to what transformations carry the (x, y) *into the* (X, Y) *coordinates such that the area is preserved. One class of such transformations is*

$$\begin{bmatrix} X \\ Y \end{bmatrix} = [S(\theta, \eta)] \begin{bmatrix} x \\ y \end{bmatrix}, \tag{5}$$

where

$$S(\theta, \eta) = R(\theta) S(\eta) R(-\theta)$$

$$= \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{ch}\,\frac{\eta}{2} + \mathrm{sh}\,\frac{\eta}{2}\cos\theta & \mathrm{sh}\,\frac{\eta}{2}\sin\theta \\ \mathrm{sh}\,\frac{\eta}{2}\sin\theta & \mathrm{ch}\,\frac{\eta}{2} - \mathrm{sh}\,\frac{\eta}{2}\cos\theta \end{bmatrix}. \tag{6}$$

The matrix $S(\theta,\eta)$ is symmetric, satisfies the symplectic condition $S^TJS = J$, and its determinant is unity. Geometrically, $S(\theta,\eta)$ elongates / contracts along the Cartesian axes tilted by angle $\frac{\theta}{2}$. It can be shown that the two-parameter scale-transformation matrices $S(\theta,\eta)$ alone cannot *form a group unless they are supplemented by a rotation matrix $R(\alpha)$, where α is determined from θ and η. This rotation does not affect the area-preservation property.*

If we introduce the matrices

$$F_1 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F_2 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad F_3 = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{7}$$

then we can write

$$S(0,\lambda) = e^{\lambda F_1}, \qquad S(\frac{\pi}{2},\eta) = e^{\eta F_2}, \qquad R(\theta) = e^{\theta F_3} \tag{8}$$

where F_1,F_2,F_3 are the generators *of the $Sp(2)$ group. They satisfy the commutation relations*

$$[F_1, F_2] = -F_3, \qquad [F_2, F_3] = F_1, \qquad [F_3, F_1] = F_2. \tag{9}$$

These generators form a system of closed *commutation relations, and are a basis for the* Lie algebra *for the $Sp(2)$ group.*

Assume

$$S(\theta,\eta) = e^{aF_1 + bF_2}. \tag{10}$$

Since F_1 and F_2 anticommute with each other, and since

$$(2F_1)^2 = (2F_2)^2 = -(2F_3)^2 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$(aF_1 + bF_2)^2 = \frac{a^2 + b^2}{4} I,$$

the expansion of the exponential (10) in a power series yields

$$S(\theta,\eta) = I\,\mathrm{ch}\,\frac{\eta}{2} + (2F_1\cos\theta + 2F_2\sin\theta)\,\mathrm{sh}\,\frac{\eta}{2} \tag{11}$$

where

$$\eta^2 = a^2 + b^2, \quad \tan\theta = \frac{b}{a}.$$

Clearly, $S(\theta,\eta)$ in (11) is identical to its form in (6).

Note that

$$F_1 = \frac{1}{2}\sigma_z, \qquad F_2 = \frac{1}{2}\sigma_x, \qquad F_3 = -\frac{i}{2}\sigma_y, \tag{12}$$

where $(\sigma_x, \sigma_y, \sigma_z)$ are the Pauli spin matrices.

It can be shown that the $Sp(2)$ group is locally isomorphic to the group of Lorentz transformations in two-dimensional space (i.e. three-dimensional pseudo-Euclidean Minkowski spacetime). This group is known as $\mathrm{SO}(2,1)$, *and is a subgroup of the group consisting of Lorentz transformations in 4-dimensional spacetime, consisting of three space- and one time-dimension. If we use (x,y,z) to specify the coordinates in this $(2+1)$ space, then* $\mathrm{SO}(2,1)$ *consists of Lorentz transformations along the x and y directions and of a rotation on the xy plane around the z axis. The generators of this group are*

$$T_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad T_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad T_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{13}$$

with the top-to-bottom rows and left-to-right columns corresponding to the x, y, t directions of Minkowski spacetime, respectively.

They satisfy the commutation relations

$$[T_1, T_2] = -T_3, \qquad [T_2, T_3] = T_1, \qquad [T_3, T_1] = T_2, \tag{14}$$

which are exactly the same like those for the generators of the group $Sp(2)$.

The element $L(\theta,\eta)$ of the Lie-group $\mathrm{SO}(2,1)$ *which corresponds to the element $S(\theta,\eta)$ of $Sp(2)$ is:*

$$L(\theta,\eta) = K(\theta)L(0,\eta)K(-\theta)$$

with

$$K(\theta) = e^{\theta T_3} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$L(0,\eta) = e^{\eta T_1} = \begin{bmatrix} \mathrm{ch}\,\eta & 0 & \mathrm{sh}\,\eta \\ 0 & 1 & 0 \\ \mathrm{sh}\,\eta & 0 & \mathrm{ch}\,\eta \end{bmatrix} = \text{ boost along the } x \text{ direction,}$$

$$L(\frac{\pi}{2},\eta) = e^{\eta T_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \operatorname{ch}\eta & \operatorname{sh}\eta \\ 0 & \operatorname{sh}\eta & \operatorname{ch}\eta \end{bmatrix} = \text{ boost along the } y \text{ direction.}$$

1870–1903 CE **Jakob Rosanes (Rosales)** (1842–1922, Germany). Mathematician. Contributed significantly to algebraic geometry and theory of invariants. Some of his important results were later proved independently by **Max Noether** and elaborated on by **Castelnuovo**.

Rosanes was born in Brody, Austria-Hungary (now the Ukraine) to a distinguished Jewish family that originated from *Castallvi de Rosanes* near Barcelona. With the expulsion of the Jews from Spain in 1492, the family went to Portugal and from there dispersed into Europe, North-Africa and the Near-East. In Portugal, they turned into Marranos, and one of their members **Immanuel Rosales** (1593–1668) became a mathematician and a famous physician.

Rosanes studied at the Universities of Berlin and Breslau, obtaining his doctorate from Breslau in 1865. In 1876 he became a professor of mathematics at the latter, where he remained for the rest of his career.

1870–1906 CE **Georg Ferdinand Frobenius** (1849–1917, Germany). Mathematician. Contributed to the theory of algebraic equations, number theory, character theory of finite groups[427] (1896) and ordinary differential equations (*Frobenius method*).

Frobenius was born in Berlin. He was educated at the University of Göttingen (1867–1870) and was a professor of mathematics at the Zürich Polytechnicum (1875–1892) and afterwards at the University of Berlin.

[427] Frobenius' character theory of finite groups was used with great effect by **William Burnside** (1852–1927, England), published by him during 1897–1911. Together they co-founded the theory of finite group representation. Burnside's ('$p^a \times q^b$ theorem') states that every finite group whose order (number of elements) is divisible by fewer than 3 distinct primes is *solvable*. In 1897 Burnside's classic work: "theory of Groups of Finite Order" was published. The second edition (1911) already included a *character theory*. While first developed for finite groups, characters were later extended to (infinite) Lie groups. The so-called *Burnside lemma*, stated in his book was actually discovered earlier by Frobenius. Burnside's conjecture that every finite group of odd order is solvable was proved only in 1962 by **W. Feit** and **J.C. Thompson**. Burnside was a student of Stokes, Maxwell and Cayley.

The 'Cult of Science' (1870–1914)

With the rapid advance in all fields of scientific research, the belief in unlimited progress that had prevailed since the enlightenment seemed happily confirmed. This created an intellectual climate and a cultural trend whose outstanding characteristic may be described as an overriding interest and a deep belief in science. Man had been interested in science before, but it was only since the second half of the 19^{th} *century that a veritable '*Cult of Science*' developed. Science offered a positive alternative to the seemingly futile idealism and Romanticism of the early* 19^{th} *century.*

Scientific research, in the past the domain of a few scientists and gentleman scholars, now became the concern of large numbers of people, especially as the application of science to industry gave an incentive for new inventions. Pure *science continued to be of fundamental importance, but* applied *science — the marriage of science and technology so characteristic of the Industrial Revolution — now took precedence in the minds of most people. A virtually endless series of scientific inventions seemed to provide tangible evidence of man's ability to unveil the secrets of nature.*

The growing concern of modern man with the material aspects of his civilization was also reflected in late 19^{th} *century thought. A few basic scientific discoveries served as a foundation for an essentially materialistic philosophy that appealed to the educated middle class. Chemists and physicists earlier in the century had declared matter and energy to be constant and indestructible.*

These scientific findings were translated by certain popularizers of science into a philosophy of materialism*. An early exponent of this philosophy was* ***Ludwig Feuerbach*** *(1804–1872,* Germany*). In his* scientific socialism*,* ***Karl Marx*** *(1818–1895,* Germany*) blended the materialistic doctrine of Feuerbach, the positivism of* ***Comte****, the Evolutionary Naturalism of* ***Spencer*** *and* ***Darwin****, and finally, the dialectic*[428] *of* ***Hegel****.*

[428] A dynamic logic which finds truth through a series of trials: thesis, antithesis, and synthesis, i.e., every fact will be understood only when related to its opposites, to those things which the thesis is not. Only by pointing out the many relationships of any one object to other objects can we establish the truth about that object. If we unite the idea to its opposite, we discover a different truth about them which transcends their previous separate meaning. It is like two conflicting forces that merge to produce a synthesis in the form of a new and greater reality. Marx applied this logical principle to socio-economic history; the two socio-economical classes which are antithetical to each other are the

According to him, history has been determined primarily by economic factors and punctuated by a series of class-struggles. The existing struggle of capital and labor was rationalized in terms of the theory of surplus value: *profit seeking capitalists pay labor subsistence wages, and take for themselves the surplus value which the workers have added to the product through their labor.*

In the second half of the 19^{th} *century, less than* 9 *generations after the trial of Galileo Galilei, science began to gain the upper hand in its long war against Christian dogma.*

As the state took over the functions of the churches in social welfare and education, and as some of the material benefits of industrialization spread among the lower classes, the need for the aid and comfort that religion had given in the past was no longer so acute. Furthermore, the tendency of the churches, to favor the political status quo *agonized many liberals, and political anticlericalism became an important issue in most countries. Finally, there was the appeal that nationalism, socialism, and materialism came to have for many people (both socialism and materialism were avowed enemies of religion).*

However, the most important reason for the decline in religious interests was the effect of modern science on Christianity; many scientific discoveries, especially in geology and biology, contradicted *Christian beliefs, and the methods of scientific inquiry, when applied to Christianity itself, produced some disturbing results [e.g.* ***Ernest Renan*** *in his book* Life of Jesus *(1863) denied that he performed miracles or had arisen from the dead].*

Far more drastic in their effect on the faithful than the attempts to humanize Christ were the findings of ***Darwin*** *and* ***Lyell***. *These scientists challenged the biblical view of creation, by making man a non-unique part of the general process of creation. Why, one might now ask, should man alone of all creatures posses an immortal soul, and at what stage of his evolution was he endowed with it?*

In 1864, *Pope Pius IX issued "A Syllabus of the Principle Errors of Our Times" which condemned most of the new scientific tendencies. In* 1870, *a general church council, in an effort to strengthen the pope's position, proclaimed* the dogma of papal infallibility, *which made the pope infallible in all statements he made officially.*

The impact of scientific discoveries on the Protestant church were felt more deeply, since its doctrine and ritual were almost entirely based on the Bible.

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property-owning class (capitalists) and the proletariat (workers who sell their labor in order to survive). The conflict between them will generate a society of people who work *and* own the means of production.

The further fact, that Protestantism was split into almost 300 *sects, made any uniform stand in the warfare between science and theology very difficult. At the same time, however, Protestant emphasis on the freedom of the individual to work out his own relations with God made it possible for many Protestants to reach their own compromise between faith and reason.*

A minority of Protestant 'fundamentalists', more influential in the United States than in Europe, continued to cling to a literal interpretation of the Bible and insisted on the validity of the account of creation as given in the Book of Genesis.

Interestingly enough, the conflict between science and theology did not seriously interfere with the progress of science. The world in 1914 *was still viewed as the intricate mechanism that Newton had supposedly shown it to be, a mechanism whose secrets would gradually yield to scientific inquiry. Only a handful of scientists realized that new developments — the discovery of X rays (1895), the isolation of radium (1898), and, most important, the formulation of the theory of special relativity and the Planck-Einstein discovery of the quantum (1905) — had opened up an infinite number of new mysteries and had brought the world to the threshold of another scientific revolution.*

The cult of science that dominated the intellectual climate at the end of the 19^{th} *century also had its impact on art and literature. In the early* 19^{th} *century, the Romantic artist escaped from the ugliness of early industrialism into an ideal world of his imagination set by his concept of natural beauty of a more glamorous past. Even before the middle of the* 19^{th} *century, some artists had begun to be interested in the world as it was, not as they felt it ought to be. This shift from Romanticism to Realism was more evident in literature, though less pronounced in painting; and there were hardly any signs of it in music.*

While the Romantic writers had been primarily interested in the unusual individual, the realistic novel was concerned with typical everyday society. Most of the great novels of the 19^{th} *century — by* ***Charles Dickens*** *(1812–1870,* England*),* ***Honore de Balzac*** *(1799–1850,* France*),* ***Gustave Flaubert*** *(1821–1880,* France*) and* ***Lev N. Tolstoy*** *(1828–1910,* Russia*) — fall into the category of social novels. Not only did authors describe the society in which they lived; they dwellt on the problems of that society. Literature increasingly became a form of social criticism.*

The trend toward Realism reached its climax in the 19^{th} *century in a literary movement called Naturalism. It represented a conscious effort on the artist's part to apply scientific principles to art.*

The naturalistic writers — men like ***Emile Zola*** *(1840–1902,* France*),* ***Henrik Ibsen*** *(1828–1906,* Norway*) and* ***Gerhart Hauptman*** *(1862–1946,*

Germany) — were not interested in the creation of beauty *but, like the scientist, they were interested in* truth. *To get at truth they discarded subjectivity and intuition and strove to describe objectively what they had learned from study and observation. The Naturalist was much impressed with the finding of modern science, especially in biology, and such new fields as sociology and psychology, and he made use of the new knowledge in his writing.*

It was the application of scientific principles to painting that characterized the school of Impressionism. Influenced by the scientific discoveries about the composition of light, painters like ***Camille Pissaro*** *(1830–1903,* France), ***Claude Monet*** *(1840–1926,* France), ***Auguste Renoir*** *(1841–1919,* France) *and others, used small dabs of color to depict nature in its ever-changing moods, not as it appeared to the logical mind but as it "impressed" the eye in viewing a whole scene rather than a series of specific objects. An Impressionist painting, examined at close range, thus appeared as a maze of colored dots which, if seen from a distance, merge into recognizable objects with the vibrant quality imparted by light.*

The prevailing school of Naturalism had little use for beauty. To the Naturalist, art had to serve a purpose and preach a message. In opposition to this view of art, a group of French poets at the end of the century claimed that art was sufficient unto itself — "art for art's sake".

*To these Symbolists [****Stephane Mallarme*** *(1842–1898,* France), ***Paul Verlaine*** *(1844–1896,* France), ***Rainer Maria Rilke*** *(1875–1926,* Germany) *and others] — art was not for the masses but only for the few to whom it spoke in 'symbols', using words not merely for their meaning but for the images and analogies they conveyed, often by sound alone. Symbolism is significant as a sign that there were people before* 1914 *who did not live in harmony with a society that glorified materialistic achievements and accepted the struggle for wealth as a sign of progress.*

The most outspoken critic of the generation before 1914 *was the philosopher* ***Friedrich Nietzsche*** *(1844–1900, Germany) who attacked almost everything his age held sacred — democracy, socialism, materialism, intellectualism, and Christianity. His wholesale condemnation of society was felt far beyond the first World War.*

Rise of Science in Germany (1870–1930)

In the 17th and 18th century, science moved relatively slowly. The number of brilliant scientists was limited. But during the 19th century scientific progress began to be made at a rapidly increasing rate. In the last three decades of the 19th century, and the first three decades of the 20th century, the center of science moved to Germany. There, due to a close working between fundamental science and its technological applications, science grew to become an integral part of the shaping of modern society in the 20th century. This monumental rise of German science and technology transformed Germany from a relative destitute and backward country into one of the great powers on earth. To understand this rapid and unparalleled growth of German science and industry, one must look at the factors that led to these developments.

When the ravages of the Thirty Years' War had been ended by the Westphalian peace treaty (1648), Germany was a devastated country. It took more than a century for it to recover intellectually[429]*, politically, and economically. In the 18th century, Germany was still quite backward compared to France and England. During the period of the Enlightenment (second half of the 18th century), the country started to recover intellectually as illustrated by the names of* ***Schiller****,* ***Goethe****,* ***Kant*** *and* ***Beethoven****. But otherwise the country was still in a depressed condition; the middle class was poor and had virtually no influence, compared to that of its counterparts in France and England. Poverty, starvation, and disease were widespread.*

An unexpected factor in the rebirth of German intellectual forces was the defeat, occupation and humiliation of Germany by Napoleon. Since the rebuilding of a strong army was, under French occupation, out of question, the leadership turned to the creation of a strong intellectual elite through the expansion of the universities. Thus, ***Wilhelm von Humboldt*** *was the driving force in the establishment of the University of Berlin, in the midst of the Napoleonic wars (1810). New universities were soon established in Breslau (1811) and Bonn (1818). In 1820, science began to develop at these universities, but reactionary forces were simultaneously at work: the revolutionary movement among students in favor of progressive ideas was suppressed by a massacre carried by troops under the command of Prince Wilhelm (1848), later to become Kaiser Wilhelm I.*

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[429] With a few exceptions such as **Johann Sebastian Bach** (1685–1750) and his family and **Gottfried Wilhelm von Leibniz** (1646–1716).

The expansion and unification of Germany under **Bismarck** *(1862–1890), created favorable conditions for the rapid growth of industry and agriculture. Unlike France and England, Germany was poor in natural resources and had no empire to exploit. Bismarck and other farsighted leaders recognized the vital importance of developing science and technology to increase the national economic wealth.*

To this end, the universities were granted strong government financial support, on a scale unprecedented in the history of any nation. During 1825–1900, a dozen of fist rate institutes of technology were established throughout the country. Many of the graduates of these excellent schools became dynamic leaders in industry*. They were fully aware that* basic science *was the main source of new inventions and improvements in* technology*. They established large research laboratories attached to their manufacturing enterprises.*

By the turn of the century, Germany had become the leading industrial country. It had a gigantic pharmaceutical and chemical industry; it had an electronic and optical industry of unmatched quality. Close collaboration between industry and universities was frequent and of mutual benefit[430].

Thus, while in 1840, with a population of about 35 million, it was plagued by poverty, misery, starvation and disease – in 1910, with a population of 70 million, it was a rich country with a highly developed middle class and a working class with better living conditions and more advanced social institutions than their counterparts in France and England, although both classes were virtually without political power.

1871 CE Parliament votes to abolish the "*religious tests*" at Cambridge University; from this year on degrees and positions could be earned without the need to adhere to the principles of faith of the Church of England. In the past, candidates for the M.A. degree and persons elected to fellowships were required to make subscriptions and declaration that, for non Christians, were equivalent to conversion[431].

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[430] The United States was the first country to follow Germany's lead in joining research and industry as evidences by DuPont, General Electric, among others. France and England soon fell behind.

[431] E.g., **James Joseph Sylvester** entered St. John's College, Cambridge in 1831. Being a Jew, and unwilling to sign the *Thirty-nine Articles*, he could not compete for one of the Smith's prizes and was ineligible for a fellowship, nor could

1871–1890 CE Years of social, economical and political instability in Western Europe: *The Paris Commune* (1871) — workers and soldiers took over the government of Paris for 3 months. The Commune was suppressed with the help of the Prussian Army. About 30,000 communards were executed by French authorities.

In 1873, there occurred the great world-wide financial crash. The next 17 years held hardship for ordinary people, great profits and consolidation for a few. Small businessmen (such as Einstein's father) were badly hit. This was a time of labor struggles, immigration, the rise of militant socialism, and above all the beginning of the age of imperialism and monopoly capitalism.

In 1878, **Otto von Bismarck** (1815–1898, Chancellor of Germany, 1871–1890) passed anti-socialist laws to suppress working-class political agitation, and said "*The great questions of the day will not be settled by revolutions and majority votes but by blood and iron*".

In 1879 **Wilhelm Marr** coined the word *anti-semitism* and found the *League of Anti-Semites*. The league blamed the Jews for the financial crisis.

It was a period of tremendous overall industrial expansion. People throughout Europe were forced off the land and into the cities. Central to Germany's industrialization was the growth of the chemical and electrical industries and the formation of the cartels of I.G. Farben, Krupp, etc. (by 1913, half of the world's trade in electro-chemical products was in German hands).

In 1887, the German government opened the *Physikalische-Technische Reichsanstalt* for research in the exact sciences and precision technology. **Werner von Siemens** (1816–1892) donated 500,000 marks to the project. His old friend, **Hermann von Helmholtz** (1821–1894) of the University of Berlin circle, was appointed head.

Arms expenditures in Germany nearly tripled between 1870 and 1890; the officer corps increased from 3000 to 22,500. Three-year military service became compulsory. Socialist literature was forbidden. Youth were subjected to intimidation and humiliation. Veteran organizations were state supported: membership increased from 27,000 in 1873 to 1,000,000 in 1900. Heads of state all appeared in military uniforms; even taxi drivers wore uniforms. The head of this military state was **Wilhelm I, Emperor of Germany** (1871–1888).

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he even take a degree; this last, however, he obtained at Trinity College, Dublin, where religious restrictions were no longer in force. Only in 1883 could he be appointed (as a Jew) a professor at Oxford.

1871–1908 CE *Discovery of the Aegean Bronze-Age Civilization.* In matters legendary and pre-Classical the antiquarian scholars confined their knowledge to the literary evidence of ancient authors: **Homer**, **Plutarch**, **Ovid**, **Pliny** and **Virgil**.

In the 18th and early 19th centuries travelers to Greece were naturally impressed by the great walls of Mycenae and Tiryns, still standing and book were written by scholars who described these impressive remains. But non of these scholars and explorers was concerned with pre-Classical history, with proving the relation of the legends of the heroic age to the visible monuments, by the clear evidence of excavation. For this an entirely new mental attitude was required, not that of antiquarianism, but of archaeology.

It came with **Heinrich Schliemann** (1822–1900, Germany), an archaeologist and linguist who realized his childhood dreams of rediscovering the Homeric world of Troy and Mycenae (1871–1884). His work was followed by **Arthur John Evans** (1851–1941, England) who conducted excavations in Crete (1894–1908), discovering pre-Phoenician script and the prehistoric Palace of Knossos, seat of an early culture he named Minoan. Evans' opening up the whole world of Mycenaean palace and state organization, has been one of the greatest contributions toward our understanding of the Bronze Age.

Scholars all over the world had struggled in vain far 50 years to decipher the Cretan so-called "Linear B". This honor fell to a young English architect[432].

1871–1909 CE **Edward Burnett Tylor** (1832–1917, England and USA). Founder of cultural anthropology. Advanced the new idea that human history is dominated by the concept of *cultures* rather than that of *Nations* that had spread across Europe after the 15^{th} century. This innovation of Modern Social Science would be the key to new ways of thinking about the meaning of history and the future. He saw all cultures as parts of a single history of human thought, and all evolution that Darwin had described in biology, Tylor too now saw in society.

Tylor was born in London, the son of a prosperous English Quaker. As Quaker he could not enter a university and so began life in the family business. Seeking a climate to cure his tuberculosis, he went to Mexico (1856) where he

[432] The Minoan Linear B script was deciphered (1952) by the architect and cryptographer **Michael George Francis Ventris** (1922–1956, England) who demonstrated that it is Greek in oldest known form, predating Homer by some 500 years. He had heard Evans' lecture in 1936 when he, Ventris, was 14. After the War, in which he served as a pilot in the Royal Air Force, he continued his efforts to decipher the linear scripts. He was killed at the age of 34 in a motor accident.

accidentally joined a study of Toltec remains. So began Tylor's lifelong study of strange and ancient societies and their relation to modern life.

Although he never studied formally at a university, he became a professor of anthropology at Oxford (1896–1909).

He wrote: *Primitive Cultures* (1871); *Early History of Mankind and the Development of Civilization* (1865).

Tylor wrote:

"The past is continually needed to explain the present, and the whole to explain the part. There seems to be no human thought so primitive as to have lost its bearing on our own thought, nor so ancient as to have lost its bearing on our own thought".

His work was continued by **F. Boas** (1911), **O. Spengler** (1918) and **A.J. Toynbee** (1934).

1872 CE **Peter Ludwig Mejdell Sylow** (1832–1918, Norway). Mathematician. Proved a key theorem in the theory of *finite groups*[433]. Sylow studied at the University of Christiania and became a high school teacher (1858–1898). Lie had a special chair created for Sylow at Christiania from 1898.

1872 CE **Ernst Mach** (1838–1916, Moravia). Physicist, philosopher and psychologist. Rejected the Newtonian concepts of absolute space and inherent inertia of material bodies. His qualitative ideas (no quantitative theory!) stemmed from the realization that Newton's observations on inertia were entirely *local*, and no references to the rest of the universe were made.

To measure the rotation of the earth, Newton used a terrestrial experiment, whereas the same rotation can be determined by global or *astronomical* measurement via the apparent motion of the stars. This coincidence, Mach argued, must stem from a causal relationship between the motion of the distant stars and the local inertial frame of reference, and it must imply that the inertia of any body is determined by the distribution of distant matter in the universe.

[433] *Sylow's Theorem*: If p^n is the largest power of the prime p to divide the order of a group G then

- G has subgroups of order p^n (called Sylow p-subgroups)
- any two such subgroups are conjugate
- G has $n_p = 1(\text{mod } p)$ such subgroups, and n_p is the order of the quotient group of G by the normalizer of any given Sylow p-subgroup

Almost all work on finite groups uses Sylow's theorem.

According to Mach there is no absolute space, and particles' inertia is due to unspecified interactions with the rest of the universe; all that matters in mechanics is the *relative* motions of *all* the masses, near and distant (i.e. an isolated mass has no inertia, centrifugal forces are physical, etc.). The totality of these ideas can be encapsulated into one statement, known as "*Mach's principle*":

A body's inertia and the local structure of space-time are determined by the mass distribution in the rest of the universe.[434]

Thus, according to Mach, stars en masse *cause* inertia[435].

Although Einstein's GTR is local and does not incorporate the 'Mach principle' in any direct way, the accepted cosmological models *based* on GTR, do

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[434] For further reading, see:
Mach, Ernst, *The Science of Mechanics*, The Open Court Publishing Co., 1989, 634 pp.

[435] On an earth covered permanently by clouds, an observer of the peculiar motions of the free gyrocompass with respect to ground would inevitably (after having searched in vain for any visible agency to which the spin axis of the gyro may appear attached) be drawn toward the Newtonian idea of some mysterious "absolute space" with respect to which the earth happens to be in rotation. However, if in this state of affairs the cloud cover were suddenly removed, revealing the stars whose average motion with respect to ground happens to correlate with the precession averaged gyro's axis, the observer would be equally inevitably drawn toward the thought that the stars are the cause of the gyro's previously inexplicable behavior. The formulation of this idea, which goes back to George Berkeley, is known under the name of *Mach's principle*: the local inertial behavior of any object is somehow determined by the entire actual distribution of masses and their motions in the universe.
Mach himself did not specify what kind of interaction ought to be held responsible for inertia. From time to time attempts have been made to put Mach's principle, which is really not more than a program, on a quantitative basis by constructing cosmological models in which a particular kind of interaction between the distant stars and local objects is invoked to explain inertial behavior. Most of these attempts invoke gravitation as the inertia-producing interaction, a rather obvious choice in view of the universal proportionality between inertial and gravitational mass (*equivalence principle* of GTR). There were also attempts to account for inertia through invention of a special kind of "field", other than gravitation, just for this purpose.
None of these attempts has been wholly satisfactory. In particular, no one has yet been able to construct a cosmological model incorporating Mach's principle and also incorporating consistently the velocity of light as the limiting speed with which physical influences may be propagated.

satisfy a limited form of Mach's principle — in the sense that at any location there is a preferred *local frame* of reference in which the distant galaxies of the universe are, on the average, *receding isotropically*. This frame is best defined by the *cosmic microwave* background radiation. Thus, in relativistic big-bang cosmology the local inertial frames of Newtonian mechanics and STR are indeed determined to some extent by distant matter — although a body's inertia is *not*.

The most "Machian" effect in GTR is *frame-dragging* — the ability of a rotating mass to (partially and locally) drag with it the inertial frame; the dragging is not rigid, and decreases with distance from the rotating mass. This is one of the effects to be tested in the "gravity probe B", a Stanford University Collaborative project with industry[436], in which an extremely round niobium coated gyroscope is set spinning in a box within a satellite, and its general-relativistic precession measured via induced quantum eddy-currents in superconducting devices. In this experiment, the frame-dragging, rotating mass is the nearby earth.

An extreme, though impractical, manifestation of this effect occurs in the **Lense-Thirring-Brill-Cohen** thought experiment: an infinitely massive, spherical shell, when set rotating, drags the inertial frames in its interior *rigidly*, at the same angular velocity as that of the rotating shell. Again, the *magnitudes* of objects' inertia are not themselves affected — only the identity of inertial *frames*[437].

There is evidently a consistent line of thought which goes through the philosophical doctrines of Spinoza-Leibniz-Berkeley-Mach, that also has a bearing on the *ideas of quantum mechanics*: namely, the non-separability of interacting systems which forces us to treat the observer as part of the physical system being observed — a *Machian notion*. Yet at the same time, in the quantum mechanical description, one is forced to consider higher-dimensional Hilbert spaces, and when proceeding to gauge theories, one must also reckon with the geometry of fiber bundles — all increasingly remote from direct observation. This latter trend in modern physics is in *contrast* to Mach's positivism: he was steadfast in his rejection of non-empirical concepts.

[436] This experiment was launched into orbit in April of 2004, and as of this writing is still taking data.

[437] However, according to the standard models of particle physics and cosmology, the rest masses of electrons and quarks – and thus of all ordinary matter existing today – were endowed, a fraction of a second after the Big Bang, by the condensation of a global, vacuum-permeating quantum scalar field; in this sense and to this extent, modern physics does incorporate the Machian notion of cosmologically determined inertia, at least as far as *rest* mass is concerned.

Mach stands out as a unique figure among physicists. He mounted a crusade against a universally accepted approach to classical mechanics and went to extremes in his zeal to purge physics of its scholastic relics[438]. Thus, he rejected the existence of atoms, and died unconvinced of the kinetic theory of gases, Brownian motion and the special theory of relativity. Einstein, however, considered him as the forerunner of the general theory of relativity.

Mach was born in Turas, Moravia and studied in Vienna. He was a professor of mathematics at Graz (1864–1867), of physics at Prague (1867–1895) and of physics at Vienna (1895–1901).

1872–1876 CE *The Challenger expedition.* The first systematic attempt to explore the depth and breadth of the world's ocean from the chemical, physical and biological points of view. Spurred by this international competition (1871), the Royal Society appointed a committee which recommended that funds be requested immediately from Her Majesty's Government for an expedition with the following objectives:

1. To investigate the physical conditions of the deep sea in the great ocean basins.
2. To determine the chemical composition of seawater at all depths in the ocean.
3. To ascertain the physical and chemical characters of the deposits at the sea floor and their origins.
4. To examine the distribution of organic life at all depths in the sea as well as on the sea floor.

To carry out these objectives, it was recommended that a sizable ship, a staff of scientists qualified to carry out the desired investigations, and an ample supply of equipment, instruments, and special apparatus be made available. As a result of these recommendations, the first great oceanographic expedition, a

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[438] Mach entered into a debate with Planck about the nature of science, at the beginning of the 20th century. Mach championed an instrumentalist philosophy of science. He lodged science in everyday *psychological* experience and exalted technology. Being a liberal democrat, Mach was intent on empowering "citizen scientists".

Plank, on the other hand, was a realist. He reduced everyday experience to the ultimate constituents of physics and promoted abstract problem solving. He was a state corporatist, who thought ordinary folks had no claim on "real science".

This debate is emblematic of all such debates since.

model for all subsequent efforts, was organized. The expedition was a bold attack upon the unknown in the tradition of the great see explorations of the 15^{th} and 16^{th} centuries.

H.M.S. Challenger was an 18-gun corvette of the British navy that had been stripped of battle gear and fitted for oceanographic research. It had a displacement of 2306 tons and was equipped with both sail and steam power. The British Admiralty provided a crew under the command of Captain George Nares;

Charles Wyville Thomson (1830–1882), professor of natural philosophy at the University of Edinburgh, headed the research staff. The 240-man expedition left Portsmouth, England, in December, 1872. The *Challenger* criss-crossed the North Atlantic, swerved down through the South Atlantic, and eastward into the Antarctic Ocean. Leaving the Antarctic, it continued its way to Australia and the Western Pacific islands, eastward to the Hawaiian Islands, on through the Straits of Magellan, and finally back to England where it landed on May 24, 1876.

In three years the *Challenger* has sailed 110,500 km and made 362 "oceanographic stations", gathering data on weather conditions, surface currents, water temperature, water composition at various depths, marine organisms, and bottom sediments. Expedition scientists charted oceanic topography on the ship's track, netted and classified 4,717 new species of marine life[439], and took a depth measurement of 8183 m in what came to be known as the *Challenger Deep*[440].

The official report of the *Challenger* expedition, filled 29,500 pages in 50 volumes and took 23 years to complete. A total of 76 authors contributed to the report, and numerous other specialists were consulted. No other expedition has made so many important contributions to oceanography.

When she left England, the ocean depths were an almost unfathomed mystery. When she returned, she had sounded the depth of every ocean except the Arctic and laid the foundation for the modern science of oceanography.

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439 One of the most interesting of those organisms is the *radiolaria*, of which the expedition collected 3508 new species to add to the 600 then known. The Challenger discoveries demonstrated that the oceans were teeming with unknown life waiting to be classified. It proved that life existed at great depth in the sea.

440 It was measured on March 23, 1875, off the Mariana Islands. The deepest known spot in all the oceans is at 11,033 m below the surface in the *Mariana Trench*.

Oceanography[441] *— The Conquest of Inner Space (1000 BCE–1927 CE)*

I Historical Background

Ancient and medieval navigation had been largely coastal; mariners did not sail many days out of sight of land. They knew the sea but not the Ocean. Indeed, even in the middle of the 15th century, at the time when the Renaissance is supposed to begin, man's knowledge at the face of the earth was still restricted to a very small portion of it, and even in that portion was very superficial. One of the great tasks to be accomplished was the discovery of planet earth.

Schematically, the earth can be regarded as being composed of three materials: rock, water and air, arranged in three layers — the lithosphere, hydrosphere, and atmosphere. The earth is also an astronomical body: the effects of nonuniform distribution of sunlight over the earth and the equally energetic but more uniform radiation of earth heat into space, acting with rotational and gravitational forces, produce a complex of interdependent fluid phenomena which characterize the world as we know it.

Oceanography may be defined as the study of the oceans. It is principally concerned with the various aspects of sea water[442]*: its motions and chemical constituents, its physical properties and behavior, its relationships to the solid earth, the atmosphere, and to living organisms of all kinds, its economic and technical potentialities, its role as part of the earth's outer covering.*

Thus, oceanographers are drawn from four large areas of science: geology, chemistry, physics, and biology. Geologic study of oceanic sediments, rock

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441 For further reading, see:

- Von Arx, W.S., *An Introduction to Physical Oceanography*, Addison-Wesley Publishing Company: Reading, MA, 1962, 422 pp.
- Weyl, P.K., *Oceanography*, John Wiley & Sons: New York, 1970, 535 pp.
- Yasso, W.E., *Oceanography*, Holt, Rinehard and Winston: New York, 1965, 176 pp.

442 Gross properties of sea water: *Characteristic density* = 1.025 gm/cm^3; *Velocity of sound at surface* = 1448.6 m/sec; *Specific heat* = 0.932 cal/gm/$^\circ$C at salinity of 35%; *Adiabatic lapse rate* $\sim$ 0.1°C/km; *Maximum surface temperature* = 32°C; *Minimum surface temperature* = -2°C.

structure, and topography is referred to as submarine geology or geological oceanography. The study of sea-water chemistry is called marine chemistry or chemical oceanography. The biological aspects of the ocean environment, such as fish populations and plant life, are studied by the marine biologist, or biological oceanographer. Atmospheric processes, circulation of the oceans, and the physical properties of ocean water fall within the scope of marine physics, or physical oceanography.

Oceans cover nearly $\frac{3}{4}$ of the earth's surface; the topography of the ocean floors is more rugged and more mountainous than the topography of the continents; the greatest oceanic depths are greater than the height of Mt. Everest. The distance from the top of this mountain to the bottom of the Mariana Trench is about 20 km. Even so, the solid surface of the earth would still be much smoother than the surface of an orange if scaled to that size. But the oceans are even smoother than that.

The volume of the oceans is roughly 1365 million cubic kilometers, with a mass of 1560 million billion tons. Although these are enormous figures, they represent only 1/790 of the volume and 1/4200 of the mass of the earth. Covering a surface area of 362 million square kilometers, the average depth of this water mass is slightly less than 3.8 kilometers, or about 1/1680 of the earth's radius. Truly, the oceans are hardly more than a film of salt water on the surface of our planet.

Yet, it was in some shallow, near-shore area of this film that life on earth began two or three billion years ago. Today it is the tremendous abundance of life in the sea that appears to present the surest solution of the world's food problems. Well over half — perhaps as much as 85 percent — of the food product of all plants is produced by marine plants.

In time the oceans will certainly become a major source of valuable minerals and chemicals and an important source of fresh water when these are no longer available in sufficient quantities on the continents. Also, in recent years, it has become clear that the oceans play a very important but little-understood role in determining the earth's weather and climate patterns. And finally, it may be that the sediment layers at the bottom of the sea contain a record going back several billion years — a record of the earth's history. For these and many other reasons, the importance of the earth's oceans can hardly be overestimated.

The heat content in the oceans does not vary appreciably from year to year, but maintains a gently shifting balance consistent with external sources and demands. Several processes serve to heat the oceans:

- Radiation absorbed from the sun and sky.

- *Condensation of water vapor as dew on the sea surface.*
- *Conduction from the atmosphere.*
- *Conduction from the sea floor.*
- *Conversion of mechanical energy into heat.*

Other processes serve to cool it:

- *Radiation from the sea surface to space.*
- *Evaporation to the atmosphere.*
- *Conduction to the atmosphere.*
- *Conduction to the sea floor.*

Of these processes, the conductive and radiative exchanges with the atmosphere are of greatest importance. The heat of conversion of the mechanical energy of winds and ocean currents is generally less than the heat of incoming radiation from the sun and the sky by a factor of 10^{-4}*, but* locally *may be somewhat greater where tides are accompanied by strong frictional retardation, as in some shallow seas.*

Conduction to the atmosphere is significant where the ocean temperature is higher than that of the atmosphere but not in the reverse case. This difference in conductive efficiency is due to the strong convection that can develop in cool air over a warm ocean, and the marked stability that develops when warm air is chilled by a cold ocean. Conduction may amount to some 10% of the evaporative heat loss of the oceans. Conductive exchanges with the sea floor are thought to be mainly upward, owing to the evolution of radiogenic heat in the earth's crust. The important influences are seen to be associated with radiative exchanges, evaporation processes, and the upward flux of heat in conduction to the atmosphere.

The flux of sensible heat is directed mostly from the oceans to the atmosphere, but not exclusively so. In middle latitudes of the Northern Hemisphere, particularly over the eastern parts of the North Atlantic Ocean, the heat flux in summer is directed from air to sea. This effect places the region of maximum atmospheric warming by conduction in the western and northern portions of the Northern Hemisphere oceans, mainly in association with the poleward transport of tropical waters in the Kuroshio of the North Pacific and the Gulf Stream of the North Atlantic Ocean. Over the middle-latitude portions of these currents the annual average upward flux of sensible heat may exceed 90 $\mathrm{cal/cm^2/day}$ *over the Kuroshio, and 120* $\mathrm{cal/cm^2/day}$ *over the Gulf*

Stream. The total energy loss of sensible plus latent heat is about four times as great.

Precipitation, in general, returns sensible but not latent heat to the oceans. Oceanic warming by latent heat can occur only when there is condensation directly on the sea surface as dew, or possibly to some extent in very shallow fogs. In other cases latent heat is released at the level of condensation, and has only the indirect effect of abating radiation losses from the sea surface.

The sea radiates to the atmosphere and space very nearly as a black body and therefore contributes outgoing energy in amounts proportional to the fourth power of its absolute surface temperature. The wavelength of maximum emission for the sea surface is nearly centered on the 10-μ "window" between the absorption bands for atmospheric water vapor and carbon dioxide. Water vapor is, however, such a strong absorber of infrared radiations even in this window that the amount of outgoing radiation from the ocean surface is more closely related to the absolute humidity of the lower air than to the ocean temperature. As the air temperature falls and the absolute humidity of the lower atmosphere is correspondingly decreased, the radiation losses from the ocean tend to increase until a skin of ice is formed. Bubble-free ice is nearly as good a black-body radiator as a free-water surface is, but as soon as bubbles are trapped in sea ice they reflect radiation back into the ocean and reduce the intensity of long wavelength emission to the atmosphere. The latter effect tends to confine heat within the water phase of ice-covered oceans, and indeed explains in part why the ice in the Arctic Ocean is relatively so thin despite the very low air temperatures that sometimes prevail.

The average energy of incoming short-wave radiation from the sun and sky usually exceeds the heat loss through radiative processes from the sea surface. The excess heat in the ocean is first communicated to the atmosphere by the process of evaporation and conduction, whereupon the atmosphere radiates this energy to space.

The movement and circulation of the oceans is tied very closely to the circulation of the atmosphere: Both are ultimately driven by the distribution of available solar energy, and their motions are linked by friction at the sea surface. There exists an imbalance in the latitudinal distribution of energy that produces an equator-to-pole temperature gradient at the surface — the driving force for the pattern of earth's surface wind. These wind patterns are responsible for the circulation of the ocean surface and the formation of the world's major ocean currents. As with the atmosphere, once the ocean starts to move, it comes under the influence of the Coriolis effect, which plays a significant role in the resulting circulation patterns.

The oceans are vertically stratified, with more dense water at the bottoms of the major ocean basins and less-dense water near the surface. The density

is controlled by the temperature and by the salt content (salinity) of the water. The deep-ocean water is separated from the surface layer of the ocean by a transition zone with sharply defined density, temperature, and salinity gradients. This deep-ocean water moves as a response to small changes in density that occur over wide areas, and the movement is largely independent of the surface-ocean circulation. Together, however, both types of ocean circulation contribute to the redistribution of available energy in the earth system, albeit over very different time scales. And both play a major role in the distribution of nutrient supplies in the oceans.

Like the circulation of the atmosphere, the circulation of the ocean is ultimately driven by solar energy. Figure 12 is a systems diagram of ocean circulation. The distribution of solar energy over space and time results in the formation of the global wind belts. These roughly latitudinal wind patterns in turn produce the ocean currents that determine the circulation patterns of the upper ocean. The distribution of surface-ocean temperatures, which is partly a result of these circulation patterns, strongly influences the density structure of the ocean. It is this density structure that drives the circulation within the deep ocean. As with the atmosphere, the feedback loops in Figure 12 are negative — surface-temperature gradients drive the circulations, but the net effect is to move warmer water poleward and cooler water toward the tropics. Both the surface-ocean and deep-ocean circulations also play a vital role in climate. The surface circulation is one of the primary factors controlling climate variability on the order of years to decades, but even the deep circulation, which generally operates on the time scale of hundreds of years, is being implicated in short-term climate change. The two different circulation systems act together in controlling the distribution of the nutrients that are essential to marine life.

The circulation in the troposphere is caused by atmospheric pressure gradients that result from vertical or horizontal temperature differences. From a global perspective, these temperature variations are caused by latitudinal differences in solar heating. But ocean surfaces are also heated by incoming solar radiation. Do the oceans, therefore, circulate for the same reason as the atmosphere? The answer is no, because the solar heating of the ocean takes place at the upper *surface of the fluid, whereas the solar heating of the atmosphere occurs largely at the* lower *surface of the fluid-near earth's surface, where clouds and the green-house effect warm the atmosphere. Solar heating results in warmer water at the surface of most of the world's oceans. But the sun's rays warm only the top few hundred meters of the ocean; 90% of the radiation that penetrates the surface is absorbed in the first 100 m. The warmer water is less dense than the cooler water below, which is not affected by the surface heating. This situation is inherently stable, so there is very little vertical movement.*

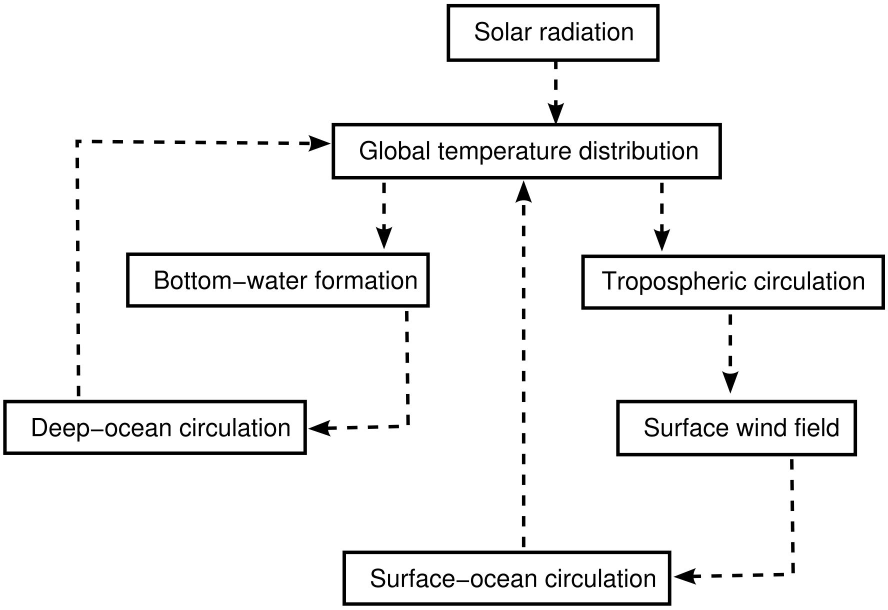

Fig. 12: Ocean circulation diagram

It is similar to the situation in the stratosphere. The atmosphere at this level is stable because the maximum solar heating occurs high in the stratosphere, the site of peak absorption of ultraviolet radiation by ozone. Where temperature increases with height, there is no density imbalance, and convection cannot take place. The fluid – water or air – remains well stratified. The true situation in the ocean is actually more complicated than this, because the density of seawater is also affected by its salt content. It remains true, however, that the ocean overturns very slowly.

At the same time, temperature changes in the ocean occur slowly. The oceans have a high heat capacity — it takes a considerable amount of heat to produce just small changes in temperature. Slight differences in incoming solar radiation from place to place thus have little impact on the surface temperature of the ocean, so lateral temperature and density differences are slight over large areas. Unlike the troposphere, therefore, the surface ocean does not circulate as a direct *response to the surface heating. Instead, surface temperature plays a more indirect role: The surface temperature influences the atmospheric circulation, and the resulting pattern of global winds determines the circulation of the upper ocean.*

The content of oceanography, like that of any other science, depends to a large extent on its history. Man's exploration of the world's oceans was at first largely limited to the surface of the sea and to the mapping of the

distribution of land and water. Later man turned to the exploration of the depths of the ocean.

Our current knowledge of the oceans developed in stages: At first the map of the world was completed. The geographic exploration of the distribution of land and sea led to an interest in the sea itself. Oceanographic expeditions were conceived and carried out, and then published scientific reports added to our store of knowledge. Finally, puzzling observations led to the development of a dynamic theory which transformed oceanography from a purely descriptive to an analytical science.

Although primitive man had explored all the oceans and most of the land areas of the world, his geographic knowledge was localized. The inhabitant of a Pacific atoll knew his immediate environment and had stick charts indicating the locations of neighboring islands. His oral traditions told of the faraway places from which his ancestors had come to settle the island. While there was extensive knowledge of the local geography, there was no geography of the earth as a whole. The gradual evolution of the picture of the world, as seen from Europe, began with the earliest roots of civilization.

*The study of the sea seems always to have been promoted by a practical rather than an abstract curiosity about the natural world. The maritime commerce of the Phoenicians, Hebrews and Greeks made the Mediterranean and adjacent seas reasonably well known even before the time of the Judean kings [**Solomon** (ca 940 BCE) II Chron* **8**, *17;* ***Jehoshaphat*** *(ca 870 BCE) II Kings* **22**, *49* and *II Chron* **20**, *36]. The waters of this sea enclosed by three continents provided a ready means of transport of goods and soldiers. The conception of the world, about* 850 *BCE, included only the land immediately adjacent to the Mediterranean, surrounded by Oceanus, the indefinite land-encircling ocean which lay everywhere beyond the frontiers of knowledge.*

*The master merchant mariners, the Phoenicians, passed through the Pillars of Hercules, the Straits of Gibraltar, into the Atlantic Ocean. They circumnavigated Africa and penetrated north to Great Britain: Indeed, about 600 BCE (according to **Herodotos**), King Necho of Egypt sent an expedition manned with Phoenician sailors down to the Red Sea and along the east coast of Africa. The ship is said to have returned to the Mediterranean,* 3 *years later by way of Gibraltar. The expedition should have established that Africa is a separate continent, but the chronicle was rejected by scholars of that time, and Greek philosophers continued to support the Homeric map.*

In the 4^{th} century BCE[443]*, man's knowledge was somewhat extended due to the Indus River expedition of **Alexander the Great** (329–325 BCE) and the voyage of **Pytheas of Massilia**. The former revealed the relationships*

443 The book of **Jonah** was composed at about that time.

of several bodies of water to one another, namely, the Caspian Sea, Persian Gulf, and Arabian Sea, and to the then known world around the Red Sea and the Mediterranean. On a map drawn about 300 BCE by ***Dinaearchos of Messina*** *(a pupil of Aristotle), parallels of latitude are used for the first time.*

The geographic knowledge of the Romans was summarized by ***Ptolemy*** *(150). He introduced the concepts of latitude and longitude, and presented a projection of the globe on a map which showed the Indian Ocean surrounded by land in part unknown.*

Although it was relatively easy to measure the latitude by noting the angle the pole star makes with the vertical, the ancients had no way to measure longitude directly. Ptolemy therefore estimated distances in the east-west direction from the time required for voyages.

The Indian Ocean was the first to be used for trade but, strangely, was one of the last to be thoroughly explored. The ancients carried on a brisk trade between the Mediterranean and the East by way of the Red Sea and Indian Ocean. This sea traffic was much influenced by the monsoon. The wet monsoon of the Northern Hemisphere summer permitted the ships of the Greek and Arab traders to penetrate the Arabian Sea and Bay of Bengal, and then during the period of the dry monsoon of the Northern Hemisphere winter to find favorable winds for the homeward journey.

It was the reversing currents associated with the monsoon winds of the Northern Indian Ocean that favored this traffic, made fruitful by the ready markets for oriental products in the Roman world. With the fall of Rome the trade dwindled, but unwittingly the knowledge that the Greeks and Romans possessed had been deposited with Arabian scholars for safekeeping during the Dark Ages[444]*. By this accident the Ptolematic view of the world was kept intact until the* 11^{th} *century CE when, as a by-product of the Crusades, Western civilization was re-educated concerning its own past.*

While southern Europe was preoccupied with matters of theology, the Norsemen were engaged in journeys of discovery, aided by improved climatic conditions, which reduced the amount of ice in Northern waters. Iceland was visited by the Picts and the Celts in 650, and settled by the Celts in 770.

In 835, a papal bull referred to Christian settlements in both Iceland and Greenland. The Vikings began to take over these Northern lands in about

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[444] The deterioration of geographic knowledge during the Middle Ages is indicated by the world map of **Cosmas Indicopleustes** (fl. 548 BCE), an Alexandrian navigator of the Indian Ocean. He insisted that the earth was a quadrilateral measuring 20,000 km × 10,000 km.

870. *In* 982 ***Eric the Red*** *crossed the Davis Strait from Greenland to Baffin Island, in Canada. Three years later he established a colony in Greenland. His son* ***Leif Ericsson*** *sailed west from Greenland in* 995 *and spent the winter in Newfoundland.*

Because of a deterioration of the climate beginning in about 1200*, the Viking colonies in Greenland became isolated; the Vikings were therefore never able to exploit their discovery of America. Were it not for the readvance of ice in the North Atlantic, the history of America might have been very different. The Vikings' conception of the Northern ocean are known to us through the map of* ***Sigurd Stefansson*** *(1570).*

While the Vikings were exploring the Northern seas, Arab traders were exploring the Indian Ocean and sailed as far as China. A map by ***Abu ar-Rayhan al-Biruni*** *(1030) reflects the geographic knowledge of his times. The Arabs brought the lodestone from China and thus introduced the magnetic compass to the West. At first it was viewed with suspicion as being under the influence of some infernal spirit. However, the mariner's need for an instrument with which he could steer a fixed course, regardless of visibility, led to the rapid adoption of the magnetic compass.*

The period from 1492 *to* 1522 *is known as the Age of Discovery because geographic knowledge expanded at a very rapid rate during these* 30 *years. The continents of North and South America were added to the globe, and the earth was circumnavigated. These daring voyages of discovery were brought about by a political event. In* 1453 *the Sultan Muhammad II captured the capital of Eastern Christendom, Constantinople. As a result, the Mediterranean ports were cut off from the riches of the East.*

At the same time, learned Greeks expelled from Constantinople brought the geographical knowledge of the ancients to Italy, and the introduction of paper permitted the wide distribution of these works. Thus the Turks indirectly revived old knowledge, which made it possible to find new sea routes while creating an economic motivation for exploration.

Meanwhile, the Portuguese and others had been making preparations for their great voyages of discovery. In 1420 *Prince* ***Henry the Navigator*** *established a maritime observatory and assembled the best Italian map-makers and Jewish astronomers to teach navigation to the Portuguese. Until that time the Portuguese had been afraid to sail out of sight of land, and all expeditions to round Africa had turned back at Cape Bojador (27°N). The Cape of Good Hope was finally rounded by* ***Bartholomeu Dias*** *in* 1488*. In* 1498*,* ***Vasco da Gama*** *extended the trip around Africa to India*[445].

[445] When **da Gama** rounded the Cape of Good Hope in April, 1498, he acquired the services of an Arab navigator, **Ahmad Ibn Majid**, to guide him across

In 1474, the Florentine astronomer ***Toscanelli*** *wrote to the King of Portugal suggesting an expedition to explore a route to the Spice Islands of the East across the Atlantic Ocean. He appended a map which greatly underestimated the distance to the east coast of Asia, placing it at a longitude just off the west coast of America. Later, on inquiry, he sent a copy of this letter to* ***Christopher Columbus****.*

In 1492, Columbus set sail westward to reach the Indies. His underestimation of the distance to China caused him to believe that he had reached the Indies when he had in fact discovered what we know as the West Indies; actually he was farther from his goal than when he had left Spain. The Spaniards and the Portuguese set out to explore the eastern shores of the Americas and the Indian Ocean. The greatest of the oceans, the Pacific, was not discovered until 1513, when ***Vasco Núñez de Balboa*** *sighted it from a mountain in Panama.*

These early voyages of discovery culminated in the circumnavigation of the globe by ***Ferdinand Magellan****. He left Spain in September 1519 with 5 ships, on a mission to find a passage between the Atlantic and Pacific Oceans. On the* 21^{th} *of October 1520 he found a passage to the great Western ocean at* $52°S$*, now known as the Straits of Magellan. In March 1521 he discovered the Philippines, and in April of that year met his death, at the hands of the aborigines of Cebu. On his trip (1521) Magellan made the first recorded attempt to measure the depth of the open ocean by lowering a weighted line to a depth of some 370 m, but failed to reach bottom, which is now known to be 3700 m*[446]*.*

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the Indian Ocean to the Malabar Coast of India. Although the route was not known to European navigators, the Arabs had an intimate knowledge of sea routes between the Indian Ocean, Arabian Sea, and Mediterranean Sea.

446 Another measuring problem faced by the mariner was how to determine the speed at which his ship was moving through the water. Since there are no fixed reference points at sea, the captain would throw a floating object overboard and time how long it took the object to drift by a measured interval marked off on the deck of the ship. An improved method of measuring speed was introduced by the Dutch near the end of the 16^{th} century. This was the so-called Dutchman's log, which has left its traces in nautical jargon.

The Dutchman's log consists of a piece of wood (the log) attached to a reel of string, with knots tied in at equal, fixed intervals. When the log is thrown overboard, an hourglass is inverted. As the sand in the hourglass runs out, the knots that pass overboard are counted. Thus one obtains the speed of the ship in *knots*. The speed is then entered in the *logbook* with information about the state of the weather and the sea.

By the 16^{th} century, the fact of the earth's rotundity was proven beyond question by Magellan's expedition.

By the year 1600 the surface of the known earth was doubled. It was not only a matter of quantity, but one of quality as well. New climates, new aspects of nature were revealed, new plants, new animals, new men and women.

The psychological reverberation of such new vistas was immense. A man of today can recall the deep emotions he felt when he found himself for the first time in the middle of the ocean, or in the heart of a tropical jungle, or when he tried to cross a desert or a glacier. These discoveries, which are fundamental for each of us individually, were made for the whole of mankind in the fifteenth and sixteenth centuries.

In the meantime, the intense rivalries of colonizing nations encouraged the progress of navigation and of the physical sciences which would increase the accuracy of sailings and minimize their dangers. The main requirements were geodetic, astronomic (better methods of taking the ship's bearings), cartographic; one needed faster ships and better instruments to navigate them. Geodetic improvements were due to ***Jean Fernel*** *(1528) and* ***Gemma Frisius*** *(1533); better maps were due to the* ***Pedro Nuñez*** *(1530), and* ***Gerhard Mercator*** *(1568), and* ***Abraham Ortelius*** *(1568). The splendid geographic atlases which were produced in 1570, provide a large mass of information of vital importance to navigators.*

One of the first fruits of oceanic navigation was a better knowledge of magnetic declination, for the compass was one of the sailor's best instruments, but its readings could not be trusted without taking occasional deviations into account. The magnetic observations and other knowledge useful for navigation were put together by Englishmen like ***Robert Norman*** *(1581) and* ***William Barlow*** *(1597) and by* ***Simon Stevin*** *(1599). At the very end of the Renaissance,* ***William Gilbert*** *published the first great treatise on magnetism (1600).*

Nearly six decades after Magellan, ***Francis Drake*** *found the gap between Tierra del Fuego and the mainland of Antarctica, the* Drake Passage *(1578). This provided the closing link in the discovery of the Southern (Antarctic) Ocean. It remained to be known, however, whether or not land lay to the south. It was* ***James Cook*** *who suggested that an Antarctic continent existed (1772–1775). He was one of the first to lead a journey intended to produce scientific discoveries. [The early discoverers set out, not to discover the secrets of nature, but rather to find the riches of the world and claim them for their royal sponsors. Their successes were measured in treasure-laden galleons rather than in scientific information recorded in expedition reports. While the gold of the Aztecs and the Incas has been largely dissipated, the*

scientific treasures of the newer explorers represent a permanent addition to our store of knowledge.]

Between 1769 and 1779 Cook commanded British vessels on three major voyages of discovery. On the last Cook set off in search of a northwest passage between the Atlantic and Pacific Oceans and successfully penetrated into the Arctic Ocean by way of the Bering Straits. A short distance beyond the Bering Strait, he was stopped by the Arctic ice pack.

Cook was the first explorer provided with the proper instruments to determine latitude and longitude accurately. On his second voyage he had four accurate clocks to help in navigation. They had been developed as a result of a naval disaster in 1707, when 2000 men were lost because of faulty navigation. To help avoid such occurrences in the future, Parliament established a prize for a method of determining longitude.

In 1000 days at sea, Cook lost only one sailor out of a crew of 118. He was the first to conquer the sailor's disease, scurvy, which results from a lack of vitamin C. By watching the diet of his sailors and giving them citrus juice, Cook showed that long sea voyages were possible without detriment to health. Because they were required to drink lime juice to avoid scurvy, British sailors were thenceforth called Limeys.

Cook's third voyage essentially completed the geographical exploration of the oceans of the world. Only the continent of Antarctica, hidden by a shield of ice, remained to be discovered. (This was accomplished in 1820 by ***Nathaniel Palmer****.) Cook's skill as a navigator set new standards and accurately fixed the location of many islands that had previously been known only vaguely. He showed that long voyages of exploration were possible without endangering the health of the crew. Finally, he demonstrated convincingly that the land was not distributed symmetrically about the equator: the great southern continent did not exist unless it was south of 70°, protected by ice.*

The depth of the sea, however, remained to be explored. The first expedition to measure the vertical extent of the ocean and one of the last to map its southern boundaries was led by **James Clark Ross** *(1800–1862, England) during the years 1839 to 1843.*

A few years before the Ross expedition, the Beagle *(1831 to 1836), with Charles Darwin aboard, made its famous voyage in which so much new knowledge of the "natural history" of the ocean islands was obtained. Darwin also looked into the geologic structure and possible origins of ocean islands. This voyage ushered in the succession of cruises devoted to scientific study of the natural history of the seas which culminated in the efforts of* ***Charles Wyville Thomson*** *in* Lightning *(1868),* Porcupine *(1869 to 1870), and finally in the* Challenger *expedition of 1872 to 1876.*

The American interest in the practical aspects of oceanography were advanced by the efforts of **Matthew Fontaine Maury** *(1806–1873). Having been injured early in his naval career, Maury was placed in charge of the depot of naval charts and instruments. There he found a collection of logbooks of ships' officers containing a wealth of information about currents and weather at sea. Maury proceeded to analyze these data systematically and from them prepared charts of winds and currents which proved to be extremely useful.*

In order to obtain even better data, Maury was instrumental in arranging for the first international oceanographic conference. At the Brussels Maritime Conference of 1853, *uniform methods of making nautical and meteorological observations at sea were agreed upon. These increased the available data and made them more reliable. In* 1855 *Maury published* The Physical Geography of the Sea, *a summary of his findings.*

The chemistry of seawater was investigated by **Johan Georg Forchhammer** *(1794–1865) of Copenhagen, a professor of geology. Over a period of* 20 *years, Forchhammer analyzed surface samples of seawater brought to him by sailors from all over the globe. When he published his findings in* 1865, *he demonstrated that while the total salt content of seawater differs from place to place, the relative amounts of the various major salts remain constant. These findings together with those of the* Challenger *expedition (1872–1876), provided the nucleus of present understanding of the architecture of the oceans. Study of the physical mechanisms that bring these features into existence has been the main concern of physical oceanography ever since*[447].

[447] Since 1953, oceanographic research has been revolutionized due to a number of technological advances:

- "*Real-time*" *recording*: Prior to 1953, most instruments were mechanical devices. A program of sampling was laid out before the ship left port, technicians collected the data, and scientists ashore analyzed it months afterwards. Today, when bathimetry can be surveyed instantly in real time, the course of the ship can be modified instantly to accommodate, say, a newly discovered undersea mountain. Moreover, oceanographers had measured the temperature and salinity of sea water at widely spaced stations by lowering instruments on vertical wires suspended from ships. Now, they trail electronic temperature sensors behind the ship, so they could detect the boundaries between ocean currents as they crossed them.

- *Expansion of instrument capabilities*: With solid-state electronics instruments which are not so sensitive to the environment, oceanographers could conduct many kinds of research that were previously impossible.

- *On-board computers* are used to process many kinds of marine data that formerly could be analyzed only after return to port. Minute by minute and

The last volume of the Challenger Report appeared in 1895 — *three years after* ***Robert E. Peary*** *discovered that Greenland was an Island, not part of a polar continent; one year before a Norwegian explorer,* ***Fridtjof Nansen****, proved that no such continent existed;* 14 *years before Peary became the first man to reach the North Pole;* 17 *years before* ***Roald Amundsen****, another Norwegian explorer, reached the South Pole.*

Although the Challenger expedition is one of the great achievements in the history of scientific exploration, it was nonetheless a candle in a vast darkness. A quarter of a century later, oceanographers had gained a general picture of the earth's oceans; and another quarter of a century after that, oceanography entered upon a new era with the Meteor expedition of 1925–1927.

During the half-century from the Challenger to the Meteor, oceanographic research had consisted mostly of isolated and widely scattered observations. The emphasis was on the amount of territory covered rather than on systematic research. The most notable exceptions to this rule were the Norwegian studies of the North Atlantic and the Norwegian Sea that begun shortly after the turn of the century.

In the field of physical oceanography proper, major contributions came from the school of thought stimulated by ***V.F.K. Bjerkens****. In* 1898 *Bjerkens published a paper which provided a basis for determining the field of motion in the sea from measurements of the vertical and horizontal distributions of pressure. Currents and volumes of water moving in the oceans are difficult to measure directly because there are no convenient reference marks at sea that are assuredly at rest. The practical methods for applying dynamical principles*

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hour by hour the ships collect data and feed them into the computer. It can print out the ships' position, its speed and much other information, as one desires.

- *SCUBA gear and research submarines*. Marine geologists and biologists can make dives to personally examine at close range the phenomena about which they had previously speculated. Deep water submersibles with remote handling (for manipulating the environment outside the submersibles) are opening new lines of research, such as detailed mapping of the Mid-Atlantic rift.

- *Special types of research vessels* can drill in the deep-sea floor, or be partially flooded so it stands on end for the measurement of the motion of the sea.

- *Artificial satellites*, coupled to the ship's on-board computer, provide a navigational system. The sensors on these satellites are capable of measuring the temperature of the ocean surface, thereby mapping ocean currents almost instantly over enormous areas. Other uses of space technology will surely further revolutionize some oceanography and the utilization of the seas.

at sea were developed during the first quarter of the 20^{th} century by ***Björn Helland-Hansen***, ***J.W. Sandström***, *and several other Norwegian, Swedish, and German oceanographers.*

II Physical Foundations of Dynamic Oceanography (1813–1969)

The important forces which drive the large-scale oceanic motions are the force of gravity ($\boldsymbol{g}$), the Coriolis force, pressure gradient force, and frictional forces ($\boldsymbol{F}$). (The centrifugal force of the earth's rotation is usually included in gravity.) Thus, the Eulerian vector equation for the acceleration of a fluid element relative to a co-rotating frame, is:

$$\frac{d\boldsymbol{V}}{dt} = \frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla \boldsymbol{V}) = (-2\Omega \boldsymbol{n} \times \boldsymbol{V}) + (-\alpha \nabla p) + \boldsymbol{g} + \boldsymbol{F}, \tag{1}$$

where $\boldsymbol{V} = (u, v, w)$ *is the velocity vector with components along x (east), y (north) and z (upward). Here* $\Omega = 0.729 \times 10^{-4}$ sec^{-1} *is the angular velocity of the earth's daily rotation, $\boldsymbol{n}$ is a unit vector pointing from south to north pole, p is the pressure* $(\mathrm{dyn/cm^2})$, $\boldsymbol{F}$ *is force per unit mass, and* $\alpha = \frac{1}{\rho} =$ *specific volume in units* $\mathrm{cm^3/gm}$. *Under conditions of* steady state $\left(\frac{\partial \mathbf{V}}{\partial t} = 0\right)$, *small advective terms* $(\boldsymbol{V} \cdot \nabla \boldsymbol{V} \doteq 0)$, *and small frictional force* $(\boldsymbol{F} = 0)$, *the motions are said to be* geostrophic *(earth-turned). In this case the Coriolis force and the pressure gradient force just balance each other. Consequently,*

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = (2\Omega \sin\phi)v, \qquad \frac{1}{\rho}\frac{\partial \rho}{\partial y} = (-2\Omega \sin\phi)u, \quad \frac{1}{\rho}\frac{\partial p}{\partial z} = g. \tag{2}$$

The above equations govern the motion of major ocean currents. These are of two general types: vertical and horizontal (both surface and deep). Lesser and variable currents may be caused by tides and storm conditions. Some currents are of short duration and cover only a small area; others, such as the great oceanic circulation systems, *are permanent.*

A few remarks are needed with regard to Eqs. (1) and (2):

- *In the gravitational term,* $\boldsymbol{g} = (0, 0, -g)$ *represents the* apparent *gravitational acceleration, or the true (central) gravitational acceleration modified by the small 'centrifugal' contribution normal to the axis of the earth's rotation. The direction of $\boldsymbol{g}$ defines the local vertical; its magnitude varies throughout the ocean from its mean value of approximately 981 $\frac{cm}{sec^2}$ by less than 0.3%, and for dynamical purposes it can be considered constant.*

- *The term $\boldsymbol{F}$ represents the resultant of all other forces acting on a unit volume of the fluid. The most important of these arises from molecular viscosity. In almost all oceanic circumstances where viscous effects are important, the water can be regarded as an isotropic, incompressible Newtonian fluid for which the stress tensor is given by*

$$\boldsymbol{T} = -pI + \mu(\nabla \boldsymbol{V} + \boldsymbol{V}\nabla), \tag{3}$$

and where μ is the fluid's viscosity. The frictional force per unit volume in therefore (with $\operatorname{div} \boldsymbol{V} = 0$)

$$\boldsymbol{F} = \mu \nabla^2 \boldsymbol{V} \tag{4}$$

If L is the differential length scale of a given motion in which the velocity varies in magnitude by U, the ratio $R = \frac{pUL}{\mu}$ (known as the Reynolds number) represents the relative magnitudes of the inertial and viscous terms in the momentum equation. In many oceanic motions, the Reynolds number is very large, and the viscous form is often quite negligible over most of the field of motion.

- *Using the vector identity $\boldsymbol{V} \cdot \nabla \boldsymbol{V} = \nabla(\frac{1}{2}\boldsymbol{V}^2) + (\boldsymbol{\omega} \times \boldsymbol{V})$, where $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{V}$, Eq. (1) becomes*

$$\frac{\partial \boldsymbol{V}}{\partial t} + 2(\boldsymbol{\Omega} + \boldsymbol{\omega}) \times \boldsymbol{V} + \frac{1}{\rho}\nabla p + \nabla\left(\frac{1}{2}\boldsymbol{V}^2\right) - \boldsymbol{g} = \boldsymbol{F}. \tag{5}$$

When ρ is in effect constant, (5) goes into

$$\rho \frac{\partial \boldsymbol{V}}{\partial t} = \rho \boldsymbol{g} + \mu \nabla^2 \boldsymbol{V} - \rho(\boldsymbol{\Omega} + \boldsymbol{\omega}) \times \boldsymbol{V} - \nabla(p + \frac{1}{2}\rho \boldsymbol{V}^2). \tag{6}$$

The Eulerian mass-acceleration is thus given as a balance between four forces. The term $\rho(\boldsymbol{\Omega} + \boldsymbol{\omega}) \times \boldsymbol{V}$ can be called total vortex force.

- *Sea water is a chemical solution; its density $\rho = \rho(p, T, S)$ where p = pressure, T = temperature and S = salinity (the mass of dissolved solids per unit mass of sea water). This dependence has no analytical form, but various empirical approximations. However, for an ordinary range of temperature and salinity encountered in the ocean, the equation of state is approximately given by*

$$\rho = 1 - aT - bS,$$

where a, b are numerical constants.

- *If in (2) we retain the Coriolis term we shall have*

$$\frac{\partial p}{\partial z} = g\rho(1 - \frac{2u\Omega}{g}\cos\theta)$$

With fast currents near the equator (e.g. zonal undercurrents of the Pacific and Atlantic Oceans flowing from west to east), $\theta \approx 0°$, $u \sim 150$ cm/sec, and therefore $\frac{2u\Omega}{g}\cos\theta \approx 2.3 \times 10^{-5}$, which may be significant.

- *In moving fluids where the velocity varies in space, frictional stresses are present as a result of momentum transfer between layers of different velocities. In the case of* laminar flow *the exchange of momentum is the result of molecular motion. However, if the fluid is stirred by some internal or external cause and individual layers are 'entangled' by macroscopic irregular displacements of water parcels, the rate of momentum exchange (as well as heat exchange, diffusion of dissolved solids, etc.) increase considerably. This is called* turbulent flow. *Turbulence, stirring, mixing and diffusion play a very important role in both oceanography and meteorology.*

The mechanical energy equation

Forming the scalar product of $\boldsymbol{V}$ *with the respective terms of (5) yields*

$$\rho\frac{\partial}{\partial t}\left(\frac{1}{2}\boldsymbol{V}^2\right) + \boldsymbol{V}\cdot\nabla p - \rho\boldsymbol{V}\cdot\boldsymbol{g} = \boldsymbol{V}\cdot\boldsymbol{F}. \tag{7}$$

Now, if ξ measures the vertical displacement *of a fluid element (measured upwards), then*

$$-\rho\boldsymbol{V}\cdot\boldsymbol{g} = \rho g w = \rho g\frac{d\xi}{dt} \tag{8}$$

and with use of the continuity equation $\frac{d\rho}{dt} + \rho\,\mathrm{div}\,\boldsymbol{V} = 0$, *we can express (8) as*

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho\boldsymbol{V}^2 + \rho g\xi\right] + \mathrm{div}\left[\boldsymbol{V}\left(p + \frac{1}{2}\rho\boldsymbol{V}^2 + \rho g\xi\right)\right] = p\,\mathrm{div}\,\boldsymbol{V} + \boldsymbol{V}\cdot\boldsymbol{F}. \tag{9}$$

Since

$$\boldsymbol{\Sigma} = \boldsymbol{V}\left(p + \frac{1}{2}\rho\boldsymbol{V}^2 + \rho g\xi\right) \tag{10}$$

is the energy flux density *vector, (9) is just a statement that the rate of change of the Hamiltonian*

$$H = \frac{1}{2}\rho\boldsymbol{V}^2 + \rho g\xi \tag{11}$$

and the divergence of the energy flux density equal the rate of working in compressing the fluid and against frictional forces. If the fluid is incompressible ($\operatorname{div}\boldsymbol{V} = 0$) *and inviscid* ($\boldsymbol{F} = 0$), *the energy balance is* $\frac{\partial H}{\partial t} + \operatorname{div}\boldsymbol{\Sigma} = 0$. *Coriolis forces do no work, since their direction is always normal to the velocity* $\boldsymbol{V}$. *They can, however, influence the energy flux* indirectly *by contributing to the pressure variation in the fluid.*

In an incompressible Newtonian fluid $\boldsymbol{F} = \mu\nabla^2\boldsymbol{V}$, *so that the rate of working against viscous forces is*

$$\boldsymbol{V}\cdot\boldsymbol{F} = 2\mu\boldsymbol{V}\cdot\operatorname{div}\boldsymbol{E} = 2\mu\operatorname{div}(\boldsymbol{V}\cdot\boldsymbol{E}) - \epsilon, \tag{12}$$

where $\epsilon = 2\mu(\epsilon_{ii})^2$. *While* $2\mu\operatorname{div}(\boldsymbol{V}\cdot\boldsymbol{E})$ *can be interpreted as a* viscous energy flux, *the quantity* ϵ *(essentially* positive*) represents the rate of energy dissipation per unit volume by molecular viscosity.*

Summary

A complete set of basic equations of dynamic oceanography is the following ($\nu = \frac{\mu}{\rho}$)

$$\frac{D\boldsymbol{V}}{Dt} + 2(\boldsymbol{\Omega}\times\boldsymbol{V}) = \frac{1}{\rho}\nabla p - \nabla\Psi - \beta\nabla\boldsymbol{g} + \nu\nabla^2\boldsymbol{V} \quad \textit{conservation of momentum} \tag{13}$$

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\boldsymbol{V}) = 0 \quad \textit{conservation of mass} \tag{14}$$

$$\rho = f(p, T, S) \quad \textit{equation of state} \tag{15}$$

$$\frac{\partial(\rho T)}{\partial t} + \operatorname{div}(\rho T\boldsymbol{V}) = -\operatorname{div}\boldsymbol{\Sigma}_T + q_T \quad \textit{conservation of heat} \tag{16}$$

$$\frac{\partial(\rho S)}{\partial t} + \operatorname{div}(\rho S\boldsymbol{V}) = -\operatorname{div}\boldsymbol{\Sigma}_S + q_S \quad \textit{conservation of salt} \tag{17}$$

These are 7 scalar equations in the 7 unknown scalar functions $(\boldsymbol{V}, p, \rho, T, S)$, *where* Ψ *is the gravitational potential,* T *is the temperature,* S *is the salinity,* $\boldsymbol{\Sigma}_T$ *is the flux of* T *due to heat conduction and diffusion, and* $\boldsymbol{\Sigma}_S$ *is the corresponding salinity flux. Note that since* $\operatorname{div}(\rho T\boldsymbol{V}) = \rho T\operatorname{div}\boldsymbol{V} + \boldsymbol{V}\cdot\nabla(\rho T)$, *the advective term replaces the divergence term for incompressible ocean. If Fick's law holds,* $-\operatorname{div}\boldsymbol{\Sigma}_T = \kappa_T\nabla^2 T$, $-\operatorname{div}\boldsymbol{\Sigma}_S = \kappa_S\nabla^2 S$, *and free* convection *ensues.*

Note that although the law of conservation of the total energy *is not included explicitly, it is* implicit *in the flux vectors* $\boldsymbol{\Sigma}_T$ *and* $\boldsymbol{\Sigma}_S$. *Also* $\beta = -\frac{1}{p}\frac{\partial\rho}{\partial T}$

is the thermal expansion coefficient of the fluid. If $\rho = \rho(p)$ only, the first 5 equations constitute a complete set for the functions $(\boldsymbol{V}, \rho, p)$. In the first equation, incompressibility was assumed for the frictional forces. In Eqs. (16)-(17), q is the total internal source of heat and salinity respectively.

It is clear from (13)-(17) that the equations of motion are quadratically nonlinear, i.e., contain products of dynamic variables. This implies that in principle it is not possible to simply superpose solutions of the equations. In physical terms, motions on one spatial scale interact with motions on other scales. There is therefore an a priori possibility that small-scale motions may influence the large-scale motions. There is in fact evidence that the small-scale motions, which appear sporadic on larger time scales, act to smooth and mix properties on the larger scales by processes analogous to molecular, diffusive transports. This will become apparent in the next section when the Reynolds stress is introduced.

Nondimensional parameters

Let us return to Eq. (9) and associate with each term a characteristic dimension by means of the correspondence

Fluid velocity	$\boldsymbol{V}$	$\rightarrow$	U
Scale-time	t	$\rightarrow$	T
Coriolis parameter	$\boldsymbol{\Omega}$	$\rightarrow$	$f = 2\Omega \sin\theta$
Frictional force	$\nu\nabla^2\boldsymbol{V}$	$\rightarrow$	$\frac{\nu}{L^2}U \quad (\boldsymbol{\nabla} \rightarrow \frac{1}{L})$
Gravitational potential	Ψ	$\rightarrow$	$gH,$

where L is some lateral characteristic length, H is a vertical characteristic length, T is a characteristic time and U is a characteristic speed. Since each term in equations (13)-(17) is of the same dimension, the relative influence of terms will be determined by dimensionless numbers. There are seven such numbers, each named after the person who first stressed its importance in some fluid system:

$$\begin{aligned}
F_r &= \textbf{Froude number} &&= \frac{\text{Inertial force}}{\text{Gravitational force}} &&= \frac{U^2}{gH} \\
R_0 &= \textbf{Rossby number} &&= \frac{\text{Inertial force}}{\text{Coriolis force}} &&= \frac{U}{fL} \\
R_e &= \textbf{Reynolds number} &&= \frac{\text{Inertial force}}{\text{Frictional force}} &&= \frac{\rho U L}{\mu} \\
E_k &= \textbf{Ekman number} &&= \frac{\text{Coriolis force}}{\text{Frictional force}} &&= L\sqrt{\frac{fp}{\mu_\nu}}
\end{aligned} \tag{18}$$

$$
\begin{aligned}
E_u &= \textbf{\textit{Euler}}\ \textit{number} &&= \frac{\textit{Inertial force}}{\textit{Pressure gradient force}} &&= \frac{\rho U^2}{|\nabla p|} \\
P_r &= \textbf{\textit{Prandtl}}\ \textit{number} && &&= \frac{\mu}{\rho \kappa_T} \\
G_R &= \textbf{\textit{Grashof}}\ \textit{number} && &&= \rho^2 \beta g L^3 \frac{T_1 - T_0}{\mu^2}
\end{aligned}
$$

The relative magnitudes of the various forces and accelerations can be compared among each other by dimensional analysis. Ocean currents are slow enough that pressure can be approximated with high accuracy by the hydrostatic equation. Consequently, pressure gradients are produced primarily by slopes of the sea surface and variations of density within the interior of the ocean. Furthermore, the forces due to horizontal gradients of pressure are balanced primarily by the Coriolis forces. The remaining forces due to accelerations and friction are generally smaller than the pressure gradient and Coriolis forces, but can become important in some regions of the ocean.

By evaluating the coefficients R_0, R_e, F_r for a given flow, we obtain an appreciation of the magnitudes of the forces involved and can decide which force needs to be taken into account for a satisfactory interpretation of the flow in terms of the equations. Conversely, we can examine the coefficients to determine the horizontal and vertical scales for which any given term of the equations becomes comparable to unity. For example, the ratio of the nonlinear accelerations to the Coriolis forces is given by the Rossby number, R_0. If we use the observation that steady velocities in the ocean do not exceed 2 or 3 m/sec, we can make the Rossby number comparable to unity only by assuming the current to be sharply confined horizontally (L small), or by assuming a current near the equator (f small). At mid-latitudes, the Rossby number approaches unity for horizontal scales of 20 to 30 km for the maximum velocities given above. Such conditions are approached only in concentrated current systems such as the Gulf Stream and Kuroshio. Another way to interpret the Rossby number is to note that the ratio U/L is the characteristic magnitude of the vertical component of relative vorticity, $\partial U_u/\partial x_1 - \partial U_1/\partial x_2$. Thus, the Rossby number can be considered as the ratio of the relative vorticity to the Coriolis parameter. Hence, if the relative vorticity approaches the Coriolis parameter, the Rossby number will be near unity and nonlinear accelerations will be of the same magnitude as the Coriolis forces.

Near the turn of the century two other major steps were taken in the formulation of the modern point of view. Maury's observation of the close relationship between surface winds and ocean surface currents was given a physical explanation by ***Vagn Walfrid Ekman*** *(1874–1954, Sweden). During* 1905–1923 *he determined the ocean's theoretical response both to a steady wind and to an impulsive horizontally uniform wind, examining particularly*

the influence of the Coriolis force on the dynamical behavior of the ice and the upper layers of the ocean.

Ekman showed not only that there should be a spiral effect (waters moving slower and further to the right with depth), but also that the wind-driven currents should extend only to a depth of about 200 *meters.*

Near the sea-bed, as the flow becomes influenced by friction *with the ocean bottom, the spiraling is such that the direction of flow moves leftward with depth (1905). He later (1923) extended his results to non-uniform winds, variable ocean-bottom topography, and variable latitude.*

It was found that major ocean currents *are basically produced by two factors: distribution of density and effect of wind stress on the sea surface. Variations in density are widely distributed due to* differential heating *and* evaporation*. They cause the waters to move both horizontally and vertically.*

Major ocean currents are set in motion by these variations and by the drag of the wind on the surface layers. Once the water mass begins to move, it is deflected due to the Coriolis force*, and the major surface current circulation is established. Upwelling and sinking of water masses are similarly caused by density differences and the blowing away or piling up of waters under the stress of* air currents*. Density gradients so light that they are difficult to measure, may nevertheless be sufficient to produce or to maintain ocean currents.*

The Coriolis effect, due to the earth's rotation, causes surface currents in the Northern Hemisphere to deflect 45° *to the right of the wind direction, and in the Southern Hemisphere,* 45° *to the left. At the depth at which the current speed is* $\frac{1}{23}$ *of the surface speed, the current moves exactly opposite to the direction at the surface, a phenomenon called the* Ekman spiral*.*

The surface circulation of the ocean is a direct result of the circulation of the atmosphere, where the pattern of gyres results from the winds and the geography of the continents. While the oceans are separated in the North, the free passage around Antarctica permits a great current to flow from west to east around the globe.

The rate of transport of water by the major ocean currents are (in 10^6 $\mathrm{m}^3\mathrm{sec}^{-1}$*): Antarctic current (200); Gulf Stream — Florida Straits (25); Gulf Stream — Cape Hatteras (100); Kuroshio (50); North Pacific to Arctic Ocean (0.7); All the world's rivers (1); Flux of atmospheric water vapor across a parallel of latitude (0.7).*

Thus, the ocean currents transport a tremendous amount of water and keep the surface water of the sea relatively well mixed. In closed seas, such as the Mediterranean Sea and the Red Sea, an arid atmosphere causes great evaporation: surface water flows in to replace the loss. In humid climate conditions, the system is reversed due to heavy rainfall.

In the Atlantic, deep water circulation takes place: below Greenland, the warm, highly saline waters of the Gulf Stream are cooled, the density increases, and the water sinks to intermediate level where it is displaced to the south. In the tropical regions of the North Equatorial Current, the surface waters are in turn heated, evaporated, and displaced by wind stress. Here the deep waters rise to the surface, completing the cycle[448].

1872–1873 CE **Elias Ney** (1844–1897, England). Explorer of Asia. Led eight major expeditions to Central Asia, all of them hazardous. His most famous journey begun in September 1872, when with a Chinese servant, a camel driver, an interpreter, six camels and two ponies he set out to cross the Gobi Desert from a location NW of Peking (114 °E; 42 °N). Traveling NW across Mongolia, he reached Uliastay, and then crossed the frozen Lake Haar Us Nuur to reach Hovd. From there he traversed the Altai range to Biysk in Siberia, on the upper waters of the Ob. Finally a horse-drawn sleigh ride in the depth of the Siberian winter took him to Nijni-Novgorod. He traveled altogether some 8000 km.

Ney was born in 1844, the son of Jewish parents. In his time he ranked with Stanley (both received the Royal Geographical Society's Founders Medal for outstanding work in 1873), but is now almost forgotten.

1872–1884 CE **Georg (Ferdinand Ludwig Philipp) Cantor**[449] (1845–1918, Germany). A great and revolutionary mathematician, whose work on set theory and the theory of the infinite created a whole new field of mathematical research and exerted profound influence on most branches of contemporary mathematics — especially the foundation of mathematics and mathematical logic. In his papers he created an arithmetic of transfinite numbers,

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[448] Movement of the deeper water masses was unknown until deep-current meters, buoys, and radioactive trace-element detectors came into use. In 1952, the *Cromwell Current* in the equatorial Pacific was discovered by scientists from the Scripps Institute of Oceanography. This great current, about 400 kilometers wide, sweeps thousands of kilometers in an eastward direction at a speed of 6.5 km/hr.

[449] For further reading, see:

- Dauben, J.W., *George Cantor*: *His Mathematics and Philosophy of the Infinite*, Harvard University Press: Cambridge, 1979, 404 pp.

analogous to the arithmetic of finite numbers, eliminating all metaphysical elements from the foundations of the exact sciences. Historians of mathematics have ranked his work as "one of the most original contributions to mathematics in the past 2500 years".

Cantor's early interests were in number theory, indeterminate equations and trigonometric series. The subtle theory of trigonometric series seems to have inspired him to look into the foundations of analysis.

Ever since the days of Zeno, men had been talking about infinity, in theology as well as mathematics, but no one before 1872 had been able to tell precisely what he was talking about. **Cauchy** and **Weierstrass** saw only paradoxes in their attempts to identify infinity in mathematics. In fact, there was a considerable '*horror infinite*' and mathematicians were reluctant to accept '*completed infinity*'. Cantor created a theory of the actual infinite which by its apparent consistency demolished the Aristotelian and scholastic "proofs" that no such theory could be found.

Some of Cantor's ideas, simply stated, are as follows:

(1) A transfinite number, unlike a finite number, can *always* be put into 1–1 (one to one) correspondence with some *part* of itself [e.g.: set of all integers with the set of all even integers].

(2) Although the set of rational numbers is *dense* (i.e. between any two rational numbers one can 'pack in' an infinity of other rational numbers), the set of all rationals can be rearranged in such a way that they can be put into 1–1 correspondence with the set of all integers, which is a *discrete set*. [A denumerable infinity is designated by the Hebrew letter $\aleph_0$. This is the "power" (cardinality) of the set of positive integers and also the "power" of the positive rationals.]

(3) Any linear continuum, no matter of what length, can be put in 1–1 correspondence with the line-segment from 0 to 1, i.e. the set of real numbers between 0 and 1 is equivalent to the set of all real numbers[450].

(4) The set of real numbers between 0 and 1 (known as a *continuum*) is not countable, in the sense that it cannot be put in a 1–1 correspondence with the aggregate of natural numbers. Its power equals to the power of the set of *all* real numbers, and is denoted by C.

(5) A 1–1 correspondence can be set up between points on a one-dimensional continuum and any finite-dimensional continuum, i.e. there are no more points in a square or cube than in a line segment $[C^n = C]$.

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[450] The mapping $z' = [1 + e^{-z}]^{-1}$ maps $(-\infty, \infty)$ onto $(0, 1)$; the inverse transformation is $z = \log_e \frac{z'}{1-z'}$.

(6) Since any finite set of m elements has 2^m subsets, one denotes, for any set A of power p, the power of the *power set* of A (set of all subsets) by 2^p. It can be shown that $2^{\aleph_0} = C$, and in fact $N^{\aleph_0} = C$, when $N \geq 2$ is any finite integer.

(7) As in ordinary arithmetic, numbers are of two kinds: *cardinal* and *ordinal* [e.g. cardinal numbers are $1, 2, 3, 4, \ldots$; ordinal numbers are $1^{st}, 2^{nd}, 3^{rd}, 4^{th}, \ldots$]. So, in the arithmetic of transfinites as well, $\aleph_0$ and C are *cardinal transfinite numbers*. Cantor conjectured that there is no cardinal number greater than $\aleph_0$ and smaller than C. This is known as the *continuum hypothesis*.

Cantor's ideas threw new light on the concept of *dimension*. This concept presents no great difficulty as long as one deals only with simple geometrical figures such as points, areas, lines, triangles, and polyhedra. A single point or any *finite* set of points has dimension zero, a line segment is one-dimensional, and the surface of a triangle or of a sphere is two-dimensional.

The set of points in a solid cube is three-dimensional. But when one attempts to extend this concept to more general point sets, the need for a precise definition arises. What dimension should be assigned to the point set R consisting of all points on the x-axis whose coordinates are *rational* numbers? The set of rational points is dense on the line and might therefore be considered to be one-dimensional, like the line itself. On the other hand, there are irrational gaps between any pair of rational points, as between any two points of a finite point set, so that the dimension of the set R might also be considered to be zero.

The problem becomes even more complex as one tries to assign a dimension to the following curious point-set, first considered by Cantor and known as the *Cantor set* C.

From the unit segment remove the middle third, consisting of all points x such that $1/3 < x < 2/3$. Call the remaining set of points C_1. Now from C_1 remove the middle third of each of its two segments, leaving a set which we call C_2. Repeat this process by removing the middle third of each of the four intervals of C_2, leaving a set C_3, and proceed in this manner to form sets C_4, C_5, $C_6, \ldots$. Denote by C the set of points on the unit segment that are left after all these intervals have been removed, i.e. C is the set of points common to the infinite sequence of sets C_1, $C_2, \ldots$. Since one interval, of length $1/3$, was removed at the first step; two intervals, each of length $1/3^2$, at the second step; etc.; the total length of the segments removed is

$$1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3^2} + 2^2 \cdot \frac{1}{3^3} + \cdots = \frac{1}{3}\Big[1 + \big(\frac{2}{3}\big) + \big(\frac{2}{3}\big)^2 + \ldots\Big].$$

The infinite series in parentheses is a geometrical series whose sum is $1/(1-2/3)=3$; hence the total length of the segments removed is 1. Still there remain points in the set C. Such, for example, are the points 1/3, 2/3, 1/9, 2/9, 7/9, 8/9, …, by which the successive segments are trisected. As a matter of fact it is easy to show that C will consist precisely of all those points x whose base 3 decimal expansions can be written in the form

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots + \frac{a_n}{3^n} + \ldots,$$

where each a_i is either 0 or 2, while the triadic expansion of every point removed will have at least one of the numbers a_i equal to 1.

What shall be the dimension of the set C? The diagonal process used to prove the non-denumerability of the set of all real numbers can be so modified as to yield the same result for the set C. It would seem, therefore, that the set C should be one-dimensional. Yet C contains no complete interval, no matter how small, so that C might also be thought of as zero-dimensional, like a finite set of points. In the same spirit, we might ask whether the set of points of the *plane*, obtained by erecting at each rational point or at each point of the Cantor set C a segment of unit length, should be considered to be one-dimensional or two-dimensional[451].

After Cantor, mathematicians based set theory on abstract postulate systems. One such axiomatization, for example, is due to **Ernst Zermelo** (1871–1956, Germany, 1922) and **Abraham Halevi Fraenkel** (1891–1965, Israel, 1927). Then, in 1938, **Kurt Gödel** demonstrated that one can safely assume Cantor's 'continuum hypothesis' as an *additional postulate* in set theory, i.e. he proved that the continuum hypothesis is consistent with the Zermelo-Fraenkel

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[451] An inductive definition of dimensionality is also contained implicitly in **Euclid**'s *Elements*, where a one-dimensional figure is something whose boundary consists of points, a two-dimensional figure one whose boundary consists of curves, and a three-dimensional figure one whose boundary consists of surfaces.

Poincaré (1912) first called attention to the need for a deeper analysis and a precise definition of the concept of dimensionality. Poincaré observed that the line is one-dimensional because we may separate any two points on it by cutting it at a single point (which is of dimension 0), while the plane is two-dimensional because in order to separate a pair of points in the plane we must cut out a whole closed curve (of dimension 1). This suggests the inductive nature of dimensionality: a space is n-dimensional if any two points may be separated by removing an $(n-1)$-dimensional subset, and if a lower-dimensional subset will not always suffice. The introduction of *fractal geometry* by **Mandelbrot** (1977) finally made the theory applicable to many physical problems.

axioms. The last word however, had not been said, since Gödel had neither proven the continuum hypothesis[452] nor shown that it is indemonstrable.

In 1963 **Paul Joseph Cohen** (b. 1934, U.S.A.) has shown that Cantor's hypothesis is independent of the other axioms of set theory. Hence, the continuum hypothesis can be assumed or denied depending on the applications one has in mind, i.e. there are at least two types of mathematics possible — one that holds that the continuum hypothesis is true, and another in which it is false.

Cantor was born in St. Petersburg, Russia, of pure Jewish descent on both sides [though his father converted to Protestantism and his mother had been born a Catholic]. In 1856 he moved with his parents to Frankfurt, Germany. He rejected his father's suggestion of preparing for a career in engineering in favor of concentrating on philosophy, physics and mathematics. He studied at Zürich, Göttingen and Berlin, where he came under the influence of **Kummer** and **Weierstrass** and took his Ph.D. degree in 1867.

In 1874 he published his path-breaking paper on the theory of infinite sets. In the same year Cantor married Vally Guttman; six children were born of the marriage. The following 10 years were the period of his most original productivity. All his active professional career was spent at the University of Halle, a distinctly third-rate institution, where he was appointed full professor in 1879. He never achieved his ambition of professorship in Berlin, which at that time was the highest German distinction. It was **Kronecker** who blocked his appointment in Berlin[453] and was instrumental in rejecting the publication of Cantor's papers in Crelle's Journal. In fact, Kronecker regarded Cantor's ideas as a dangerous type of mathematical insanity, and attacked the hypersensitive author vigorously and viciously with every weapon that came to his hand. The tragic outcome was that Kronecker's attack broke the creator of the theory, who died in a mental hospital in Halle.

David Hilbert (1862–1943), himself one of the greatest mathematicians of recent times, considered Cantor's achievement to be: "*the most wonderful flowering of the spirit of mathematics and indeed one of the greatest achievements of human reason*" (1926).

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[452] The *continuum hypothesis* problem was the first of **Hilbert**'s famous 23 problems delivered to the Second International Congress of Mathematicians in Paris (1900).

[453] Kronecker's vicious animosity toward Cantor was basically very personal. It was motivated by a combination of jealousy and fear, disguised under the hypocritical cover of an academic controversy and enhanced by the fact that both belonged to an intellectual elité of an assimilated convert minority.

Infinity, Transfinity and Set Theory[454]

The idea of infinity has been the subject of deep thought from the time of the Greeks. **Zeno of Elea** *(ca 450 BCE), with his famous 'paradoxes', made an early major contribution. Other thinkers who had adduced ideas on the concept of infinity include* **Aristotle**, **Descartes**, **Berkeley** and **Leibniz**. **Albert of Saxony** *in his Questiones subtilissime in libros de celo et mundi (1365) proves that a beam of infinite length has the same volume as 3-space. He proves it by sawing the beam into imaginary pieces which he then assembles into successive concentric shells which fill space. Thus, by the Middle Ages, discussion of the infinite had led to comparisons among infinite sets of objects.* **Nicolas of Cusa** *(1440) studied the infinitely large and the infinitely small and had an intuitive feel for the procedure of the limit.*

By the beginning of the 19th century a clear distinction had been established between analysis (the study of infinite processes) and algebra, which deals with operations on discrete entities such as the natural numbers and polynomials. A major objective of much of the 19th century mathematical effort was to unify — or, at any rate, to build bridges between — these two branches of mathematics. This endeavor was termed 'the arithmetization of analysis'. It was realized that the prime task was to construct a sound logical foundation of the real number system. Although the basic concepts of analysis — function, continuity, limit, convergence, infinity and so on — were progressively clarified and refined during the first half of the 19th century, mathematicians failed to consider the precise structure and properties of the real numbers. Even **Cauchy** *lacked the understanding of the structure of the number system. Indeed — the theory of the arithmetic continuum was needed.*

A step in the direction of an improved understanding of irrational numbers was the mid-19th century work on algebraic and transcendental numbers[455].

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454 For further reading, see:

- Aczel, A.D., *The Mystery of the Aleph*, Washington Square Press, 2001, 258 pp.
- Kaplan, R. and E. Kaplan, *The Art of the Infinite*, Oxford University Press, 2003, 324 pp.
- Lieber, L.R., *Infinity*, Reinhart and Company, 1953, 359 pp.

455 Real numbers can be divided into *algebraic* and *transcendental* numbers. An algebraic number is defined as a number which is a root of a polynomial equation

The interest in this distinction was heightened by the 19th-century work on the solution of equations, because this work revealed that not all irrationals could be obtained by finite sequences of algebraic operators on rational numbers[456]. *This grew out of the question of whether there are indeed any transcendental numbers at all. In 1844,* ***Liouville*** *answered the question in the affirmative by actually constructing such numbers. He proved, e.g. that all numbers of the form*

$$\frac{a_1}{10^{1!}} + \frac{a_2}{10^{2!}} + \frac{a_3}{10^{3!}} + \cdots = 0.a_1a_2000a_3\ldots$$

where a_i are arbitrary integers in the range 0–9, are nonalgebraic, and therefore transcendental.

B. Bolzano *considered sets with the following definition:*

"An embodiment of the idea or concept which we can conceive when we regard the arrangement of its parts as a matter of indifference".

Bolzano defended the concept of an infinite *set. At this time many believed that infinite sets could not exist. He gave examples to show that, unlike for finite sets, the elements of an infinite set could be put in one-to-one correspondence with elements of one of its proper sets. This idea eventually came to be used in the definition of a finite set. It was with* ***Cantor's*** *work, however, that set theory came to be put on a proper mathematical basis.*

In 1873 ***Hermite*** *proved that e is transcendental; and in 1882* ***Lindemann*** *did the same for π. Finally in 1934,* ***A. Gelfond*** *discovered that if a is an algebraic number (not equal to 0 or 1) and b is an algebraic irrational number, then a^b is transcendental (e.g. $3^{\sqrt{2}}$). However, the Mascheroni-Euler constant*

$$\gamma = \lim_{n\to\infty}(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n) = 0.577216\ldots$$

is not known to be rational, irrational, algebraic or transcendental.

Dedekind *(1858) realized that the system of real numbers lacked a firm logical foundation: the Greeks, with their geometrical predilections, identified real numbers (rational and irrational) with line segments.*

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of the form: $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$, where n and the a_i are integers. A transcendental number is, therefore, any irrational number that is not algebraic (a rational number is also algebraic, of course). It can be proved that even if a_i are any algebraic numbers (as opposed to integers or rationals), all roots of $\sum_{i=0}^{n} a_ix^{n-i}$ are, nevertheless, algebraic.

456 Nor could all algebraic numbers be expressed as finite successions of arithmetical operations upon integers, *even* when these operations include the radical operations $\sqrt[n]{}$, n integer.

They tacitly assumed that any such number could be represented by a unique point on an infinitely extended straight line with a specific origin. The Greeks, however, while accepting irrational geometric entities *(such as the diagonal of the unit square), could not accept the concept of* irrational numbers*, because it was less intuitive and required confrontation with the concept of* infinity*, whether by sequences, decimals, or continued fractions.*

Western mathematics seemed to follow this trend and until the 19th century this seemed to be a good reason for considering geometry *to be a better foundation continuum for mathematics than arithmetic. Then the problems of geometry came to a head with the advent of non-Euclidean geometry, and mathematicians began to fear geometric intuition as much as they had previously feared infinity.*

***Dedekind** then set out to construct irrational numbers from scratch using sets of* rationals*. The number $\sqrt{2}$ is determined by the two sets of positive rationals:*

$$L_{\sqrt{2}} = \text{ all rational numbers } r \text{ whose square } r^2 < 2$$

$$U_{\sqrt{2}} = \text{ all rational numbers } r \text{ whose square } r^2 > 2$$

He decided to identify *$\sqrt{2}$ with this pair of sets. In general, any partition of the positive rationals into sets L, U such that any member of L is less than any member of U* and *elements of L can get arbitrary close to elements of U, defines a positive real number. This idea (known as a* Dedekind cut*) gives a complete and uniform construction of all real numbers, or points on a line, using just rationals. It is an explanation of the* continuous *in terms of the* discrete*, finally resolving the fundamental conflict in Greek mathematics.*

The assumption that the points on a line can be put in one-to-one correspondence with the real numbers is now known as the Dedekind-Cantor axiom*. It must be realized that the logical definition of an irrational number is rather sophisticated — being not just a single symbol or a pair of symbols but an* infinite collection*.*

Finally, Dedekind's theory shows that the 'arithmetic continuum' of real numbers is closed *under infinite processes (Dedekind theorem): with real numbers we reach, as it were, the end of the road.*[457]

Although Dedekind retained the Greek geometrical model of the number system as an aid to thought and exposition, the aim of most 19th century

[457] However, it turned out there are *other* roads via which infinite sets of rationals may be used to define a continuum; these result in the so-called *p-adic number systems*, studied by **Hensel** and others.

mathematicians was to exclude geometrical considerations altogether, to base virtually the whole of mathematics on the concept of number. Broadly speaking, this objective had been achieved by the mid 20th century, although some foundational difficulties (e.g. with the logical basis of set theory) still remain. The next step in the erection of foundations for the number system was the definition and deduction of the properties of the rational numbers. ***Peano*** *(1889) began this process with five axioms for the* natural numbers:

- *1 is a natural number*
- *1 is not the successor of any other natural number*
- *Each natural number a has a successor*
- *If the successors of a and b are equal, then so are a and b*
- *If a set S of natural numbers contain 1, and if when S contains any number a it also contains the successor of a, then S contains all the natural numbers (axiom of* mathematical induction*)*

On these axioms he built all the familiar properties of natural numbers, and then established the properties of the negative whole numbers and the rational numbers as ordered pairs *of integers. Again, suitable definitions of the operations of addition and multiplication of pairs lead to the usual properties of the rational numbers.*

Thus, once the logical approach to the natural numbers was attained, the problem of building up the foundations of the real number system was completed.

Sets

Dedekind seemed to have settled the ancient problem of explaining the continuous *in terms of the* discrete*, but in penetrating as far as he did, he also uncovered deeper problems. The central problem is the relationship between two concepts:* completeness *and* countability*. To this end,* ***Cantor*** *(1874) introduced the notion of a* set*, which is one of the basic primitive mathematical concepts which does not lend itself to an accurate definition.*

Set is the name for an aggregate, ensemble, or collection of objects that are combined under a certain criterion or rule, e.g. the set of planets of our solar system, the set of all roots of a given equation, the set of all natural numbers,

the set of all points of a line, etc. The mathematical discipline that studies general properties of sets, i.e. properties that do not depend on the nature of the constituent objects, is called the theory of sets. The ideas and concepts of this theory penetrated into all branches of mathematics and changed its face entirely. It is of particularly great significance for the theory of functions of real variable.

A countable *set is one that can be put in one-to-one correspondence with the set of all natural numbers $N = \{1, 2, 3, 4, \dots\}$. If both sets are infinite and such a correspondence can be set up, then we say that they have the same* cardinality*. If not, then we say that one of them contains more elements than the other, or that one has a* greater *cardinality than the other.*

Cantor discovered that the set of rationals *and the set of* algebraic numbers *are countable.*[458]*He proved, however, that the set of all real numbers (equivalent to the set of all points of the segment $0 < x < 1$) is not countable.*[459] *Thus the* non-countability of the continuum *was established. Since the real numbers are uncountable and the algebraic numbers are countable, there must be* transcendental irrationals *– and in fact all reals but a subset of vanishing small relative size must be transcendental! This is Cantor's nonconstructive existence proof.*

Having demonstrated the existence of infinite sets with the same size (cardinality) and different sizes, Cantor introduced the theory of cardinal and ordinal numbers *(1879–1884). He expressed the size (or 'power' as he called it) of an infinite set by means of a* transfinite *number*[460]*. He started with the infinite set of the natural numbers (i.e. the positive integers), and denoted its 'size' by the transfinite number $\aleph_0$, which is the* cardinal number *of this set. Since the real numbers cannot be put into one-to-one correspondence with*

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458 To the layman the former may seem rather counterintuitive for the following reason: although the integers consist an infinite set, one cannot "pack in" any integers *between* two successive integers. Yet one *can* "pack in" an *infinity* of other rational numbers *between any two rationals* by simply taking the arithmetic mean between them and repeating the process indefinitely [e.g. take $\frac{1}{2}$ and $\frac{1}{3}$; the first average is $\frac{5}{12}$; the second mean is $\frac{1}{2}(\frac{1}{2} + \frac{5}{12}) = \frac{11}{24}$ and so on.] The rationals are thus called a *dense* set. Intuitively, there are infinitely 'more' rationals than there are integers, yet a 1:1 correspondence is possible between the two sets — e.g. by a lexicographical ordering of positive rationals $\frac{m}{n}$ (m, n positive) by $m + n$ and m, followed by "pruning" from the ordered list all rationals in which m and n possess a common factor.

459 E.g. by means of the homeomorphism $x \to \tan \pi(x - \frac{1}{2})$.

460 This theory is *distinct* from the concept of *infinity* wherein one speaks of a *variable becoming* infinitely large in a limit.

the natural numbers, the set of real numbers must have a *greater* cardinal number which is denoted by C (first letter of the word continuum). Thus $C > \aleph_0$. In Cantor's theory of sets, there is a whole *hierarchy* of transfinite numbers. Thus:

- The power of the set of positive rational numbers is also $\aleph_0$.

- Since $k + \aleph_0$ is also denumerable[461], we can write $k + \aleph_0 = \aleph_0 \quad (k > 0$ integer).

- Since $k\aleph_0$ is also denumerable, we can write $k\aleph_0 = \aleph_0$, where k is any positive integer (e.g. the union of the sets of *even* and *odd* integers, each countable, is just the set of integers, also countable).

- We construct the table

$\rightarrow i$

$j \downarrow$	1	2	3	4	5	6	...
1	1	2	4	7	11		
2	3	5	8	12			
3	6	9	13				
4	10	14					
5	15						
⋮							

$$i = 1, 2, \ldots, \infty$$
$$j = 1, 2, \ldots, \infty$$

where the entries in the table are assigned natural numbers along the consecutive *diagonals*, each starting from the upper right and sweeping toward the lower left. Thus, any pair of natural numbers is uniquely associated with a definite box (ij). Clearly, the totality of such pairs is represented by the numbered boxes, and this set of boxes is countable. Hence we see that

$$\aleph_0 \cdot \aleph_0 = \aleph_0$$

In general $\aleph_0^n = \aleph_0$ (n = finite positive integer).

- Any real number (say between 0 and 1) can be written as the base-2 decimal fraction $0.a_1a_2a_3a_4\ldots$. Each a_i can be filled in by any of the two binary digits 0 or 1. Hence $2^{\aleph_0} = C$.

Another way of demonstrating this is as follows: Let S be a *finite* set, and let its number of elements be n. The number of distinct subsets is 2^n.

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461 e.g. $\begin{array}{ccccccccc} \{a\} & \bigcup & \{ & 1, & 2, & 3, & 4, & \ldots & \} \\ \downarrow & & & \downarrow & \downarrow & \downarrow & \downarrow & & \\ 1 & & & 2, & 3, & 4, & 5, & \ldots & \end{array}$

Extending this notion to infinite sets *leads to the result (or actually, definition) that the set of all subsets (known as the* power set*) of N (the set of natural numbers) has cardinality $2^{\aleph_0}$. But this also equals the cardinality of the reals in the interval $(0,1)$, since each such real number is uniquely represented by* the set of its binary digits that equal 1 *and these sets are precisely the distinct subsets of the set of decimal positions to the right of the decimal point – i.e., the subsets of $N = \{1, 2, 3, \ldots\}$. Thus we again obtain $C = 2^{\aleph_0}$.*

Furthermore, $C \cdot C = 2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{2\aleph_0} = 2^{\aleph_0} = C$, and in general

$$C^n = C \qquad\qquad (n \text{ any positive integer})$$

The geometrical interpretation of this statement is that the 'number' of real points in a unit square (or unit hypercube of any dimension) is the same *as the 'number' of points in one of its sides. Indeed, it is easy to demonstrate that there exists a 1-1 correspondence between the points on a* line segment *and the points in a* square*. Likewise, a 1-1 correspondence can be set between points on a line segment and points in a cube (C^3), and in fact in an n-dimensional continuum C^n for any $n > 1$.*

- *A 1-1 correspondence can be set up between all points of a line segment and any* part *of itself. This equivalence is demonstrated* analytically *via the transformation $z' = a + (b - a)z$, where $0 \le z \le 1$ and $0 \le a \le 1$, $0 \le b \le 1$, which maps $[0, 1]$ onto $[a, b]$. Likewise, the transformation $z' = \frac{e^z}{1+e^z}$ transforms z in the range $-\infty < z < +\infty$ to $0 < z' < 1$. This means that the set of real numbers between 0 and 1 is equivalent (in its cardinality) to the set of* all *real numbers.*

- $$C^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = C$$

means that a continuum of a denumerable (countable) infinity *of dimensions (such as e.g. the set of all Fourier Series on a real interval) still has the same 'power' as a* one dimensional continuum.

- $${\aleph_0}^{\aleph_0} = 2^{\aleph_0} = C$$

Ordinal numbers

The concept of infinity is perplexing to mathematicians and nonmathematicians alike. The ancient Greek philosophers and mathematicians were

quite wary of infinities except potential infinities. Their lines were never completed, always what we call line segments. Aristotle distinguished two kinds of infinity. One can be described by the unlimited extent to which a line segment can be extended. This is a potential infinity, an unboundedness. Aristotle's other infinity can be found within a line segment since that line segment can be divided without bound. Again a potential infinity.

Through the centuries actual infinities remained suspect. Calculus (infinitesimal analysis) would probably have developed sooner had actual infinities been accepted. Actual infinities gained slow acceptance. For instance, completed lines became standard objects in geometry. Infinity in calculus remained a problem, though, until Cauchy found a way to define limits as a foundation for calculus.

***Cantor** discovered that there were different kinds of infinities in set theory: one was that of the* cardinal numbers *(sizes of sets). The other is that of* ordinal numbers *(counting numbers: first, second, third, etc.). For finite sets, they are the same. They differ, however, for infinite sets.*

A set is simply ordered *if an ordering relation ("<") can be defined on it such that any two elements have a definite order — i.e., given* m_1 *and* $m_2 \neq m_1$*, either* m_1 *precedes* m_2 *or* m_2 *precedes* m_1*; the notation is* $m_1 < m_2$ *or* $m_2 < m_1$*. Further, if* $m_1 < m_2$ *and* $m_2 < m_3$*, then simple order also implies* $m_1 < m_3$*; that is, the order relationship is* transitive*. An ordinal number of a simply ordered set* M *is the* order type *of the order in the set.*

Two ordered sets are similar if there is a 1-1 correspondence between them and if when m_1 *corresponds to* n_1 *and* m_2 *corresponds to* n_2 *and* $m_1 < m_2$*, then* $n_1 < n_2$ *(and vice versa). Two similar sets have the same type of ordinal number. The algebra of ordinal numbers is best expounded in the following words of Morris Kline.*[462]

> *"The ordinal number of the set of positive integers* in their natural order *is denoted by* ω*. On the other hand, the set of positive integers in decreasing order, that is* $\ldots, 4, 3, 2, 1$ *is denoted by* ${}^*\omega$*. The set of integers with zero in the usual order has the ordinal number* ${}^*\omega + \omega$*.*
>
> *Next Cantor defined the addition and multiplication of ordinal numbers. The sum of two ordinal numbers is the ordinal number of the first ordered set plus the ordinal number of the second ordered*

[462] *Mathematical Thought from Ancient to Modern Time*, pp. 1001–1002 (Oxford University Press, 1972).

set taken in that specific order. Thus, the set of positive integers followed by the first five integers, that is,

$$1, 2, 3, \ldots, 1, 2, 3, 4, 5,$$

has the ordinal number $\omega + 5$*. Also the equality and inequality of ordinal numbers is defined in a rather obvious way.*

He next introduced the full set of transfinite ordinals, partly for their own value and partly to precisely define higher transfinite cardinal numbers. To introduce these new ordinals he restricted the simply ordered sets to well-ordered sets. A set is well-ordered if it has a first element in the ordering and if every subset has a first element[463]*. There is a hierarchy of ordinal numbers and cardinal numbers. In the first class, denoted by* Z_1 *, are the finite ordinals*

$$1, 2, 3, \ldots.$$

In the second class, denoted by Z_2 *are the ordinals*

$$\omega, \omega+1, \omega+2, \ldots, 2\omega, 2\omega+1, \ldots, 3\omega, 3\omega+1, \ldots \omega^2, \omega^3, \ldots, \omega^\omega, \ldots$$

Each of these ordinals is the ordinal of a set whose cardinal number is $\aleph_0$.

The set of ordinals in Z_2 *has a cardinal number. The set is not denumerable and so Cantor introduces a new cardinal number* $\aleph_1$ *as the cardinality of the set* Z_2. $\aleph_1$ *is then shown to be the next cardinal after* $\aleph_0$*. The ordinals of the third class, denoted by* Z_3 *are*

$$\Omega, \Omega + 1, \Omega + 2, \ldots, 2\Omega, \ldots.$$

These are the ordinal numbers of the well-ordered sets, having $\aleph_1$ *elements. However, the set of ordinals* Z_3 *has more than* $\aleph_1$ *elements, and Cantor denoted the cardinal number of the set* Z_3 *by* $\aleph_2$*. This hierarchy of ordinals and cardinals can be continued indefinitely.*

Now, Cantor had also shown that given any set, it is always possible to create a new set, the set of subsets (power set) of the given set, whose cardinal number is larger than that of the given set. If $\aleph_0$ *is the given set, then the cardinal number of its power set is* $2^{\aleph_0}$*. As noted above, Cantor proved that* $2^{\aleph_0} = C$*, where* C *is the cardinal number of the continuum. On the other hand he*

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463 *Zermelo* proved that in the version of Zermelo-Fraenkel set theory where the *axiom of choice* is included, every set if well-ordered.

introduced $\aleph_1$ through the ordinal numbers and proved that $\aleph_1$ is the next cardinal after $\aleph_0$.

Hence $\aleph_1 \leq C$, but the question naturally arises as to whether $\aleph_1 = C$. The conjecture that this holds, known as the continuum hypothesis, *Cantor, despite arduous efforts, could neither prove nor disprove.*

For general sets M and N it is possible that M cannot be put into one-to-one correspondence with any subset of N and N cannot be put into one-to-one correspondence with a subset of M. In this case, though M and N have cardinal numbers α and β, say, it is not possible to say that $\beta = \alpha$, $\alpha < \beta$, or $\alpha > \beta$. That is, the two cardinal numbers are not comparable.

For well-ordered sets, Cantor was able to prove that this situation cannot arise. But it seemed paradoxical that there should be non-well-ordered sets whose cardinal numbers cannot be compared. But this problem, too, Cantor could not solve.

***Ernst Zermelo** (1871–1953) took up the problem of what to do about the comparison of the cardinal numbers of sets that are not well-ordered. In 1904 he proved, and in 1908 gave a second proof, that* every set can be well-ordered *(in some rearrangement). To construct the proof he had to use what is now known as the* axiom of choice *(Zermelo's axiom), which states that given any collection of nonempty, disjoint sets, it is possible to choose just one member from each set and so make up a new set.*

The axiom of choice, the well-ordering theorem, and the fact that any two sets may be compared as to size (that is, if their cardinal numbers are α and β, either $\alpha = \beta$, $\alpha < \beta$, or $\alpha > \beta$) are all equivalent principles."

*The issue of whether there are any transfinite numbers between $\aleph_0$ and C was finally resolved by **Paul Cohen** (1963) when he proved that the question is* undecidable *in the sense that consistent theories of infinite sets can be constructed which* either accept or deny *the assumption of the continuum hypothesis. Its status is, therefore, that of an independent axiom of set theory, just as Euclid's parallels axiom was shown by Gauss and others to be an independent axiom of classical geometry. Cohen* also *proved that the axiom of choice (and thus also the cardinality-comparability of any two sets) is itself undecidable.*

*In 1930, **Kurt Gödel** proved that no formal axiomatic system adequate to embrace arithmetic (and thus number theory) can be both consistent and complete. If such a system is consistent, then there must be some true statements which can be neither proved nor disproved within this formal system;*

they are undecidable within the system, and so some problems are logically unsolvable within a give axiomatic framework. (Cohen's discoveries, mentioned in the last section, established both the continuum hypothesis and the axiom of choice as such undecidable propositions.) In 1933 Gödel proved a second negative theorem: that there is no constructive procedure whereby an axiomatic system can establish its own consistency, i.e. freedom from internal contradictions.

The study of the properties of infinite sets, with their paradoxes and apparent contradictions, has been a major mathematical activity of the twentieth century. We cannot go into the details of this highly technical field, but to give a glimpse of the kinds of problems that arise, we end this section with a well-known story.

A village has only one barber, who claims that he shaves every man in the village who does not shave himself. The question is: who shaves the barber? Either answer leads to a contradiction. This paradox is a simple example of a situation where the use of such words as 'all' or 'every' can set a trap for the unwary. Indeed, the argument as to how such difficulties are to be resolved has divided mathematical logicians into contending camps for most of the 20th century.

1872–1897 CE **Julius Wilhelm Richard Dedekind** (1831–1916, Germany). One of the most original mathematicians of the 19^{th} century. He was educated at Göttingen and became one of **Gauss**' last students. His most influential teachers were **Riemann** and **Dirichlet**. In 1858 Dedekind was appointed professor of mathematics in Zürich, and from 1862 he taught at the polytechnical school in his native city, Brunswick.

Dedekind's work is associated with 4 main topics: Theory of positive integers (1887), theory of irrational numbers (1872)[464], the theory of algebraic numbers (1871) and the idea of the modular grid (1897). His revolutionary contribution to the first topic is the establishment of the integer concept on exclusive theoretical-logical basis, using set-theoretic concepts and the introduction of the 'definition via induction'.

[464] **W.R. Hamilton** made in 1833 one of the first attempts to define irrationals as partitions of rationals, thus presaging the *Dedekind cut.*

In 1872 Dedekind presented a theory of *real numbers* based on the concept of *Dedekind cut*[465], in which he *proved* that every cut in the domain of rational numbers defines a real number. The principal contribution of Dedekind to mathematical science is his creation of the theory of *algebraic numbers* and the introduction of the concept of the 'ideal', which is fundamental to ring theory.

Dedekind's brilliance was expressed not only in the theorems and concepts that he studied: Because of his ability to formulate and express his ideas so clearly, he introduced a whole new style of mathematics that has been a major influence on mathematicians ever since.

Richard Dedekind was born in Brunswick (the natal town of Gauss), the youngest of the four children of Julius Levin Dedekind, a professor of law of Jewish origin. In 1848 he entered, in the footsteps of Gauss, the Caroline College and from there he went in 1850 to Göttingen, to become one of Gauss' last pupils. He stayed there for 7 years and in 1857 was appointed an ordinary professor at the Zürich polytechnic, returning in 1862 to Brunswick as professor at the technical high school. He stayed there for the next 50 years. Nobody has as yet been able to explain why Dedekind occupied a relatively obscure position for half a century, while men who were not fit to lace his shoes filled important and influential university chairs. Dedekind never married.

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[465] **Dedekind**'s idea is this: all *rationals* can be divided into 2 classes, such that all the terms in one class are less than all the terms in the other. There are 3 possibilities:

(1) There may be a maximum to the lower section (L) *and* a minimum to the upper section (U).

(2) There may be a maximum to L and no minimum to U or vice versa.

(3) There may be *neither* a maximum to L *nor* a minimum to U. Example: The series of decimal approximants to $\sqrt{2}$ where L is the class of all rational numbers whose square is less than 2 and U is the class of all rational numbers whose square is greater than 2. In this case, since L has no maximum rational and U has no minimum rational, there is a *hole* (cut) in the rational series, which must be filled, if one desires continuity of the number line. This is done by *postulating* that every such 'cut' *defines* an irrational number.

Algebraic Numbers and Dedekind's Ideals

The failure of 18^{th} and 19^{th} century mathematics to resolve Fermat's Last "Theorem" led to the development of the new arithmetic of algebraic numbers.

An algebraic number is a complex number that is a root of an algebraic equation $p(x) = 0$, where $p(x) = a_0x^m + \cdots + a_m$ is a polynomial with rational coefficients with $a_0 \neq 0$ and $m > 1$. An algebraic number is a root of infinitely many equations of various degrees: e.g. $\alpha = \sqrt{3}$ satisfies the equations $x^2 - 3 = 0$; $x^3 - x^2 - 3x + 3 = 0$; $\quad x^4 - 9 = 0$ etc. But the polynomials for the last two equations are reducible over the field of rationals, i.e. they can each be factored into lower-degree polynomials with rational coefficients. If it is impossible to factor a polynomial $p(x)$ over the rationals into non-constant factors of lower degree, again with rational coefficients, then p is called irreducible over the rationals. Thus $p(x) = x^2 - 3$ is irreducible.

For any algebraic number α there is exactly one irreducible polynomial $\phi(x)$ over the rationals with leading coefficient 1 such that $\phi(\alpha) = 0$. The degree of the algebraic number α is defined as the degree of $\phi(x)$. For example, every rational number r is algebraic of the 1^{st} degree, being the root of $x - r = 0$; $\frac{1}{2}(1 + i\sqrt{3})$ is of degree 2, being the root of $x^2 - x + 1 = 0$; and $\sqrt[n]{2}$ is of degree n as a root of $x^n - 2 = 0$. If all the coefficients in $\phi(x) = 0$ are integers, then α is called an algebraic integer. It can be shown that the roots of a polynomial with algebraic coefficients, are also algebraic.

It is customary to denote the ring of integers by the symbol $\mathbb{Z}$; a ring is a set of numbers in which operations of addition, subtraction and multiplication are performed without restriction. Addition in a ring is always commutative, but multiplication need not be. If it is, the ring is said to be commutative. A commutative ring in which every nonzero element has a multiplicative inverse, is called a field. The symbol $\mathbb{Q}$ stands for the field of rational numbers. The symbol $\mathbb{Q}(\sqrt{d})$ denotes the field of numbers of the form $a + b\sqrt{d}$, where (a, b) are arbitrary rational numbers: such a field is known as a quadratic field. If $d > 0$, we call it a real quadratic field. If $d < 0$, we call it complex quadratic field. If d itself is a square of a rational number, $\mathbb{Q}(\sqrt{d})$ is just $\mathbb{Q}$. Just as $\mathbb{Q}$ is the set of all ratios of elements of $\mathbb{Z}$, so $\mathbb{Q}(\sqrt{d})$ is the set of ratios $\frac{a}{b}$ with $a \in \mathbb{Z}(\sqrt{d})$, $b \in \mathbb{Z}(\sqrt{d})$, $b \neq 0$, where $\mathbb{Z}(\sqrt{d})$ is the ring-subset of $\mathbb{Q}(\sqrt{d})$ consisting of $a + b\sqrt{d}$ with a, b integers. All the rings to be considered here are commutative, have an identity (an element $1 \neq 0$ such that $a \cdot 1 = a$ for all a in the ring) and have no zero divisors (i.e. $a \cdot b = 0$ implies $a = 0$ or $b = 0$). Such a ring is called an integral domain.

A number α *of* $\mathbb{Q}(\sqrt{d})$ *is called a quadratic integer (or just integer for short) if either* α *is in* $\mathbb{Z}$ *or* α *is irrational and the coefficient of* x^2 *in the defining integer-coefficient polynomial equation for* α *is* 1*. The numbers in* $\mathbb{Z}$ *will be called rational integers. Thus* $\frac{1}{2}(-1+\sqrt{-3})$ *is an integer because its defining equation is* $x^2+x+1=0$*, while* $\frac{1}{2}(-3+6\sqrt{-3})$ *is not an integer because its defining equation is* $4x^2+12x+117=0$*. Note that if* d *belongs to the ring* $\mathbb{Z}(\sqrt{d})$ *in* $\mathbb{Q}(\sqrt{d})$*, it is a quadratic integer (but the converse need not hold).*

Consider the Diophantine equation $y^2+2=x^3$ *that has exactly two solutions in positive integers:* $x=3$*,* $y=\pm 5$*. We wish to show that there are no more solutions in ordinary integers. Although the l.h.s. has no real polynomial factors, we can still factor it into algebraic integers of the form* $a+b\sqrt{-2}$*, namely* $(y+\sqrt{-2})(y-\sqrt{-2})=y^2+2$*.*

Let us assume for the moment that this factorization has broken y^2+2 *into two mutually prime factors in* $\mathbb{Z}(\sqrt{-2})$*. If that is so, then by the unique factorization theorem*[466]*, each of* $(y+\sqrt{-2})$ *and* $(y-\sqrt{-2})$ *must be a cube if their product is to be* x^3*. That is,*

$$y+\sqrt{-2}=(u+v\sqrt{-2})^3=(u^3-6uv^2)+(3u^2v-2v^3)\sqrt{-2},\ \text{with } u,v \text{ integers.}$$

Equating the imaginary parts of both sides we find $1=3u^2v-2v^3=v(3u^2-2v^2)$*. Therefore* v *can only be* 1 *and* $u=\pm 1$*. Matching the real parts then yields* $y=\pm 1\mp 6=\mp 5$*, and thus* $x=\sqrt[3]{y^2+2}=3$*.*

This proof has two major gaps: The first is the assumption that $(y\pm\sqrt{-2})$ *are prime in the ring* $\mathbb{Z}(\sqrt{-2})$*, which can be proved. The second missing step is more serious: how do we know that prime factorization is unique in* $\mathbb{Z}(\sqrt{-2})$*? The answer to this question is long and interesting, and involves a complicated story. It began with* 19^{th} *century attempts to prove Fermat's Last Theorem.*

Among those who thought for a time that they had proved it was ***E. E. Kummer*** *(1810–1893). He assumed as a matter of course that factorization into primes was always unique, even when the integers were of the form* $a+b\sqrt{-5}$ *[*a,b *being regular (rational) integers, i.e. belonging to* $\mathbb{Z}$*]. But this happens to a ring* $\mathbb{Z}(\sqrt{-5})$ *in which the Fundamental Theorem of Arithmetic fails, e.g.*

$$6=2\cdot 3=(1+\sqrt{-5})(1-\sqrt{-5}).$$

Each of 2*,* 3*,* $(1+\sqrt{-5})$ *and* $(1-\sqrt{-5})$ *can be shown to be a prime number in this ring. So we have two different prime factorizations of the number* 6*, and it appears that no simple arithmetic theory of the algebraic integers could be possible.*

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[466] This theorem turns out to hold for $\mathbb{Z}(\sqrt{-2})$.

In order to resolve this difficulty, Kummer (1846) created a new kind of entity that he called an "ideal number". Although he failed to prove Fermat's Last Theorem, he laid the foundation to the theory of algebraic numbers.

***Dedekind** (1871) reformulated the concept of "ideal number" and generalized it to other rings of algebraic numbers. He replaced algebraic integers by a new concept which he coined: that of an ideal, one example of which is the ring R of quadratic integers in a particular quadratic number field. We now consider ideals and their properties in some detail.*

By the use of Euclid's algorithm it is shown that the factorization of a rational integer into prime factors is unique, apart from the order of the factors and their sign. The same is not true of every integral domain. The first difficulty that arises is due to the fact that in some integral domains there exist numbers besides 1 *and* -1 *which have reciprocals. Thus in* $\mathbb{Z}(\sqrt{2})$ *[i.e., numbers of the form* $a + b\sqrt{2}$ *with* a, b *integers],* $(\sqrt{2}+1)(\sqrt{2}-1) = 1$. *If such factorizations were taken into account, then no number could be prime, e.g.* $3 = (3\sqrt{2}+3)(\sqrt{2}-1)$. *Moreover, we have* $6 = 3 \cdot 2 = (3\sqrt{2}+3)(2\sqrt{2}-2)$ *which renders a non-unique factorization. The difficulty is surmounted by regarding* $(\sqrt{2}+1)$ *and* $(\sqrt{2}-1)$ *as units. Since* $3\sqrt{2}+3 = 3(\sqrt{2}+1)$ *and* $2\sqrt{2}-2 = 2(\sqrt{2}-1)$, *then* $3 \cdot 2$ *is considered to be equivalent to* $(3\sqrt{2}+3)(2\sqrt{2}-2) = 3 \cdot 2(\sqrt{2}+1)(\sqrt{2}-1)$ *and unique factorization is restored*[467].

In $\mathbb{Z}(\sqrt{5})$ *[i.e., numbers of the form* $a + b\sqrt{5}$ *with* a, b *in* $\mathbb{Z}$*] it is found that* $(\sqrt{5}-1)(\sqrt{5}+1) = 4 = 2 \cdot 2$. *Though it might appear at first sight that unique factorization has failed here (and with it, the theory of congruences and residues!), such is not the case. For* $\frac{1}{2}(1 \pm \sqrt{5})$ *are not only algebraic integers [satisfying* $x^2 - x - 1 = 0$*] but also units, on account of* $\frac{\sqrt{5}+1}{2}\frac{\sqrt{5}-1}{2} = 1$. *Therefore both* $\sqrt{5}+1$ *and* $\sqrt{5}-1$ *are equivalent to* 2. *A complete failure of unique factorization (and of Euclid's algorithm), however, occurs, as we saw, in* $\mathbb{Z}[\sqrt{-5}]$. *Thus, the two factorizations of* 6, *namely,* $6 = 3 \cdot 2 = (1+\sqrt{-5})(1-\sqrt{-5})$ *cannot be reconciled by any convention.*

Dedekind evaded the difficulty by the introduction of a new entity, which restored uniqueness of prime factorization to domains for which it fails. Let R *be a ring whose elements* r *may be real or complex. Consider a subset* I *of* R *with the following properties:*

(i) If a *and* b *are numbers in* I, *so is* $a - b$.

(ii) For every number r *in* R *and every number* a *in* I, *the product* ra *also lies in* I.

467 Note that since $(\sqrt{2}+1)^n(\sqrt{2}-1)^n = 1$, there exist here an *infinite* number of units!

The subsets I of R are called ideals in R. For example, if m is a natural number, the totality of numbers $\{0, \pm m, \pm 2m, \pm 3m, \dots\}$ is an ideal in $\mathbb{Z}$ and will be denoted by $[m]$. Clearly, the difference of two integer multiples of m is again a multiple of m, and every multiple of a multiple of m is itself a multiple of m. Thus, $[6] = \{\dots, -18, -12, -6, 0, 6, 12, 18, \dots\}$.

Ideals consisting of all the multiples of a single element of a ring R (in the present case m) and no other elements, are called principal ideals. It is easy to verify that in the ring $\mathbb{Z}$, every ideal is principal, so that all ideals $[m]$ in $\mathbb{Z}$ are obtained by setting in turn $m = 0, 1, 2, \dots$ (in other rings, however, there exist also ideals which are not principal). The elements of a principal ideal are all multiples of a single element a which is said to generate the ideal (m in the above example). Thus, the principal ideal $[a]$ is the set of all elements xa, when a is a fixed number (element) of the ring and x is any other element.

The product of two ideals is the set of numbers which have the form of a product of two numbers, one from each ideal, together with all numbers that can be formed from these by addition and subtraction. The product is easily shown to itself be an ideal. Thus the product of the principal ideals $[2] \times [3]$ contains the numbers $2 \times 3 = 6$, $4 \times 3 = 12$, $2 \times 9 = 18$, etc., and forms the ideal $[6]$, in full analogy to arithmetic multiplication. In particular, $[1]$ is the unit ideal since it clearly satisfies $[1][m] = [m]$; $[1]$ is in fact the entire ring ($\mathbb{Z}$ in this case). If an ideal cannot be expressed as a product of two ideals in which neither is the unit ideal, we shall say that it is a prime ideal, again in analogy to prime factorization of integers in ordinary arithmetic[468].

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[468] Consider the two principal ideals over $\mathbb{Z}$:

$$[3] \;= \{\dots, -3,\; 0, 3, 6, 9, 12, \dots\}$$
$$[10] = \{\dots, -10, 0, 10, \dots, \}$$

When *any* number of [3] is factorized into two factors a, b of $\mathbb{Z}$ it is found that either a or b belong to [3]. The ideal [10] does not share this property since, for example, $20 = 4 \cdot 5$, where neither 4 nor 5 is in [10]. Dedekind therefore said that [3] is a *prime ideal*. [In general, an ideal P in a ring R is called a *prime ideal* if $ab \in P$ implies either $a \in P$ or $b \in P$, for all elements of P.]
Thus, in the ring $\mathbb{Z}(\sqrt{-5})$, where the elements are of the form $(m + n\sqrt{-5})$, the ideal $[2, 1 + \sqrt{-5}]$ (the sub-ring of elements $2m + (1 + \sqrt{5})n$ with m, n in $\mathbb{Z}$) is not expressible as a principal ideal as there is no element which divides both $(1 + \sqrt{-5})$ and 2. In fact

$$[2] = [2, 2 + 2\sqrt{-5}] = [2, 1 + \sqrt{-5}][2, 1 + \sqrt{-5}]$$

by the law of ideal multiplication, where $[2, 1 + \sqrt{-5}]$ is a *prime ideal* in $R(\sqrt{-5})$, but [2] is not!

The concept of divisibility can also be defined for ideals. One says that an ideal A is divisible by an ideal B if every element a of A is also element b of B. When applied to $\mathbb{Z}$, it is seen that $[a]$ is divisible by $[b]$ iff a is divisible by b.

We next consider a type of non-principal ideal generated by two elements of the integral domain (ring) and denoted as $[a, b]$. It consists of all elements $xa + yb$, where a and b are fixed members of the domain, and x, y are any elements whatsoever therein. In general, given an integral domain D, the set of elements of D which are linear combinations of α_1,$\alpha_2, \dots, \alpha_i$ with any coefficients in the domain, constitutes an ideal which is denoted $[\alpha_1, \alpha_2, \dots, \alpha_i]$.

Multiplication of such ideals is defined by the relation

$$[\alpha_1, \alpha_2, \dots, \alpha_i][[\beta_1, \beta_2, \dots, \beta_j] = [\alpha_1\beta_1, \alpha_1\beta_2, \dots, \alpha_1\beta_j, \alpha_2\beta_1, \dots, \alpha_i\beta_j].$$

As stated earlier, a principal ideal is an ideal with but one symbol in the bracket, such as $[\alpha_1]$, and consists of the products of α_1 with all elements of D. Ideals in $\mathbb{Z}$ are all principal ideals, for any pair of rational integers m, n have a g.c.d. (greatest common divisor) h, such that $Mm + Nn = h$ for some M, N in $\mathbb{Z}$, and m and n are multiples of h. Thus $[m, n] = [h]$. Factorization of ideals in the ring of rational integers is therefore the same as factorization of integers.

With $s = \sqrt{-5}$ we define

$$p = [2, 1 + s]; \qquad q = [3, 1 + s]; \qquad r = [3, 1 - s]$$

and find that

$$pq = [1 + s]; \qquad pr = [1 - s]; \qquad pp = [2, 2 + 2s] = [2, 2s]; \qquad qr = [3, 3s].$$

Hence the principal ideal [6] can be expressed as either the product of the two non-prime ideals [2] and [3] or uniquely as the product of the prime ideals

$$[6] = p^2 qr.$$

In general, Dedekind proved his main theorem:

"Every ideal in a ring R, other than R itself and $[0]$, can be represented as a product of prime ideals, uniquely apart from the order of the factors."

For the ring $\mathbb{Z}$ of rational integers, this means that every principal ideal $[m]$ is uniquely (apart from the order of the factors) a product of prime ideals $[p_1][p_2][p_3]\dots[p_m]$, which is another way of stating the fundamental theorem of elementary number theory: $m = \pm p_1 p_2 \dots p_n$ for any $m \in \mathbb{Z}$, where p_j are some (possibly repeating) prime numbers.

1872–1912 CE **Felix Klein** (1849–1925, Germany). Mathematician. A central figure in world mathematics during his lifetime. Known for his novel approach to geometry and his decisive influence upon the development of mathematics in the 19^{th} century.

At the age of 23 he was a full professor at the University of Erlangen. His inaugural lecture there had made mathematical history as the *Erlangen Program* — a bold proposal to use the group concept to classify and unify the many diverse and seemingly unrelated geometries which had developed since the beginning of the 19^{th} century.

Early in his career he had shown an unusual combination of creative and organizational abilities and a strong drive to break down barriers between pure and applied science. His mathematical interests were all-inclusive: geometry, number theory, group theory, invariant theory, algebra — all were combined for the development and completion of the Riemannian ideas on geometric function theory. He made significant contributions to the theory of automorphic functions, topology ("*Klein's bottle*"), group theory (*Klein's group*, *Cayley-Klein parameters*) and gyroscopic theory. Klein solved (1877) the *icosahedral equation* in terms of hypergeometric functions. This allowed him to give a closed-form solution of the quintic.

Klein was born in Düsseldorf, Prussia. At 17 he was chosen by **Julius Plücker** (1801–1868) as his assistant in his physics laboratory at Bonn [the same laboratory where Plücker had invented what today we call the *Geissler tube*]. Plücker had reverted in his later years to his early interest in geometry. When he died in 1868, he left an unfinished manuscript, entitled "*New geometry of space, founded on the straight line as element*".

The task of completing the work and issuing the second half of the book was entrusted to Plücker's young assistant, Felix Klein. During 1869/70 he was at the University of Berlin, where he hoped to profit from personal contact with Weierstrass, but the latter was not receptive to Klein's ideas. Klein participated in the Franco-German war, serving in the ambulance corps.

Klein married the beautiful Anna Hegel, granddaughter of the philosopher **Hegel**. In 1886 he migrated to the University of Göttingen, where he stayed for the rest of his life.

"*All things near and far*
Hiddenly linked are.
Thou canst not stir a flower
Without the troubling of a star".

William Blake (1757–1827)

1872 CE **Francois Felix Tisserand** (1845–1896, France). Astronomer. Suggested that the gravitational force exerted by a moving body might obey the same laws as the electric and magnetic forces exerted by a moving charge. His results predicted that the planets would deviate slightly from their Newtonian orbits around the sun. Such a deviation had been detected in the motion of Mercury by **LeVerrier** in 1845, but Tisserand was unable to match his theory with the observations. The motion of Mercury remained a mystery until 1915.

This was the first effort at a gravitodynamical extension of Newtonian theory.

1872–1921 CE **Gabriel Jonas Lippmann** (1845–1921, France). Physicist and inventor. A multi–talented researcher best known for his contributions to optics, electricity and thermodynamics:

- Invented (1872), at Heidelberg, the *capillary electrometer*, which measures small differences in voltage and was used by **Weller** and **Einthoven** in their early *electrocardiographs*; it is an instrument in which small electric currents are detected by movement of a mercury meniscus in a capillary tube. The instrument consists of a thin glass tube with a column of mercury beneath sulphuric acid. The mercury meniscus moves with varying electrical potential and is observed through a microscope.

- Invented the *coelostat*, a new astronomical tool that compensated for the earth's rotation and allowed a region of the sky to be photographed without apparent movement. Essentially, it allows long-exposure photographs of the sky by compensating for the earth's motion during the exposure: The device consists of a flat mirror that is turned slowly by a motor to reflect the sun's continuously into a fixed telescope. The mirror is mounted to rotate about an axis through its front surface that points to a celestial pole and is driven at a rate of one revolution in 48 hours. The telescope image is then stationary and non-rotating. The coelostat is particularly useful for eclipse expeditions when elaborate equatorial mounting of telescopes is impossible.

- Studied induction in *superconductor* circuits (precursor of **Kamerlingh–Onees'** validations)

- Did early important work in *piezoelectricity* (precursor of **Pierre Curie**'s work). His research furthered developments in this field.

- The beginning of photography came before 1849 due to the efforts of such as Niepce, Daguerre, and Talbot. Even though these men made the foundation for photography, they did not know how to obtain proper color from the early examples. Edmond Becquerel came close to discovering this, but like many others, failed. After the theories and experiments of other men such as Wilhelm Zenker and Otto Weiner, color could finally be duplicated, more or less.

  Lippmann had evolved the general theory of his process for the photographic reproduction of color in 1886, but the practical execution presented great difficulties. However, after years of patient and skillful experiment, he was able to communicate the process to the Academy of Sciences in 1891, although the photographs were somewhat defective due to the varying sensitivity of the photographic film. In 1893, he was able to present to the Academy photographs taken by A. and L. Lumiere in which the colors were produced with perfect orthochromatism. He published the complete theory in 1894.

  Lippmann's color photographic technique was based on interference, the combining of different light waves arriving simultaneously at the same point – the same phenomenon that causes color to appear in colorless substances such as soap bubbles. To receive the image, Lippmann used a glass plate coated on one side with light-sensitive emulsion, a mixture of gelatin, grains of silver nitrate, and potassium bromide. In the camera, the emulsion side of the plate faced a plate holder coated with mercury, which acted as a mirror.

  When the camera lens was opened, light was reflected from the objects in the lens's field of view through the lens to the emulsion-coated plate and through the plate to the mirror; the various wavelengths of this light corresponded to the various colors of the objects in the field of view. The incoming light was then reflected back into the emulsion by the mirror. When the incoming light waves and the light waves reflected by the mirror met on the surface of the emulsion, they created interference patterns in the silver grains of the emulsion. These patterns were then fixed on the plate by chemical baths. When the plate dried, the interference patterns reflected light in various wavelengths corresponding to the original colors of the photographic objects. Lippmann's process was an important experimental milestone although it proved impractical in photography: because exposure times were too lengthy, the image had

to be viewed at a precise angle to a light source, and it could not be reproduced.

Gabriel Lippmann was born of Jewish parents at Hollerich, Luxembourg.The family moved to Paris and he received his early education at home. He entered the Lycée Napoleon (1858) and in 1868 was admitted to the Ecole Normale. In 1873, he was appointed to a Government scientific mission visiting Germany to study methods for teaching science: he worked with **Kirchhoff** in Heidelberg and with **Helmholtz** in Berlin.

He joined the Faculty of Science in Paris (1878), was appointed Professor of Mathematical Physics at the Sorbonne (1883) and became a Professor of Experimental Physics[469] there (1886). He then became Director of the Research Laboratory and retained this position until his death.

Lippmann became a member of the Academy of Sciences (1883) and served as its President (1912). He was awarded the 1908 Nobel Prize in Physics for his method of reproducing colors photographically based on the phenomenon of interference, known as the *Lippmann plate*.

He died at sea on July 13, 1921, during his return from a journey to North America.

1873 CE **Louis Joseph May** (England) and **Willoughby Smith** (1828–1891, England) discovered *electrical photoconductivity*, thus enabling to transform images into electrical signals. They found that the electrical conductivity of the element *selenium*[470] changes when light falls on it, i.e. when a selenium bar is exposed to light it becomes a strong conductor of electricity and the ensuing current is proportional to the amount of light hitting the bar. May then used selenium to send a signal through the Atlantic cable (laid in 1865).

Both inventors were at the time telegraph operators in Valentin, Ireland.

In the same year Maxwell published his book *Treatise on Electricity and Magnetism*, expounding the theory of electromagnetic radiation.

Due to the *photoconductive effect*, selenium would become the basis for the manufacture of *photoelectric cells*[471].

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[469] He was **Marie Curie**'s thesis advisor at the Sarbonne. Lippmann let her use his laboratory for her thesis work and helped her find other sources of support. At that time Lippmann did early studies in a field of electrical effects in crystals. It was he that introduced Marie to one of his best students, **Pierre Curie**.

[470] Discovered (1818) by **J.J. Berzelius**.

[471] The first photocell was built by **Charles Sumner Tainter** (1880). The first practical *photoelectric cell* was devised through 1900–1904 by **Julius Elster**

Willoughby Smith was born in Great Yarmouth. During 1850 he superintended the manufacture and laying of a telegraph cable from Dover to Calais, and later assisted Charles Wheatstone with his experiments. In 1865 he was on board the *Great Eastern* and assisted in laying the transatlantic cable from Ireland to Newfoundland.

Photoconductivity and photoelectric cells (1873–1973)

Photoconductivity is an internal photoeffect. The absorption of a photon by an intrinsic photoconductor results in the generation of a free electron excited from the valence band to the conduction band, and a corresponding free hole in the valence band. The application of an electric field in the material results in the transport of both electrons and holes through the material and the consequent production of an electric current in the electrical circuit of the detector.

Photoelectric cell is a device that converts light into electricity. Two main types of photoelectric cell are in use today: the phototube and the solid-state photodetector. The phototube is an electron tube in which electrons initiating an electric current originate through photoelectric emission. In its simplest form the phototube is composed of a cathode coated with a photosensitive material, and an anode. Light falling upon the cathode causes the liberation of electrons, which are then attracted to the positively charged anode, resulting in a flow of electrons (i.e., current) proportional to the intensity of the light.

Phototubes may be highly evacuated, or filled with an inert gas at low pressure to achieve greater sensitivity. In a modification called the multiplier phototube, or photomultiplier, a series of metal plates are shaped and arranged so that the photoelectric emission is amplified by secondary electron emission. The multiplier phototube is capable of detecting radiation of extremely low intensity; it is an essential tool for nuclear research, astronomy, and space guidance systems.

(1854–1920) and **Hans Geitel** (1855–1923). They studied together in Heidelberg and Berlin, and worked as high school teachers of mathematics and physics in Wolfenbüttel.

The second type of photoelectric cell, the solid-state photodetector, has replaces the phototube for many applications because it is small, inexpensive, and uses little power.

The simplest type of solid-state photodetector is the photoconductor — a semiconductor whose resistance changes when it is exposed to light — that is, to a flow of photons. Semiconductors are characterized by an energy gap that separates the electron valence band from the conduction band. When an electron in the valence band absorbs a photon of energy greater than the energy gap, it can move from the valence band into the conduction band and increase the conductivity of the semiconductor. Moving the electron into the conduction band leaves an excess positive charge, or hole, in the valence band, which can also contribute to conductivity.

The conductivity of a photoconductor increases (while its resistance decreases) as the number of photons increases. When the photoconductor is connected in an electric circuit, the current through it therefore increases in proportion to the intensity of the light striking it.

The photoconductor, popularly known as the electric eye, is employed in operating burglar alarms, traffic-light controls, and door openers. A light source (which may be infrared and invisible to the human eye) at one end of the circuit falls on the photocell located some distance away. Interrupting the beam of light breaks the circuit. This in turn causes a relay to close, which energizes the burglar alarm or other circuit. Other common uses for photoconductors include light switching and dimming, and light meters for cameras.

A more sophisticated photodetector, the CCD (charge-coupled device) is a small capacitor, composed of metal, oxide, and semiconductor layers, capable of both photodetection and memory storage. When a positive voltage is applied to the metal layer (called the gate), electron-hole pairs created in the semiconductor by the absorption of a photon are separated by an electric field, and the electrons become trapped in the region under the gate. This trapped charge represents a small piece of an image known as a pixel. The complete image can be recreated by reading out a sequence of pixels from an array of CCDs. These arrays are used to capture images in video and digital cameras.

More stable and precise than a simple photoconductor, the photodiode is a p-n junction formed by placing a p-type semiconductor against the surface of an n-type semiconductor The region around the interface between the two types of semiconductors, called the depletion region, contains an electric field. If a photon of sufficient energy is absorbed in this region, it creates an electron-hole pair; the electric field sweeps the electron toward the n region and the hole toward the p region. If the p and n terminals are connected, or a reverse voltage is applied, an external current is created.

The photovoltaic cell, or solar cell, is a well-known application of the photodiode. "Avalanche" diodes are used to amplify the signal from a light source. In these devices, a large reverse voltage is applied so that a photon-created electron in the conduction band gains enough energy to bounce against atoms in the semiconductor and thus liberate additional electrons. A large current is therefore produced when light strikes the diode.

Phototransistors are also used to amplify light signals. Their construction is similar to conventional transistors except that one of the transistor's junctions is exposed to radiation. In bipolar phototransistors, it is the base-emitter junction that is exposed to radiation; in field-effect phototransistors it is the gate junction.

1873 CE **Johannes Diderik van der Waals** (1837–1923, Holland). Theoretical physicist. Established an improved ideal gas law which accounts for the finite size of gas molecules and for the intermolecular attraction forces.

In 1910, he won the Nobel prize in physics for developing the equation of state which bears his name.

Van der Waals was born at Leyden, The Netherlands. He served as professor of physics at Leyden from 1877 until his retirement in 1907.

1873–1876 CE **Carl Paul Gottfried von Linde** (1842–1934, Germany). Engineer. Introduced the first practical cooling compression system (refrigerator), utilizing the Joule-Thomson effect with liquid ammonia as coolant. Earlier experimental cooling systems were produced by **Jacob Perkins** (U.S.A., 1834) who developed the first compression machine, and **Ferdinand Carré**, a French engineer who built the first absorption system using ammonia (1854).

1873–1878 CE **William Kingdon Clifford** (1845–1879, England). Mathematician and philosopher. Generalized the quaternions of Hamilton to biquaternions. These are used for the study of both Euclidean and non-Euclidean spaces. Determined the topological equivalence of many-sheeted surfaces.[472]

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[472] He showed, for example, that the Riemann surface of an n-valued function with w branch-points can be transformed into a topological equivalent of a sphere with p holes where $p = \frac{w}{2} - n + 1$ (i.e. a sphere with p handles).

In his endeavor to graft Hamilton's quaternions on to Grassmann's extensive algebra, he discovered '*Clifford algebra*'. In 1878 he coined the words '*divergence*' and '*curl*'[473]. The term Nabla was contributed by **Robertson Smith**. Clifford confessed his belief (1870) that "*matter is only a manifestation of curvature in a space time manifold*" (!!) He made advances in non-Euclidean geometry. *Clifford parallels* and *Clifford surfaces* are named after him.

Clifford derived his theory of *biquaternions* (through a generalization of quaternions), associating them specifically with linear algebra. In this way he represented motions in three-dimensional non-Euclidean space. He then suggested (1876) that motion of matter may be due to changes in the geometry of space.

Clifford was born at Exeter. He was educated at Kings' College, London and at Trinity College, Cambridge. In 1871 he was appointed professor of mathematics at University College, London and in 1874 he became Fellow of the Royal Society. He died of pulmonary consumption at Madeira.

Clifford Algebras

William Clifford invented his algebras (1876–1879) as an attempt to generalize the quaternions to higher dimensions. To begin with, he started from quaternions over the complex number field with 1 and 0 as its unity and zero elements, i.e. $q = a + bj + ck + dl$, *where* (a, b, c, d) *are allowed to be complex numbers. Here, one must distinguish between the* quaternion conjugate $q^* = a - bj - ck - dl$ *and the ordinary* complex conjugate $\bar{q} = \bar{a} + \bar{b}j + \bar{c}k + \bar{d}l$. *If* $q^* = \bar{q}$, *the quaternion is* Hermitian. *A quaternion for which* $\|q\| \equiv a^2 + b^2 + c^2 + d^2 = 1$ *is called a* unit quaternion. *If* $\|q\| = 0$ *the quaternion is* singular. *Hermitian* biquaternions *(with a real scalar part) are used to represent space-time in the theory of relativity.*

[473] The corresponding *concepts* are, however, due to **Maxwell**.

On setting

$$a = a_0, \qquad b = ia_1, \qquad c = ia_2, \qquad d = -a_3, \qquad i^2 = -1,$$

$$j = -ie_1, \qquad k = -ie_2, \qquad l = -e_1e_2,$$

$$e_1^2 = e_2^2 = 1; \qquad e_1e_2 = -e_2e_1; \qquad (e_1e_2)^2 = -1$$

and remembering that the quaternion units obey the relations

$$j^2 = -1, \qquad k^2 = -1, \qquad l^2 = -1;$$

$$jk = l, \quad kl = j, \quad lj = k, \quad kj = -l, \quad lk = -j, \quad jl = -k,$$

the quaternion q is transformed into the new form $q \to q'$ where

$$q' = a_0 + a_1e_1 + a_2e_2 + a_3e_1e_2.$$

The entity q' has a base $\{1, e_1, e_2, e_1e_2\}$ with a 'multiplication table':

	1	e_1	e_2	e_1e_2
1	1	e_1	e_2	e_1e_2
e_1	e_1	1	e_1e_2	e_2
e_2	e_2	$-e_1e_2$	1	$-e_1$
e_1e_2	e_1e_2	$-e_2$	e_1	-1

This new algebra, known as C_2, is isomorphic to quaternion algebra such that

$$\left.\begin{array}{l} q \to q' \\ p \to p' \end{array}\right\} \text{ implies } \qquad \begin{array}{c} q + p \Rightarrow q' + p', \qquad qp \Rightarrow q'p' \\ \lambda q \Rightarrow \lambda q' \quad (\lambda = \text{ complex scalar}) \end{array}$$

The algebra can be represented by square matrices A, B such that

$$\left.\begin{array}{l} q \to A \\ p \to B \end{array}\right\} \text{ implies } \qquad \begin{array}{c} \lambda q' + \mu p' \Rightarrow \lambda A + \mu B \\ p'q' \Rightarrow AB \end{array}$$

$$(\lambda, \mu = \text{ elements of the complex field})$$

To this end we infer from the above 'multiplication table' that the unit elements $\{1, e_1, e_2, e_1e_2\}$ *can be represented by the matrices*

$$1 \Rightarrow E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$e_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad e_1e_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The algebra C_2 *may be generalized to an algebra* C_n *generated by* n *symbols* e_r $(r = 1, 2, \ldots, n)$ *satisfying the relations*

$$e_re_s + e_se_r = 2\delta_{rs}$$

These algebras are known as the Clifford-Dirac *algebras, with applications in quantum mechanics.*

To see the connection we now 'translate' our former 4×4 *matrix representation of the quaternions (*C_2 *algebra) into the language of the Pauli (spin) matrices* σ_k*:*

$$j = -ie_1 \Rightarrow -i\sigma_1 \qquad e_1 \Rightarrow \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$k = -ie_2 \Rightarrow -i\sigma_2 \qquad e_2 \Rightarrow \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$l = -e_1e_2 \Rightarrow -i\sigma_3 \qquad e_1e_2 \Rightarrow i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We then have the isomorphism*:*

$$\begin{aligned} q = q_0 + q_1j + q_2k + q_3l &\Rightarrow q_0E - iq_1\sigma_1 - iq_2\sigma_2 - iq_3\sigma_3 \\ &\Rightarrow \begin{bmatrix} q_0 - iq_3 & -q_2 - iq_1 \\ q_2 - iq_1 & q_0 + iq_3 \end{bmatrix} = M, \text{ say} \end{aligned}$$

with $\det M = q_0^2 + q_1^2 + q_2^2 + q_3^2$. *For unit quaternions* $\det M = 1$. *For real unit quaternions* $M = U =$ *unitary matrix:* $U\bar{U} = E$.

If we denote $a = q_0 - iq_3$, $b = -q_2 - iq_1$, *then for a real quaternion*

$$q_0 = \frac{1}{2}(a + \bar{a}); \quad q_1 = \frac{i}{2}(b - \bar{b}); \quad q_2 = -\frac{1}{2}(b + \bar{b}); \quad q_3 = \frac{i}{2}(a - \bar{a})$$

If on the other hand q *is a complex unit quaternion , the elements of* M *are not complex conjugate of each other and we can only write*

$$q \to M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q_0 - iq_3 & -q_2 - iq_1 \\ q_2 - iq_1 & q_0 + iq_3 \end{pmatrix},$$

where (a, b, c, d) *only satisfy the condition* $ad - bc = +1$.

Clifford C_3 algebra (biquaternions)

An associative noncommutative 8-dimensional algebra can be generated by 3 symbols e_1, e_2, e_3, satisfying the relations

$$e_1^2 = e_2^2 = e_3^2 = 1; \qquad e_r e_s + e_s e_r = 0, \quad s \neq r \quad (r, s = 1, 2, 3)$$

The 8 basis elements of the algebra are

$$1, \quad e_1, \quad e_2, \quad e_3, \quad e_1e_2, \quad e_1e_3, \quad e_2e_3, \quad e_1e_2e_3,$$

and their linear combinations are known as biquaternions. *The algebra can be represented by the 2×2 Pauli matrices in the following way:*

$$e_1 \to \sigma_1; \qquad e_2 \to \sigma_2; \qquad e_3 \to \sigma_3$$

$$e_1e_2 \to \sigma_1\sigma_2 = i\sigma_3; \qquad e_2e_3 \to \sigma_2\sigma_3 = i\sigma_1; \qquad e_3e_1 \to \sigma_3\sigma_1 = i\sigma_2$$

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = E$$

$$-ie_1e_2e_3 \to -i\sigma_1\sigma_2\sigma_3 = -i(i\sigma_3)\sigma_3 = \sigma_3^2 = E.$$

Clifford C_4 algebra ('Clifford numbers')

An associative noncommutative 16*-dimensional algebra is generated by the 4 symbols e_1, e_2, e_3, e_4 satisfying the relations*

$$e_r^2 = 1; \qquad e_r e_s + e_s e_r = 2\delta_{rs} \qquad (r, s = 1, 2, 3, 4)$$

$$\begin{cases} e_k \to \alpha_k = \sigma_1 \times \sigma_k & k = 1, 2, 3 \\ \qquad e_4 \to \alpha_4 = (\sigma_3 \times E) \end{cases}$$

$$\alpha_k^2 = (\sigma_1 \times \sigma_k)(\sigma_1 \times \sigma_k) = (E \times E) = E_4, \quad \alpha_4^2 = E_4,$$

where $\times$ is the Kronecker direct product. Hence

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \alpha_2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix},$$

$$\alpha_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad \alpha_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

These 4×4 matrices are sometimes recast in the shorthand

$$k = 1, 2, 3 \qquad \alpha_k = \begin{bmatrix} (0) & \sigma_k \\ \sigma_k & (0) \end{bmatrix}; \qquad \alpha_4 = \begin{bmatrix} E & 0 \\ 0 & -E \end{bmatrix}$$

where $(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *is the zero 2×2 matrix. The 16 basis elements of the algebra can be grouped into five sets as follows:*

$$\begin{array}{llllll} 1, & & & & & \\ e_1, & e_2, & e_3, & e_4, & & \\ e_1e_2, & e_2e_3, & e_3e_1, & e_1e_4, & e_2e_4, & e_3e_4, \\ e_1e_2e_3, & e_2e_3e_4, & e_3e_4e_1, & e_4e_1e_2, & & \\ e_1e_2e_3e_4. & & & & & \end{array}$$

Note that the anti-commutation relations $\alpha_r\alpha_s + \alpha_s\alpha_r = 2\delta_{rs}$ *are satisfied by an infinite number of other matrix representations, but they must be at least 4×4. It is remarkable that* ***Clifford*** *(1876) forged the mathematical tools used by* ***Dirac*** *(1928) in his formulation of the special relativistic free electron quantum mechanical wave equation:*

$$[\alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3} + \frac{i\alpha_4 m_0 c}{\hbar} + \frac{1}{c} E_4 \frac{\partial}{\partial t}]\Psi = 0$$

with x_j, $j = 1, 2, 3$ *the spatial coordinates,* t *time,* m_0 *the electron rest mass,* c *the speed of light in vacuum, and* Ψ *the electron 4-component complex spinor wavefunction.*

Clifford algebras in linear vector spaces

The quaternions, biquaternions and Clifford numbers are examples of hypercomplex numbers. But just as we have previously found it convenient to order sets of scalar components together to form vectors, so it becomes convenient to define hypercomplex vectors *as linear arrays of components which*

are hypercomplex numbers. We shall next show that the Pauli matrices can be considered as a vector base in a linear vector space L which obeys Clifford algebra in 2^n dimensions.

Let $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \dots)$ be vectors in L with base $(e_1, e_2, \dots, e_n)$. Then (employing the summations convention)

$$\boldsymbol{\alpha} = \alpha_i e_i, \qquad \boldsymbol{\beta} = \beta_j e_j$$

If the symbols e_i are ordinary vectors, say in real 3D Euclidean space, we define the wedge product as the antisymmetric dyadic

$$\begin{aligned}\boldsymbol{\alpha} \wedge \boldsymbol{\beta} &= \frac{1}{2}(\boldsymbol{\alpha}\boldsymbol{\beta} - \boldsymbol{\beta}\boldsymbol{\alpha}) \\ &= \frac{1}{2}(\alpha_1\beta_2 - \beta_1\alpha_2)(e_1e_2 - e_2e_1) + \frac{1}{2}(\alpha_2\beta_3 - \beta_2\alpha_3)(e_2e_3 - e_3e_2) \\ &\qquad + \frac{1}{2}(\alpha_3\beta_1 - \beta_3\alpha_1)(e_3e_1 - e_1e_3) = -\frac{1}{2}\left[I \times (\boldsymbol{\alpha} \times \boldsymbol{\beta})\right]\end{aligned}$$

also known as a bivector.

If however the symbols e_i are the Pauli spin matrices

$$e_1 \Rightarrow \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad e_1e_2 - e_2e_1 = 2ie_3$$

$$e_2 \Rightarrow \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad e_2e_3 - e_3e_2 = 2ie_1$$

$$e_3 \Rightarrow \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e_3e_1 - e_1e_3 = 2ie_2,$$

then the wedge product assumes the form

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \Rightarrow i\boldsymbol{\sigma} \cdot (\boldsymbol{\alpha} \times \boldsymbol{\beta})$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a hypercomplex vector. Now, the formal scalar product of $\boldsymbol{\alpha}$ and $\boldsymbol{\sigma}$ is the hypercomplex vector

$$(\boldsymbol{\alpha} \cdot \boldsymbol{\sigma}) = \begin{bmatrix} i\alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & -i\alpha_3 \end{bmatrix} = a \quad \text{(defined)}$$

It then follows that

$$ab = (\boldsymbol{\alpha} \cdot \boldsymbol{\sigma})(\boldsymbol{\beta} \cdot \boldsymbol{\sigma}) = (\boldsymbol{\alpha} \cdot \boldsymbol{\beta})E + i\boldsymbol{\sigma} \cdot (\boldsymbol{\alpha} \times \boldsymbol{\beta}).$$

Introducing the notation

$$(a \cdot b) \equiv (\boldsymbol{\alpha} \cdot \boldsymbol{\beta})E, \qquad a \wedge b = i\boldsymbol{\sigma} \cdot (\boldsymbol{\alpha} \times \boldsymbol{\beta}),$$

we find that the 'product' rule of two hypercomplex vectors in their vector space is

$$ab = (a \cdot b) + a \wedge b.$$

The wedge product has the usual distributive *and* associative *properties (λ a scalar)*

$$a \wedge (b + c) = a \wedge b + a \wedge c; \qquad (a + b) \wedge c = a \wedge \ + b \wedge c$$

$$\lambda(a \wedge b) = a \wedge (\lambda b) = (\lambda a) \wedge b; \qquad (a \wedge b) \wedge c = a \wedge (b \wedge c).$$

It is however anti-commutative*: $a \wedge b = -b \wedge a$.*

The wedge product of two vectors can be written symbolically in the determinant form

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = \frac{1}{2}(\boldsymbol{\alpha}\boldsymbol{\beta} - \boldsymbol{\beta}\boldsymbol{\alpha}) = \frac{1}{2}\sum_{i,j} t_{i,j}(\boldsymbol{e}_i \wedge \boldsymbol{e}_j)$$

$$t_{ij} = \alpha_i\beta_j - \beta_i\alpha_j = \begin{vmatrix} \alpha_i & \beta_i \\ \alpha_j & \beta_j \end{vmatrix}$$

One may then extend the concept of the wedge product to antisymmetric tensors of higher order. For example, the totally antisymmetric 3$^{\text{rd}}$ rank tensor has the determinant form in m dimensions ($i, j, k = 1, \ldots, m$):

$$\begin{aligned} t_{ijk} &= \frac{1}{6}\begin{vmatrix} \alpha_i & \beta_i & \gamma_i \\ \alpha_j & \beta_j & \gamma_j \\ \alpha_k & \beta_k & \gamma_k \end{vmatrix} \\ &= \frac{1}{6}(\alpha_i\beta_j\gamma_k + \alpha_j\beta_k\gamma_i + \alpha_k\beta_i\gamma_j - \gamma_i\beta_j\alpha_k - \gamma_j\beta_k\alpha_i - \gamma_k\beta_i\alpha_j) \end{aligned}$$

$$\text{or} \qquad \begin{aligned} \mathfrak{T} &= \frac{1}{6}(\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\gamma} + \boldsymbol{\gamma}\boldsymbol{\alpha}\boldsymbol{\beta} + \boldsymbol{\beta}\boldsymbol{\gamma}\boldsymbol{\alpha} - \boldsymbol{\gamma}\boldsymbol{\beta}\boldsymbol{\alpha} - \boldsymbol{\alpha}\boldsymbol{\gamma}\boldsymbol{\beta} - \boldsymbol{\beta}\boldsymbol{\alpha}\boldsymbol{\gamma}) \\ &= \frac{1}{6}[(\boldsymbol{\alpha}\boldsymbol{\beta} - \boldsymbol{\beta}\boldsymbol{\alpha})\boldsymbol{\gamma} + (\boldsymbol{\gamma}\boldsymbol{\alpha} - \boldsymbol{\alpha}\boldsymbol{\gamma})\boldsymbol{\beta} + (\boldsymbol{\beta}\boldsymbol{\gamma} - \boldsymbol{\gamma}\boldsymbol{\beta})\boldsymbol{\alpha}], \end{aligned}$$

where t_{ijk} reverses sign under the interchange of any two indices (i.e., interchange of any two rows in the determinant). This tensor is known as a trivector. It can be represented by the triple wedge product

$$\mathfrak{T} = \boldsymbol{\alpha} \wedge \boldsymbol{\beta} \wedge \boldsymbol{\gamma} = \frac{1}{6}\sum_{i,j,k} t_{ijk}(\boldsymbol{e}_i \wedge \boldsymbol{e}_j \wedge e_k).$$

This can be naturally expressed in terms of a suitable Clifford algebra. Thus, for $m = 4$ *(e.g. tensors in spacetime) we have in* C_4*:*

$$(\boldsymbol{\alpha}\cdot\boldsymbol{\Gamma})(\boldsymbol{\beta}\cdot\boldsymbol{\Gamma})(\boldsymbol{\gamma}\cdot\boldsymbol{\Gamma}) = (\boldsymbol{\alpha}\cdot\boldsymbol{\beta})\,\boldsymbol{\gamma}\cdot\boldsymbol{\Gamma}+(\boldsymbol{\beta}\cdot\boldsymbol{\gamma})\,\boldsymbol{\alpha}\cdot\boldsymbol{\Gamma}-(\boldsymbol{\gamma}\cdot\boldsymbol{\alpha})\,\boldsymbol{\beta}\cdot\boldsymbol{\Gamma}+\boldsymbol{\alpha}\cdot\boldsymbol{\Gamma}\wedge\boldsymbol{\beta}\cdot\boldsymbol{\Gamma}\wedge\boldsymbol{\gamma}\cdot\boldsymbol{\Gamma},$$

where

$$\Gamma_i\Gamma_k + \Gamma_k\Gamma_j = 2\delta_{jk}, \qquad \Gamma_i \wedge \Gamma_j \wedge \Gamma_k \equiv \epsilon_{ijkl}\Gamma_5\Gamma_l.$$

Here ϵ_{ijkl} *is the totally antisymmetric Levi-Civita tensor (*$\epsilon_{ijkl} \equiv 1$*), and* $\Gamma_5 = \Gamma_1\Gamma_2\Gamma_3\Gamma_4$.

C_4 *relations such as this are quite useful in particle-physics quantum field theory calculations (such as in QED or the Standard Model).*

Note that if $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ *are orthogonal*

$$\mathfrak{T} = -\frac{1}{6}[I \times (\boldsymbol{\alpha}\boldsymbol{\alpha} + \boldsymbol{\beta}\boldsymbol{\beta} + \boldsymbol{\gamma}\boldsymbol{\gamma})]$$

and if they are orthonormal

$$\mathfrak{T} = -\frac{1}{6}(I \times I) = \frac{1}{6}\epsilon_{ijk}.$$

The above results can be generalized to entities of the form

$$\begin{aligned} a &= \lambda I + (\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}), \\ b &= \mu I + (\boldsymbol{\beta}\cdot\boldsymbol{\sigma}), \qquad \textit{for which} \\ ab &= \{\lambda\mu + (\boldsymbol{\alpha}\cdot\boldsymbol{\beta})\}I + [\lambda\boldsymbol{\beta} + \mu\boldsymbol{\alpha} + i(\boldsymbol{\alpha}\times\boldsymbol{\beta})]\cdot\boldsymbol{\sigma}. \end{aligned}$$

In this case

$$\begin{aligned} a\cdot b &= \{\lambda\mu + (\boldsymbol{\alpha}\cdot\boldsymbol{\beta})\}I, \\ a\wedge b &= [\lambda\boldsymbol{\beta} + \mu\boldsymbol{\alpha} + i(\boldsymbol{\alpha}\times\boldsymbol{\beta})]\cdot\boldsymbol{\sigma}. \end{aligned}$$

Note that

$$\boldsymbol{\nabla}\wedge\boldsymbol{a} = \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{a} - \boldsymbol{a}\boldsymbol{\nabla}) = -\frac{1}{2}I \times \operatorname{curl}\boldsymbol{a}.$$

The development of a Clifford calculus is completed by defining the differential operator $\boldsymbol{\nabla} \equiv \sigma^k\partial_k$, $\partial_k = \frac{\partial}{\partial x^k}$, *where* σ^k *are base vectors.*

We call $\boldsymbol{\nabla}$ *the gradient operator, and decompose it into a symmetrical and antisymmetrical parts:*

$$\boldsymbol{\nabla}a = \boldsymbol{\nabla}\cdot a + \boldsymbol{\nabla}\wedge a = \boldsymbol{\nabla}\cdot a + i\boldsymbol{\nabla}\times a$$

[in line with the vector analog

$$\frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{a} - \boldsymbol{a}\boldsymbol{\nabla}) = -\frac{1}{2}\mathfrak{I} \times \text{curl } \boldsymbol{a}].$$

In conclusion: Clifford algebra over the field of real numbers is a linear vector space, closed w.r.t. the multiplication operation ab, defined for two hypercomplex vectors a and b.

Localization of Cerebral function (1810–1876)

As the 19th century progressed, the problem of the relationship of mind *and* brain *became especially acute as physiologists and psychologists began to focus on the nature of cerebral function. In a diffuse and general way, the idea of functional localization had been available since antiquity. A notion of 'soul', globally related to the brain, can be found in the works of* ***Pythagoras****,* ***Hippocrates****,* ***Plato****,* ***Aristotle****,* ***Herophilos*** *and* ***Galen****. The pneumatic physiologists of the Middle Ages thought that mental capacities were located in the fluid of the ventricles.*

As belief in animal spirits died, so too did the ventricular hypothesis. By 1784, when ***Jiri Prochaska*** *published his* De functionibus systematis nervosi*, interest had shifted to the brain stem and cerebellum. Despite these early views, the notion that specific mental processes are correlated with discrete regions in the brain and the attempts to establish localization by means of empirical observation – were essentially 19th century achievements.*

The first critical steps toward those ends can be traced to the work of ***Franz Josef Gall*** *(1758–1828,* Germany*). His work was followed by* ***Marie-Jean-Pierre Flourens*** *(1794–1867,* France*) who provided the first experimental demonstration of localization of function in the brain (1824). However, Flourens concluded, erroneously, that while* sensorimotor *functions are differentiated and localized sub-cortically, higher mental functions operate together, spread throughout the entire cerebellum.*

For more than 30 years this was the established view. Then in 1861 the first studies appeared that would lead to the rejection of this idea and to the establishment of patterns of functional localization in the cortex.

In the period between 1861 and 1876, ***Paul Broca*** *(1824–1880),* ***Gustav Theodor Fritsch*** *(1838–1927),* ***Eduard Hitzig*** *(1838–1907),* ***David Ferrier*** *(1843–1928) and* ***John Hughlings Jackson*** *(1835–1911) provided conclusive evidence that circumscribed areas of the cortex are involved in movement of the contralateral limbs. Their findings established* electrophysiology *as a preferred method for the participation of the hemispheres in motor function. Ferrier also examined the functions of the spinal cord, the medulla*[474]*, the corpora quadrigemina, and the cerebellum (1876).*

[474] For further reading, see:

- Bruun, R.D. and B. Bruun, *The Human Body Random House*, New York, 1982, 96 pp.
- Netter, F.H., *Atlas of Human Anatomy*, CIBA-GEIGY Corporation, 1993, 514 pp.

1873–1893 CE **Camillo Golgi** (1843–1926, Italy). Physician and neuropathologist. Discovered dendritic nerve cells called 'Golgi cells' and Golgi tendon spindle (1880). First to use silver nitrate to stain nerve tissue for study (1873). Shared with **Santiago Ramon y Cajal** the 1906 Nobel Prize for physiology or medicine.

Golgi was born at Corteno near Brescia. He studied medicine at the University of Pavia. He was a professor of General Pathology at Pavia (1881–1918) and senator (1900). In 1886 he demonstrated the life cycle and structure of malarial parasites.

1873–1893 CE **Ernst Abbe** (1840–1905, Germany). Physicist. Established the theoretical basis for the design of microscopes and laid the foundation of *Fourier optics* already in 1873, ahead of Rayleigh (1896).

Although the inventions of the telescope and microscope date back to the 16^{th} and 17^{th} centuries — with such eminent scientists as **Galileo**, **Huygens**, and **Newton** contributing to their development — their design was not placed on a strictly scientific basis until the beginning of the 19^{th} century, with the work of **J. Fraunhofer**.

In 1869, Abbe started to develop his theory of *image formation* which gave new insight into the laws underlying the formation of an image in the microscope in terms of light wave amplitudes. Abbe's study revealed that sharp imaging of surface elements perpendicular to the optical axis is achieved only under specific conditions (even though these elements were located close to the axis). Consequently he discovered an unambiguous relationship between the angles of rays of an arbitrarily wide bundle on the object side and those of corresponding rays on the image side (*Abbe sine condition*). The fulfillment of this condition can provide an optical system (capable of imaging an *axial* point without *spherical aberration*) with the additional ability of imaging the points of a small surface element lying perpendicular to the axis without the asymmetrical aberration called *coma*.

He also formulated the theoretical limits of *resolving power* (minimum distance between two points in an object that can be resolved in the image) and found that the resolving power is limited by the wavelength of the light used for producing the image, by the angular aperture 2α of the objective

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- Hole, J.W. Jr., *Human Anatomy and Physiology*, Wm.C. Brown Publishers: Dubuque Iowa, 1987, 966 pp.

- Sherwood, L., *Human Physiology – From Cell to Systems*, Wadsworth Publishing Co., 1997, 947 pp.

and by the refractive index of the medium filling the space between specimen and objective. As a measure for this, he established the *numerical aperture* $N = n\sin\alpha$.

Later, he searched for the basic laws governing the efficiency of optical instruments in general. Going beyond the findings of **Gauss**, **Listing** (1808–1882), **Helmholtz**, and **Neumann** (1832–1925), Abbe established the essential characteristics of any optical image formation in a form of a collinear relationship between image space and object space, in the geometric-optical sense. No one before Abbe had dealt with the basic theory of light intensity in optical instruments.

An illuminated object in front of a thin lens, becomes a source of secondary Huygens wavelets, and is diffracted by the finite lens.

Under the Fraunhofer approximation (far field), the amplitude distribution in the focal plane of a lens is the *spatial Fourier transform* of the amplitude distribution for the light field on the surface of the object, with an error not exceeding a certain insignificant phase multiplier, and a scale factor. The total diffracted field in the *focal plane* is given explicitly by

$$\Phi(x,y;k,L,f) = Be^{i\frac{k}{2f}(1-\frac{L}{f})(x^2+y^2)}\iint_{-\infty}^{\infty}\psi(x_0,y_0)e^{-\frac{ik}{f}(xx_0+yy_0)}dx_0dy_0,$$

where B is an arbitrary amplitude. Here $k=\frac{\omega}{c}$ is the wavenumber of the monochromatic plane wave that falls on the object, f is the focal length of the lens, L is the distance of the object plane in front of the lens, and ψ is the amplitude *aperture function* in the *object plane*.

If $L = f > 0$ there is no phase distortion in the focal plane of the lens. If, on the other hand, the amplitude distribution to be imaged is not in the lens' focal plane, but at arbitrary distance ℓ, the condition $\frac{1}{L}+\frac{1}{\ell}=\frac{1}{f}$ will secure (under the *geometrical optics approximation*, and more specifically the *stationary phase* method in the xy integral) that the diffraction pattern intensity distribution $|\Phi(x,y)|^2$ will be proportional to $|\psi(x_0,y_0)|^2$ with $(x,y) = -M(x_0,y_0)$ and M the magnification factor predicted by the ray theory. We then say that an *image* of the object has been formed on a plane normal to the optical axis at ℓ. The relation between them is

$$|\Phi(x,y)|^2 = |B|^2\left|\psi\left(\frac{-x}{M},\frac{-y}{M}\right)\right|^2,$$

where $M=\frac{\ell}{L}$.

Thus, the diffraction formation of an image can be split into two stages:

(1) The formation of a *diffraction pattern* of the object in the focal plane of a lens, and

(2) the transformation of the diffraction pattern in the focal plane of the lens into an *image* of the object in the image plane.

The entire information contained in the image of the object is also contained in the diffraction pattern of the object in the focal plane of the lens. If the diffraction pattern in the focal plane is altered [for example, if some maxima are eliminated or attenuated], the image of the object will change accordingly. The variation in the image of an object through a modification of its diffraction pattern from which the image is subsequently formed is called the *spatial filtering of the image* (Abbe, 1893).

Another model of optical imaging (influenced by earlier work of **Airy** and **Helmholtz**) was proposed by **Rayleigh** in 1896. The model visualizes an image as the superposition of *Airy patterns* (or more complicated patterns if aberration is present). The Airy pattern is the image (response) of the entire optical system to a point light source in the object. The wavefronts from it are limited in their entry into the imaging system by the finite aperture of the imaging lens, and the diffraction pattern of that aperture is formed in the image plane. Each point of the object is therefore imaged *not as a point* but as the Airy pattern of the aperture of the imaging lens.

The advantage of the Rayleigh method is in its validity even for *incoherent* illumination. For then, the Airy intensity patterns due to all the object points are simply additive. If it is coherent, there is interference, and mathematically one deals with the combination of the complex-amplitude Airy patterns.

Ernst Abbe was born in Eisenach. In 1861 he received his doctorate at the University of Jena, where he became a lecturer. In 1866 he met **Carl Zeiss** (1816–1888) and started a relationship which shaped his entire life. Zeiss was at that time running a small microscope factory in Jena. He realized the shortcomings of the trial-and-error development techniques of that era, and therefore employed Abbe to help him to improve the construction of microscopes. They later became partners. After Zeiss' death in 1888, Abbe established the Carl Zeiss foundation and transferred to it his entire fortune, worth millions. In his statute governing the foundation, Abbe introduced the 8-hour work day (before 1900!) and social benefits to his workers, far ahead of his time. He pondered the problems of human co-existence as deeply and as methodically as he approached the natural sciences[475].

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[475] An account of Abbe's life and work is given by H. Volkmann, *Applied Optics* **5**, 1720–1731, 1966.

Whatever Abbe initiated, be it as an economist, as a social scientist, or as a man of pure science, all his innovations led to products of unsurpassed quality.

1873–1895 CE **Friedrich Wilhelm Nietzsche**[476] (1844–1900, Germany). Philosopher and classical scholar. Professor of classical philology, Basel (1869–1879). Worked chiefly on philology, music, Greek antiquity and especially philosophy: *The Birth of Tragedy* (1872); *Essays* (1873–1878); *The Genealogy of Morals* (1887); *The Antichrist* (1888); *Thus Spoke Zarathustra* (1883–1892); *Twilight of Idols* (1889); *Ecce Homo* (1888); *Beyond Good and Evil* (1886); *Will to Power* (1888).

Nietzsche did not produce his own systematic doctrine. However, by virtue of his insight into the existential condition of modern man, his perception of the culture flattening of the industrial era, and his idea of breeding a new aristocracy — he has brought about a considerable impact on 20th century thought; many philosophers, writers and psychologists have been deeply influenced by him.

Nietzsche criticized religion. In his proclamation '*God is dying*' he meant that religion, in his time, had lost its meaningfulness and power over people. Thus, he argued, religion could no longer serve as the foundation for moral values.

Nietzsche sought a re-evaluation of all values; he said that the warriors who originally dominated society had defined their own strength and nobility as "good", and the weakness of the common people as "bad". Later, when the priests and the common people came to dominate society, they redefined their own weakness and humility as "good" and the strength and cruelty of the warriors whom they feared as "evil". Nietzsche criticized this second set of values because it was based on fear and resentment, and he associated these values with the Judeo-Christian tradition, repeatedly criticizing Christianity and Judaism.

Nietzsche's major psychological theory is that all human behavior is basically motivated by the will of people to overpower each other and gain control over their unruly passions. He thought that the self-control exhibited by ascetics and artists was a higher form of power than the physical bullying of the weak by the strong. Nietzsche's ideal, the *overman*, is the passionate man who learns to control his passions and use them in a creative manner.

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[476] For further reading, see:

- *A Nietzsche Reader*, Penguin Books, 1977, 286 pp.
- Strathern, P., *Nietzsche in 90 minutes*, Ivan R. Dee: Chicago, 1996, 83 pp.

Nietzsche unjustly suffered notoriety as a racist and forerunner of Nazism. This is largely due to the editing and misinterpretation of his ideas by Nazi propagandists.[477]

Nietzsche was born in Saxony, the son and grandson of Protestant ministers. He studied at the Universities of Bonn and Leipzig. He became a professor of classics at the age of 24 and retired from academic life (1879) because of poor health. Collapsing under the weight of the questions that he posed for himself, he suffered a mental breakdown (1889) from which he never recovered. Spent his last years in care of his mother at Naumburg and his racist sister Elizabeth at Weimar. A strong opponent of Wagner in art and Schopenhauer in philosophy.

Worldview XXIV: Nietzsche

* * *

"We live in the age of atoms, in an atomistic chaos. In the Middle Ages the opposite forces were held together by the Church, to some extent assimilated

[477] Nietzsche not only admired Jews for their spiritual mastery and grandeur, but vehemently dissociated himself from the '*damnable German antisemitism*', detesting '*the stupidity, crudity and pettiness of German nationalism*'. At the same time he criticized the Jews, whose historic legacy he denounced as being responsible for 'the slave-revolt in morals'. This aspect of his approach to Judaism was posthumously distorted in an effort to turn him into a spiritual godfather of German Nazism. This was mainly due to the making of his sister Elizabeth, who 'edited' (from various notes and rough drafts) and issued, posthumously, a book which Nietzsche had abandoned, *The Will to Power*; she thus recruited her brother to the Nazi antisemitic and racist propaganda machinery, promoting him falsely into a Nazi thinker. Nietzsche, however, detested all kinds of nationalism. Nevertheless, in his book *The Antichrist* (1888) he launched a vicious crusade against Judaism, which had undoubtedly great impact on German Nazis' ideology.

into each other under the strong pressure it exerted. Since this pressure has diminished, the opposing forces have rebelled against each other."

* * *

"If you want to achieve piece of mind and happiness — then believe, but if you want to be a disciple of truth — search."

* * *

"There are no facts, only interpretations."

* * *

"Nobody dies nowadays of fatal truths: there are too many antidotes to them."

* * *

"Everything that lives — suffers. There you have the essence of existence: To live is to want, to want is to suffer. We are fugitive, doomed to sickness, nostalgia and death."

* * *

"It is all over with priests and gods when man becomes scientific. Science is the first sin, seed of all sin, the original sin. This alone is morality: 'Thou shalt not know' — the rest follows."

* * *

"Do you believe that the sciences would ever have arisen and become great if there had not beforehand been magicians, alchemists, astrologers and wizards, who thirsted and hungered after recondite and forbidden powers?"

* * *

"What is it: is man only a blunder of God, or God only a blunder of men?"

* * *

"Morality is the best of all devices for leading mankind by the nose."

* * *

"In former times, fanaticism of the lust for power was inflamed by the belief that one was in possession of the truth. This lust bore such beautiful names that one could thenceforward venture to be inhuman with a good conscience *(to burn Jews, heretics and good books and exterminate entire higher cultures such as those of Peru and Mexico). The means employed by the lust for power have changed, but the same volcano continues to glow, the impatience and the immoderate love demand their sacrifice: and what one formerly did 'for the sake of God', one now does for the sake of money, that is to say, for the sake of that which now gives the highest feeling of power and good conscience."*

* * *

"The Jews are the most remarkable nation of world history because, faced with the question of being and not being, they preferred being at any price. Considered psychologically, the Jewish nation is a nation of the toughest vital energy, placed in impossible circumstances, voluntarily, from the profoundest shrewdness of its self-preservation, took the side of all decadence instincts — not *as being dominated by them, but because it divined in them a power by means of which one can prevail* against *'the world'."*

* * *

"The Jews are indeed the strongest, toughest, and purest race now living in Europe, who could gain mastery over it if so they wished. Yet, they desire nothing but accommodation and absorption, to put an end to their centuries of wandering — to which purpose it might be useful and fair to expel the antisemitic screamers from the country."

* * *

"There was only one Christian and he died on the cross."

* * *

"The deepest sense of existence is to be found in suffering and only art enables us to face this suffering and not run away from it."

* * *

"Insanity in individuals is rare — but in groups, parties, nations, and epochs, it is the rule."

* * *

"A politician divides mankind into two classes: tools and enemies."

* * *

"The Christian conception of God is one of the most corrupt conceptions of God arrived at on earth; perhaps it even represents the low-water mark in the descending development of the God type. God degenerated into the contradiction of life, *instead of being its transfiguration and eternal* Yes!, *into a declaration of hostility toward life, nature and the will to life, into a formula for every calumny of 'this world', for every lie about 'the next world', — into a God of nothingness denied, into a will to nothingness sanctified..."*

* * *

"When on a Sunday morning we hear the bells ringing, we ask ourselves: is it possible that this is going on because of a Jew crucified 2,000 years ago who said he was the son of God? A God who begets children on a mortal woman,

a sage who calls upon us no longer to work, no longer to sit in judgment, but to heed the signs of the imminent end of the world; a justice which accepts an innocent man as a substitute sacrifice; someone who bids his disciples to drink his blood; prayers for miraculous interventions; sins perpetrated against a god atoned for by a god; fear of a Beyond to which death is a gateway; the figure of the Cross as a symbol in an age which no longer knows the meaning and shame of the Cross — how gruesomely all this is wafted to us, as if out of the grave of a primeval past."

Can one believe that things of this sort are still believed in?"

* * *

"One day there will be associated with my name the recollection of something frightful — of a crisis like no other before on earth, of the profoundest collision of conscience, of a decision evoked against everything that until then had been believed in, demanded, sanctified."

* * *

"There will be wars such as there have never yet been on earth."

* * *

"The philosopher is not interested in truth, but only in 'my truth'."

1873–1903 CE *The years of the chromosome*[478]: elucidation of the essential facts of *mitosis* (the division of the cell nucleus into two daughter nuclei, during cell multiplication) *meiosis* (the formation of sex cells, i.e. eggs and sperms) and their relation to *heredity*. About ten main actors took part in this drama, most of them Germans, born in the interval 1831–1862. These were:

- **Anton Schneider** (1831–1890, Germany). Cytologist. Discovered visible changes in the nucleus during cell division. His account was the first accurate description of mitosis in animal cells (1873).
- **Eduard Strassburger** (1844–1912, Germany). Botanist. Observed in plant cells (1875) all the phenomena seen by others in more transparent animal cells. He noted the difference between *mitosis* and *meiosis* and its meaning for heredity. He coined *haploid* (a cell with half the usual complement of chromosomes and *diploid* (with the normal number). Discovered maturation (reduction division) of plant cells (1888).
- **Walther Flemming** (1843–1905, Germany). Coined the name *mitosis* (1879) and made first accurate accounts of chromosome numbers and accurately figured the *longitudinal splitting* of chromosomes (1882). Determined chromosome number as 24 in man (1898).
- **Edouard van Beneden** (1846–1910, Belgium). Zoologist. First studies *meiosis* (1883) and stressed the importance of the qualitative and quantitative equality of chromosome distribution to daughter cells (1887).
- **Wilhelm von Waldeyer-Hartz** (1836–1921, Germany). Anatomist. Coined the name *chromosome* (1888). Proposed the *neuron* theory of the nervous system (1891).
- **August Weismann** (1834–1914, Germany). Biologist. Proposed a theory of heredity (1892) whereby the germ-plasm, located in the sex cells, is the carrier of the heredity endowment, half the germ-plasm for an offspring coming from the mother and half from the father. The germ-plasm is transmitted unmodified to offspring such that acquired characteristics are not inherited. Described the process of meiosis, whereby the number of chromosomes is halved.
- **Oscar Hertwig** (1849–1922, Germany), zoologist, and **Theodore Boveri** (1862–1915, Germany), biologist. Showed independently that

478 *Chromosomes*: Dark staining strands in the cell nucleus comprising the material of *heredity* and containing two forms of *nucleic acid*, mostly *DNA* and some *RNA* combined with protein. Lengths of chromosomes constitute the *genes* and carry the *genetic code*. Chromosomes occur in pairs.

the nature of cell-division was one of maturation, i.e. pairs of chromosomes split, replicating each member before dispersing into four separate nuclei.

- **Walter Stanborough Sutton** (1877–1916, U.S.A.), biologist, and **Theodore Boveri** (1862–1915, Germany). Pointed out (1902) the parallelism between chromosome behavior and *Mendelism*, closing the gap between cytology and heredity. Sutton coined the name *gene* (1902), and proposed that chromosomes carry genes (factors which **Mendel** said could be passed from generation to generation).

1874 CE The first practical mechanical typewriter, by present-day standards, was commercially produced[479].

In 1868, **Christopher Latham Sholes** (1819–1890, USA), a Milwaukee senator and former postmaster, with his friend **Carlos Glidden**, an attorney, presented an improved version of their earlier (1867) invention. In 1873, E. Remington and Sons, a gun manufacturer, became interested in Sholes' typewriter and the company put the machine on the market in 1874. The first key-shift model was produced in 1878. The first successful *portable* typewriter appeared in the early 1900's. The *electric* typewriter came into use during the 1920's. Sholes devised the QWERTY keyboard which has been used from 1874 to the present day. It is the dominant survivor of dozen of keyboard designs that competed during the early years of the typewriter. The name derives from the arrangement of the letters in three rows:

Q W E R T Y U I O P
A S D F G H J K L
Z X C V B N M

[479] The first recorded typewriter patent was filed in 1714 by the British engineer **Henry Mill**, but there is no evidence that Mill actually built his proposed machine. The Italian **Pellegrino Turri** constructed a typewriter (1808), which allowed blind people to write more easily. Over the next six decades, several dozen inventors filed patents or built prototypes, but none of the machines entered mass production or attained commercial success.

1874 CE **George Johnstone Stoney** (1826–1911, Ireland). Physicist. Eccentric and original thinker. The first person to show how to deduce whether or not other planets in the solar system possessed a gaseous atmosphere, like the earth, by calculating whether their surface gravity was strong enough to hold on to one.

Postulated that an electric oscillator exists within the atom which generates its characteristic spectra. He called this oscillator "*electron*" and asserted that the magnitude of its charge is the same as that on a hydrogen atom during electrolysis.

In 1899, following the discovery of **J.J. Thomson** in 1897, **Lorentz** suggested that the name electron be given to the newly discovered particle.

In 1874, Stoney first discussed the possibility that there exist particular systems of units picked out by nature itself, what we might term 'natural units'[480]. To this end, he advocated the selection of natural constants that prevail throughout the universe such as the velocity of light, c (because it connects electrostatic and electromagnetic units); Newton's gravitation constant, G, and lastly, e, the unit of electric charge deduced from Faraday's Law. From these entities, a length, a mass, and a time can be constructed.

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[480] Compare with Planck's units (1906):

$$l_P = (\frac{G\hbar}{c^3})^{1/2} \sim 10^{-33} \text{ cm}; \qquad t_P = (\frac{G\hbar}{c^5})^{1/2} \sim 5 \times 10^{-44} \text{ s};$$
$$m_P = (\frac{c\hbar}{G})^{1/2} \sim 10^{-5} \text{ gm}.$$

Since the ratio $\frac{e^2}{\hbar c}$ is dimensionless and approximately equals $\frac{1}{137}$, we see that each Stoney's unit just differs from the corresponding Planck quantities by a numerical factor $\sim \frac{1}{\sqrt{137}}$.
For further reading, see:

- Davies, Paul, *The Accidental Universe*, Cambridge University Press, 1993, 139 pp.
- Cohen-Tannoudji, G., *Universal Constants in Physics*, McGraw-Hill, 1993, 116 pp.
- Rees, Martin, *Just Six Numbers*, Basic Books, 2000, 195 pp.
- Barrow, J.D., *The Constants of Nature*, Vintage, 2003, 352 pp.
- Seife, C., *Alpha and Omega*, Penguin Books, 2003, 294 pp.

The values of the units Stoney evaluated from these three standards were:

$$L_s = (\frac{Ge^2}{4\pi\epsilon_0 c^4})^{1/2} \sim 10^{-34} \text{ cm}; \qquad T_s = (\frac{Ge^2}{4\pi\epsilon_0 c^6})^{1/2} \sim 3 \times 10^{-45} \text{ s};$$

$$M_s = (\frac{e^2}{4\pi\epsilon_0 G})^{1/2} \sim 10^{-6} \text{ gm}.$$

These new natural units attracted little attention. There was no practical use for them and their significance was hidden to everyone, even Stoney himself. Natural units needed to be discovered all over again in the 20^{th} century.

Stoney was a professor at Queen's College, Galway (1852–1857) and Queen's University (1857–1882). His work included also investigations related to physical optics, molecular physics, kinetic theory of gases and planetary atmospheres.

Stoney was the uncle of **Geoge FitzGerald** and also an older distant cousin of **Alan Turing**.

1874–1888 CE **Henry Morton Stanley** (1841–1904, England). Explorer of Africa, discoverer of the course of the Congo River. With **David Livingstone** (1813–1873, England) made the African continent known to the world. Accomplished more geographical discoveries in Africa than any other explorer.

Stanley was born at Denbigh, Wales, and was baptized John Rowlands. His father died when the boy was two, and he spent most of his youth in an orphanage. At 18, he sailed as a cabin boy to New Orleans, La, where he was adopted by a merchant, Henry Morton Stanley, who gave him his name. When the Civil War began (1861), Stanley joined the Confederate Army but was soon captured. He later joined the Union Army.

After the war, Stanley became a newspaper reporter, and the New York Herald sent him to find Livingstone (1869). After many hardships he met Livingstone on Lake Tanganyika (1871), greeting him with the famous words: "Dr. Livingstone, I presume?" They stayed together until March 1872. In 1874 Stanley heard of Livingstone's death and returned to Africa to carry on his work. In November 1874 he left Zanzibar on a dangerous trip down the Congo River from its source to its mouth. The Congo region was rich in rubber and ivory, and Stanley tried to interest the British in the area. But he did not succeed, and, instead, the Belgian colonized the region as the Congo Free State. Stanley then led another expedition for the Belgians (1879–1883). In 1887 he made his last trip to Africa to rescue **Emin Phasha** during the African uprising. Stanley returned to Britain and served in Parliament until 1900.

The main achievements of Stanley are:

- Traced the Nile's source.
- Circumnavigated Lakes Victoria and Tanganyika, showing that the latter was an isolated lake.
- Traced the whole source of the Congo River.
- First to cross Africa from ocean to ocean.
- Traced Lake Albert's source to Lake Edward and identified the Ruwenzori Range as the fabled *Mountains of the Moon*.

1874–1895 CE **Jacobus Hendricus van't Hoff** (1852–1911, Holland). Chemist and physicist whose chief aim was the application of mathematics to chemistry. A founder of *stereochemistry* (1874). Discovered the laws of *chemical kinetics of weak solutions* and *osmotic pressure*[481] (1885–1886), for which he received the Nobel prize for chemistry in 1901.

Van't Hoff was born in Rotterdam. During 1869–1871 he studied at the polytechnic at Delft, in 1871 at the University of Leyden, in 1872 under F. Kekulé at Bonn, in 1873 at Paris and in 1874 at Utrecht. In 1878 he was appointed professor of chemistry in Amsterdam University, and in 1896 he went to Berlin, as professor at the Prussian Academy of Sciences.

In 1848 **Louis Pasteur** (1822–1895, France) discovered molecular asymmetry and demonstrated the existence of *optical isomers*: he had shown that a compound called sodium ammonium tartrate existed in two different crystalline forms. The two crystal types were identical to each other, except that they were mirror images, like right and left hands. They had identical properties, but solutions of one crystal would rotate polarized light in one direction and the other type in the opposite direction. This was among the earliest works dealing with 3-dimensional structure of molecules.

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[481] He stated that the *osmotic pressure* is given by $\delta p = (c_2 - c_1)\frac{kT}{v}$, where $\{c_2, c_1\}$ are the concentrations of solutions on both sides of the neutral semi-permeable membrane, k is the Boltzmann constant, T is the absolute temperature, and v is molecular volume of the pure *solvent*. In particular, if there is pure solvent on one side of the membrane $(c_1 = 0,\ c_2 = c)$, one arrives at van't Hoff's formula: $\delta p = \frac{nkT}{V}$, where n is the number of molecules of the *solute* in a volume V of solvent. (It is similar to the *Clapeyron formula*, if we replace gas pressure by osmotic pressure, volume of gas by volume of solution, and number of particles of gas by number of molecules of the solute.)

Osmosis plays an extremely important role in the world of animals and plants. Most of the partitions in living organisms and plants are semi-permeable. For example: the osmotic pressure in plant cells reaches several atmospheres, owing to which ground water can rise along the trunk of a tree to a large height. In the human body, osmosis plays an important part in the function of the kidneys. It also results in the transfer of water and various nutrients between the blood and the fluid of cells. Chemists use *reverse osmosis* to purify water.

The practical applications of osmosis were apparently known to **Moses** as early as ca 1230 BCE, for it is written in *Exodus* **15**, 23–25: "*And when they came to Marah they could not drink of the waters for they were bitter. And the people murmured against Moses, saying. What shall we drink? And he cried unto the Lord; and the Lord showed him a tree, which when he had cast into the waters, the waters were made sweet*".

Stimulated by Pasteur's work, van't Hoff, and independently **Joseph Achille Le Bell** (1847–1930, France), proposed an explanation for this phenomenon in 1874: the bonds formed by carbon could be considered as pointing to the corners of a regular tetrahedron, where it is attached to 4 different substituents. The molecule thus formed is *nonsuperposable* on its mirror image.

The discovery meant that in order to explain the constitution of certain organic compounds, the tridimensional arrangement of atoms in space must be taken into account.

When more than one C atom is considered, the possibilities become more complex, but they still remain mirror images of each other, identical in all properties except that they rotate polarized light in opposite directions. Such optically active compounds are of vital importance to the chemistry of life. This problem, however, could not be successfully attacked from the theoretical side until knowledge of the structure of atoms had been gained.

From 1874 to 1884 van't Hoff's attention was mainly given to the law of mass action; he classified and defined *orders of reactions*[482] in terms of number of molecules actively involved in the reaction. In 1884 he defined an index of *chemical affinity* as the maximum external work generated from a reversible isothermal reaction.

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[482] Reaction-rate (velocity) is the change in concentration of a reactant or product per unit time. A formula in brackets, for example $[A]$, represents the concentration of the indicated species in moles per liter. The reaction velocity is designated by $\frac{d}{dt}[A]$, and measured in mole/liter/min. Knowledge of factors which influence reaction velocity has practical consequences; such information includes its velocity and the temperature and concentration dependence of the velocity. An equation relating reaction rate $d[p]/dt$ and concentrations is called a *rate law*. The exponent of a concentration factor in the rate is the *reaction order* for that species. For example, the rate of oxidation of bromide ion by bromate ion in acidic aqueous solution

$$5Br^- + BrO_3^- + 6H^+ \rightleftharpoons 3Br_2 + 3H_2O,$$

is given by the law

$$\frac{d}{dt}[Br_2] = k[Br^-][BrO_3^-][H^+]^2,$$

where the rate constant k is fixed for a given temperature. The above reaction is of second order in hydrogen ions. The sum of the exponents is the total reaction order, namely 4. A rate law is a *differential equation*, that can be integrated. The order, and powers of individual species' concentrations in the rate, can be modified from their naive-counting values, e.g. by pre-equilibrium of fast steps in a reaction chain; Thus, in this example, we would have expected the powers of $[Br^-]$ and $[H^+]$ to be 5 and 6, respectively, had the reaction proceed in a *single* step. Empirical orders can be fractional and even *negative*.

From 1885 to 1895 he was engaged in the theory of solutions, and developed an analogy between dilute solutions and gases. He showed (1886) that the *osmotic pressure* of a solution is equal to the gas pressure which the solute would exert if all the solvent were removed, and the dissolved substance were left in the space in the condition of an ideal gas. The consequences of this theory had a remarkable influence on the progress of the science of biology.

1874–1896 CE **Marie-Esprit Léon Walras** (1834–1910, France). Mathematical economist. Made outstanding original contributions to modern economic theory; first to apply comprehensive mathematical analysis to general *economic equilibrium* in 'Elements d'economie politique pure' (1874–1877). Professor at Lausanne (1871–1892).

He showed how to formulate an independent system of equations relating to prices and quantities in all markets for the economy as a whole. These reflected conditions under which the market mechanism maximizes benefits and minimizes costs for the economy generally. Walras' model only recently became a source of inspiration for the new fast-developing specialization of mathematical economists.

1874–1909 CE **Karl Ferdinand Braun** (1850–1918, Germany). Physicist and inventor. Discovered the *crystal rectifier* (1874), and used it for the detection of radio waves (1901). Invented the *cathode*[483]*-ray tube* (1897). Discovered method in *wireless telegraphy* of boosting the outgoing signal at the sending station.

Another example is the reaction:

$$2HI \rightleftharpoons H_2 + 2I \rightleftharpoons H_2 + I_2$$

with its rate law

$$\frac{d[I_2]}{dt} = k_f[HI]^2 - k_r[H_2][I_2].$$

At equilibrium

$$k_f[HI]^2 = k_r[H_2][I_2],$$

yielding the equilibrium constant

$$K = \frac{k_f}{k_r} = \frac{[H_2][I_2]}{[HI]^2}.$$

In this example, the empirical reaction orders of the three relevant species are the same as their naive-counting values.

483 He painted the inside end of a glass tube with fluorescent paint; a cathode inside the tube emitted electrons which made the paint glow.

Braun was a professor of physics at the University of Tübingen (1885–1895), and director of the Physical Institute at Strasbourg (from 1895). Shared the 1909 Nobel prize for physics with G. Marconi.

One Earth — One Language (1670–1983)

1875 CE, May 20 *The Treaty of the Meter. An international conference signed a treaty to adopt* new *measurement standards for the kilogram and meter. Seventeen nations, including the United States, took part in the conference. The treaty set up a permanent organization, the International Bureau of Weights and Measures in Sèvere, France (IBWM).*

In the original metric system (1790), the unit of length equaled 10^{-7} *of the distance from the North Pole to the equator along the line of longitude going through Dunkirk, France and Barcelona, Spain. It was named* metre, *from the Greek word metron, meaning a measure.*

The unit of mass, the gram, was defined (1790) as the mass of a cubic centimeter of water at the temperature where it weights the most, namely at $4°C$ *(39°F).*

Before the development of the metric system, every nation used measurement units that had grown from local customs. However, the rapid development of science and technology made scientists realize that the 'tower of science' could not be built unless "the whole earth was of one language and of one speech" (Genesis **XI**, *1–6).*

Indeed, already in 1670, **Gabriel Mouton**, *the vicar of St. Paul church in Lyons, France proposed a decimal measurement system, namely the length of a minute (1/21,600) of the earth's circumference. In 1671* **Jean Picard** *(1620–1682), a French astronomer, proposed the length of a pendulum that swung once per second as a standard of length. Through the years, other people suggested various systems and standards of measurement.*

In 1790, the National Assembly of France requested the French Academy of Sciences to develop a standard system of weights and measures. A commission appointed by the Academy (including **Laplace**, **Lagrange**, **Lavoisier** *and* **Monge***) proposed a system that was both simple and scientific. This became*

known as the metric system*, and France officially adopted it in 1795 [but the government did not require the French people to use the new units until 1840].*

Thomas Jefferson (1743–1826), then the U.S. Secretary of State, recommended that the United States use a decimal system of measurement, but Congress rejected the idea.

In 1792, **Jean Baptiste Joseph Delambre** *(1749–1822, France) and* **Pierre Mechain** *(1744–1804, France), began their measurement of the arc of the meridian from Dunkirk to Barcelona.*

In 1821, **John Quincy Adams** *(1767–1848), the U.S. Secretary of State, proposed conversion to the metric system. Congress again rejected the proposal. In 1866 Congress made the metric system* legal *in the United States.*

The 1875 conference decided that units based on the size of the earth and mass of water are inaccurate for scientific purposes, and replaced it with a standard length demarced on a platinum-iridium bar and a standard mass of platinum-iridium.

In 1899, the new meter and kilogram standards, based on those adopted by the 1875 conference, were made and sent to all countries who signed the treaty. The kilogram standard was established in the form of cylinder made of a platinum-iridium alloy.

In 1960, the meter was redefined on the basis of the frequency of light emitted by a particular isotope of Krypton.

In 1975, the United States Congress passed the 'Metric Conversion Act' which called for voluntary *change over to the metric system. At that time, almost every country in the world had either converted to the system or planned to do so.*

In 1983, IBWM *settled on a new basis for the standard of length by making the meter exactly the distance traveled by light in vacuum in 1/299,792,458 seconds. From then on, every measurement of light's speed in vacuum, is by definition really a measurement of the* length *of one's laboratory distance-yardsticks in terms of the new* meter*!*

In 1992 it was discovered that the standard kilogram changed its mass by the amount 23 microgram. Consequently, a new mass standard will be established in the beginning of the 21st century.

1875–1876 CE **John Kerr** (1824–1907, Scotland). Physicist. Discovered the *electro-optical* Kerr effect (1875) and the *magneto-optical* Kerr effect (1876), among the first *non-linear* optical phenomena.

Kerr was born in Ardrossan, Ayrshire, the son of a fish merchant. He was educated at Glasgow University in theology and became a lecturer in mathematics at the Free Church Training College for Teachers, Glasgow (1857–1901), and set up a modest laboratory there. He was one of the first research students of **Kelvin**.

Amorphous substances (e.g. glass and other insulators, liquids with inversion symmetry such as carbon disulphite, paraffin oil, nitrobenzene etc.) which are isotropic under ordinary conditions, become *doubly refracting* (birefringent) when subjected to intense electric fields[484]. They then resemble uniaxial crystals with their optic axes parallel to the applied field.

Kerr showed that the effect was strongest when the plane of polarization was 45° to the field and zero when perpendicular or parallel. He found that the extent of the effect is proportional to the *square* of the applied field strength.

In the magneto-optical effect, a beam of plane polarized light was reflected from the polished pole of an electromagnet. The beam became elliptically polarized (with the major axis *rotated* from the original plane), when the magnet was switched on. The effect depended on the position of the reflecting surface w.r.t. the direction of magnetization and to the plane of incidence of the light.

In 1893 **F. Pockels** discovered a similar but much weaker effect in several crystals. Isotropic (cubic) crystals became uniaxial, and uniaxial crystals became biaxial, in a steady electric field of sufficient intensity, but the effect is *linearly*[485] proportional to the applied electric field.

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[484] Because the molecules tend to align under the influence of the electric field. The existence of the Kerr-effect makes it possible to construct an electrically controlled “light valve”: a cell with transparent walls contains a liquid between a pair of parallel plates. The cell is inserted between crossed *Nicols*; light is transmitted when an electric field is set up between the plates and is cut off when the field is removed. Thus one may *modulate* the intensity of a light beam.

[485] The change in the refractive index n due to an applied electric field $\boldsymbol{E}$ is expressed as $\delta(\frac{1}{n^2}) = r|\boldsymbol{E}| + g|\boldsymbol{E}|^2$. The first term is linearly proportional to the applied electric field and is known as the *Pockels effect*. The second term, which has quadratic dependence on the applied electric field, is known as the *Kerr electrooptic effect*. While the Pockels effect depends upon the *polarity* of the applied electric field, the Kerr effect does not.

The most popular crystals which display the electrooptic effect and are used for electrooptic devices are:

1875–1885 CE **Eduard Suess** (1831–1914, Austria). Geologist. Argued (1875) that mountains and continents were formed not by vertical uplift but by thrusting movements that crumpled and broke outer portions of the earth's crust. Postulated (1883) the existence of *Gondwana*, a great southern continent that broke up to form Africa, Antarctica, Australia, India, and South America. Coined the name *biosphere* as that part of the earth in which life exists.

In his five-volume work *Das Antlitz der Erde* he attempted to explain many geological features in terms of the earth's contraction as it cooled.

Suess was born in London. Professor at Vienna (1857–1901). Liberal[486] member of the *Landtag* of Lower Austria (1869–1896) and of the *Reichsrat* (1872–1896).

Suess' tectonic synthesis is one of the most remarkable achievements of the beginning of the 20th century. For the first time, science acquired an elaborate survey of the *whole earth*, a description of all the irregularities of its crust, the mountains, the seas and lakes, the valleys, the river beds and deltas — an attempt to explain the deformations and foldings which led to the earth's present appearance.

1875–1920 CE **Luther Burbank** (1849–1926, U.S.A.). Naturalist, plant breeder and horticulturist. Developed many new trees, fruits, flowers, vegetables, grains and grasses. He also improved many plants and trees already known. Many common foods we eat every day come from his experiments. Among the plants he developed are the *Burbank potato*, the *Shasta daisy*, the *spineless cactus*, and the *blackberry*.

Burbank was born in Lancaster, Mass. He became a gardener to support his widowed mother. In 1875 he moved to California, and settled in Santa

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- *Lithium niobate* ($LiNbO_3$)
- *Lithium tantalate* ($LiTaO_3$)
- *Potassium dihydrogen phosphate* (KH_2PO_4), known as KDP.
- *Ammonium dihydrogen phosphate* ($NH_4H_2PO_4$), known as ADP.
- *Gallium arsenide* (GaAs).

[486] Amidst the surging wave of antisemitism in Austria, Suess had the courage and the integrity to speak firmly in the Reichstrat (April 1894):
"What has been spoken, written, and done against the Jewish people during the last few years, has been a flagrant violation not only of our Constitution, but of the principles of human justice and Christianity."

Rosa. Stimulated by the works of Charles Darwin, he improved his crops by *crossing* and *selection*. He usually grew thousands of plants in the effort to produce one improved species.

The vast range of his experiments (he handled more than a million plants each year, and sometimes would perform more than a thousand simultaneous experiments), and his ability to detect and exploit even the smallest variation in a plant for a fast development of new properties — made him the greatest plant-creator ever. It is amazing that he could do so much without grasping the true *genetic significance* of his achievements, and despite his erroneous belief in the inheritance of acquired characteristics.

1876–1880 CE **Wilhelm Lexis** (1837–1914, Germany). Statistician-economist. A pioneer in the application of statistics to the social sciences.

Graduated from the University of Bonn (1859) in mathematics. Went to Paris (1861) to study social science and subsequently held positions at Strasbourg (1872), Dorpat (1874), Freiburg (1876), Breslau (1884) and finally Göttingen (1887).

1876–1886 CE **Eugen Goldstein** (1850–1931, Germany). Physicist. Performed many valuable experiments upon discharges through gases. Coined the names '*cathode rays*' (1876) and '*canal rays*' (1886). These were eventually shown to be *electrons* and *ions* respectively by Thompson (1897).

Goldstein was born at Gleiwitz. He was a pupil of Helmholtz and worked at the Potsdam Observatory from 1888. He was first to suggest that 'cathode-rays' emanating from the sun produce the *northern lights* and affect the magnetic field of the earth.

1876–1891 CE **Francois Edouard-Anatole Lucas** (1842–1891, France). Mathematician. Best known for his results in number theory (e.g. the converse of Fermat's little theorem[487]). In particular, he studied (1878) the *Fibonacci*

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[487] *Fermat's little theorem* (1640), known as FLT, states that for every number a not divisible by the prime p, the congruence $a^{p-1} = 1(\text{mod } p)$ is satisfied. The examples

$$2^{340} \equiv 1(\text{mod } 341), \qquad 3^{90} \equiv 1(\text{mod } 91)$$

where $341 = 11 \cdot 31$ and $91 = 7 \cdot 13$ are sufficient to show that the converse of FLT is not generally valid. However, it was shown by Lucas (1876) that by *imposing additional restrictions* on the number a in the above congruence, it is possible to express a converse form of FLT.

The theorem of Lucas states: When for some number a the congruence $a^{n-1} \equiv 1(\text{mod } n)$ holds, while no similar congruence with a lower exponent $a^t = 1(\text{mod } n)$, $0 < t < n-1$ is fulfilled, the module n is prime.

sequence and the associated *Lucas sequence*[488] named after him. Devised methods of testing primality, and used them (1876) to prove that the Mersenne number $2^{127}-1$ is prime (*Lucas test*[489]).

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[488] The *Lucas sequence*: $L_n = a_{n-1} + a_{n+1}$, where a_n is the general term of the Fibonacci sequence. It follows that $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, and consequently

$$L_n = \alpha^n + \beta^n, \qquad \alpha = \frac{1}{2}(1+\sqrt{5}\,), \qquad \beta = \frac{1}{2}(1-\sqrt{5}\,).$$

One can also show that if

$$F(x) = \sum_{n=1}^{\infty} a_n x^{n-1} \equiv (1-x-x^2)^{-1}$$

then

$$L(x) = \sum_{n=1}^{\infty} L_n x^{n-1}.$$

[489] If p is an odd prime, and $N = M_p = 2^p - 1$ is the corresponding Mersenne number, let $\{r_i\}$ be the sequence defined by $r_1 = 4$, $r_i = r_{i-1}^2 - 2$. Then N is prime if $r_{p-1} \equiv 0 (\mathrm{mod}\ N)$ and otherwise composite.

Example: Suppose that we want to know if the 5^{th} Mersenne number $2^5 - 1 = 31$ is prime. We start with the number 4 and we repeatedly square it, and subtract 2. At each step, however, we reduce the result by taking only the remainder when it is divided by 31. So our test goes like this

$$\begin{aligned}
r_1 &= 4\\
r_2 &= 4^2 - 2 = 14\\
r_3 &= 14^2 - 2 = 194 = 6\cdot 31 + 8\\
r_4 &= 8^2 - 2 = 62 = 2\cdot 31 + 0.
\end{aligned}$$

There is no remainder at the *fourth step*, and so the *fifth* Mersenne number is indeed a prime.

Lucas also sought algorithms for testing the primality of arbitrary numbers, not necessarily of the Mersenne type. He noticed that the sequence A_n defined by the recursion relation $A_{n+1} = A_{n-1} + A_{n-2}$ (known as the *Padovan sequence*) with initial values $A_0 = 3$, $A_1 = 0$, $A_2 = 0$ has the following bizarre property: Whenever n is a prime number, it divides A_n exactly! For example, $A_{19} = 209$ and $\frac{209}{19} = 11$. Until 1982 it was an open question whether or not the converse statement was true, since nobody had found such numbers (known as *Perrin pseudoprimes*).

Unfortunately Perrin pseudoprimes do turn out to exist! Adams and Shanks (1982) discovered the smallest one, $521^2 = 271441$ and J. Grentham proved

He used the algebraic identity

$$4x^4 + 1 = (2x^2 - 2x + 1)(2x^2 + 2x + 1)$$

to effect the factorization

$$2^{4n+2} + 1 = (2^{2n+1} - 2^{n+1} + 1)(2^{2n+1} + 2^{n+1} + 1)$$

with $x = 2^n$.

Lucas is also well known for his invention of the *Tower of Hanoi*[490] (1883) and other mathematical recreations.

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that there are many such pseudoprimes. The conjecture that no Perrin pseudoprimes exist was important, because the remainder on dividing A_n by n can be calculated very rapidly. If the conjecture were true this would have provided a speedy primality test and useful application to secret codes, which nowadays often hinge on properties of large primes.

490 Three pegs are fastened to a stand. There are n (usually 8) wooden discs, each with a hole in the center. The discs are of different radii, and at the start of the game all are placed on one peg in order of size, the biggest at the bottom. The problem is to shift the pile from one peg to another by a succession of steps, at each moving just one disc, and seeing to it that *at no stage is any disc underneath a larger one*. All three pegs may be used.

Let T_n be the minimum number of moves that will transfer the n discs from one peg to another under Lucas' rules. Then obviously $T_0 = 0$ (*no* moves are needed to transfer *no* disc), $T_1 = 1$ and $T_2 = 3$. Experiments with three discs show that the winning idea is to transfer the top two discs to the middle peg, then move the third, then bring the other two onto it. This gives us a clue for transferring n discs in general: We first transfer the $n-1$ smallest to a different peg (requiring T_{n-1} moves), then move the largest (requiring one move), and finally transfer the $n-1$ smallest back onto the largest (requiring another T_{n-1} moves). Thus we can transfer n discs (for $n > 0$) in *at most* $2T_{n-1} + 1$ moves:

$$T_n \le 2T_{n-1} + 1, \qquad \text{for } n > 0.$$

But there is no better way! At some point we must move the largest disc. When we do, the $n-1$ smallest must be on a single peg, and it has taken at least T_{n-1} moves to put them there. We might move the largest disc more than once, if we are not too alert. But after moving the largest disc for the last time, we must transfer the $n-1$ smallest discs (which must again be on a single peg) back onto the largest; this too requires T_{n-1} moves. Hence

$$T_n \ge 2T_{n-1} + 1, \qquad \text{for } n > 0.$$

These two inequalities, together with the trivial solution for $n = 0$, yield

$$\begin{aligned} T_0 &= 0, \\ T_n &= 2T_{n-1} + 1, \qquad \text{for } n > 0. \end{aligned}$$

He was wounded as a result of a freak accident at a banquet when a plate was dropped and a piece flew up and cut his cheek, and he died of Erysipelas a few days later.

Lucas Sequences and Primes

The number-sequences of ***Fibonacci*** *(1202 CE),* ***Fermat*** *(1637 CE), and* ***Pell*** *(1668), among many others, occupy a central role in modern number theory. Many particular facts were known about these sequences; however, the general theory was first developed by* ***Lucas*** *in a seminal paper which appeared in Volume I of the American Journal of Mathematics (1878). It is a long memoir with a rich content, relating Lucas sequences to many interesting topics, like trigonometric functions, continued fractions, the greatest common divisor and primality tests.* ***R. D. Carmichael*** *(1913) corrected errors and generalized results.*

Consider the polynomial $f(t; P, Q) = t^2 - Pt + Q$ *where* (P, Q) *are nonzero integers. From its roots*

$$\alpha = \frac{1}{2}(P + \sqrt{P^2 - 4Q}) \qquad \text{and}$$
$$\beta = \frac{1}{2}(P - \sqrt{P^2 - 4Q})$$

Lucas constructed the sequence of numbers

$$U_n(P, Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad V_n(P, Q) = \alpha^n + \beta^n \qquad n \geq 0$$

which are called the Lucas sequences associated with the pair (P, Q). *They*

Substituting $U_n = T_n + 1$ we get $U_0 = 1$, $U_n = 2U_{n-1}$ for $n > 0$. A solution to this recurrence relation is $U_n = 2^n$, leading to $T_n = 2^n - 1$ for $n \geq 0$, which can easily verified by mathematical induction. Thus, the problem can always be solved in $2^n - 1$ steps. Assuming that the player can make one transfer every second, with never a mistake, he must work more than 500,000 *million years* for $n = 64$.

obey the recurrence relations

$$U_n = PU_{n-1} - QU_{n-2}; \qquad V_n = PV_{n-1} - QV_{n-2}$$

$$U_0 = 0, \qquad U_1 = 1; \qquad V_0 = 2, \qquad V_1 = P.$$

Special cases:

(1) $P = 1,\ Q = -1;\quad U_n = U_{n-1} + U_{n-2}$

$$U_n = 0, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$$

$$\alpha = \frac{1}{2}[1 + \sqrt{5}], \quad \beta = \frac{1}{2}[1 - \sqrt{5}]$$

The $\{U_n\}$ *are immediately recognized as the Fibonacci numbers. They have the properties:*

- $F(x) = \sum_{n=1}^{\infty} U_n x^{n-1} \equiv (1 - x - x^2)^{-1}$
- $G(x) = \sum_{n=1}^{\infty} V_n x^{n-1} \equiv -\log(1 - x - x^2)$
- $(U_m, U_n) = U_{(m,n)}$, *where* $(,)$ *indicate the operation of taking the greatest common divisor:*

$$\textit{e.g.} \quad (U_{45}, U_{30}) = (1,134,903,170;\ \ 832,040) = 610 = U_{15}.$$

The associated series $V_n(1, -1)$ *yields the Lucas numbers*

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots$$

(2) $P = 3,\ Q = 2;\qquad U_n = 3U_{n-1} - 2U_{n-2}$

$$U_n = 2^n - 1 \quad (\textit{Mersenne numbers}); \quad V_n = 2^n + 1.$$

(3) $P = 2,\ Q = -1;\qquad U_n = 2U_{n-1} + U_{n-2}$

$$\alpha = 1 + \sqrt{2}, \qquad \beta = 1 - \sqrt{2} \quad (\textit{Pell numbers})$$

The Pell numbers $(n = 0, 1, 2, 3, \ldots)$

$$U_n = 0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, \ldots$$

and their companions $(n = 0, 1, 2, 3, \ldots)$

$$V_n = 2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, 6726, 16238, 39202, \ldots$$

are associated with the solutions of the Pell equation

$$z_n^2 - 2x_n^2 = (-1)^n,$$

where $z_n = \frac{1}{2}V_n, \qquad x_n = U_n$ *and*

$$z_n = \frac{1}{2}\{[1+\sqrt{2}]^n + [1-\sqrt{2}]^n\},$$

$$x_n = \frac{1}{2\sqrt{2}}\{[1+\sqrt{2}]^n - [1-\sqrt{2}]^n\}.$$

(4) $P = 4$, $Q = 3$;

$$\alpha = 3, \qquad \beta = 1; \qquad\qquad U_n = \frac{1}{2}(3^n - 1), \qquad V_n = 3^n + 1.$$

(5) $P = 11$, $Q = 10$;

$$\alpha = 10, \qquad \beta = 1; \qquad\qquad U_n = \frac{1}{9}(10^n - 1), \qquad V_n = 10^n + 1.$$

The numbers $10^n + 1$ *are known as* repunits *(repeated units).*

Having defined his sequences, Lucas asked: for what values of p *and* a *do the sequences* $\frac{a^p-1}{a-1}$ *and* $\frac{a^p+1}{a+1}$ *yield prime numbers. We know, for example, that* $10^n + 1$ *yields prime numbers for* $n = 1, 2$*:*

$$10^1 + 1 = 1; \qquad\qquad 10^2 + 1 = 101.$$

Clearly, if n *contains an odd factor,* $10^n + 1$ *cannot be prime because* $10^{(2k+1)d} + 1$ *is divisible by* $10^d + 1$*. Factorizations for* $n > 2$ *then yield*

$$\begin{aligned}
10^3 + 1 &= 11 \times 91,\\
10^5 + 1 &= 11 \times 9091,\\
10^7 + 1 &= 11 \times 909091,\\
10^6 + 1 &= 101 \times 9901,\\
10^{10} + 1 &= 101 \times 99009901,
\end{aligned}$$

and

$$
\begin{aligned}
10^4+1 &= 73\times 137,\\
10^8+1 &= 17\times 5882353,\\
10^{16}+1 &= 353\times 449\times 641\times 1409\times 69857,\\
10^{32}+1 &= 19841\times 976193\times 6187457\times 834427406578561,\\
10^{64}+1 &= 1265011073\times 15343168188889137818369\\
&\quad \times 515217525265213267447869906815873,\\
10^{128}+1 &= 257\times 15361\times 453377\times \text{ a prime of 116 digits.}
\end{aligned}
$$

In general, machine calculations have yielded the following results for the Lucas sequences:

- $\frac{a^p-1}{a-1}$ *is prime for:*

$a=2$; $p=3$; 5; 7; 13; 17; 19; 31; 61; 89; 107; 127; 521; 607; 1279; 2203; 2281; 3217; 4253; 4423; 9689; 9941; 11,213; 19,937; 21,701; 23,209; 44,497; 86,243; 110,503; 132,049; 216,091; 756,839; 859,433; 1,257,787; 1,398,269; 2,976,221; 3,021,377; 6,972,593

$a=3$; $p=3$; 7; 13; 71; 103

$a=5$; $p=3$; 7; 11; 13; 47; 149; 181

$a=6$; $p=3$; 71; 127

$a=7$; $p=5$; 13; 131; 149

$a=10$; $p=2$; 19; 23; 317; 1031

$a=11$; $p=17$; 19; 73

$a=12$; $p=3$; 5; 97; 109

- $\frac{a^p+1}{a+1}$ *is prime for:*

$a=2$; $p=3$; 5; 7; 11; 13; 17; 19; 23; 31; 43; 61; 101; 127; 167; 191; 199; 313; 347; 701

$a=3$; $p=3$; 5; 7; 13; 23; 43; 281

$a=5$; $p=5$; 67; 101; 103

$a = 6; \quad p = 3;\ 11;\ 31;\ 43;\ 47;\ 59;\ 107$

$a = 7; \quad p = 3;\ 17;\ 23;\ 29;\ 47;\ 61$

$a = 10; \quad p = 5;\ 7;\ 19;\ 31;\ 53;\ 67$

$a = 11; \quad p = 5;\ 7$

$a = 12; \quad p = 5;\ 11$

1876–1916 CE **John William Strutt** (**Lord Rayleigh**, 1842–1919, England). Distinguished physicist. In 1904 he was awarded the Nobel prize in physics for his discovery (with **William Ramsay**) of Argon (1894).

Rayleigh's life-work included numerous contributions on a wide range of subjects in chemical physics, capillarity and viscosity, theory of gases, optics, photography, color vision, acoustics, electromagnetism, elasticity, hydrodynamics and mathematical physics. His treatise on sound includes much original work on diffraction and scattering (1877–1878).

In 1885 he predicted the existence of elastic surface-waves produced by natural and artificial sources in the earth that now bear his name: *Rayleigh waves.*

In 1900 he derived the blackbody radiation formula for long waves, known as *Rayleigh's radiation formula*, which later became the starting point for **Planck**'s quantum theory[491]. In 1892 he generalized the principles of *dimen-*

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[491] *Rayleigh scattering* applies where the size of the scatterer is much smaller than the wavelength of the radiation, e.g.: scattering of light by molecules. For gases with a moderate number density N and refractive index n, the total scattering intensity per unit volume of material is given by $I = \frac{32\pi^3(n-1)^2}{3N\lambda^4}\langle S_0\rangle$, where $\langle S_0\rangle$, λ are respectively the mean energy flux density and the wavelength of the incident wave. It means that air molecules are more effective scatterers of the shorter wavelength (blue and violet) portion of the 'white' sunlight then the longer wavelength (red and orange) portion.

Thus, when we look in a region of the sky away from direct solar rays, we see predominantly *blue light* which was more readily scattered. On the other hand, the sun appears to have yellowish to reddish tint when viewed near the horizon; the solar beam must travel through a great deal of atmosphere before it reaches the observer. Hence most of the blue and violet will be scattered out, leaving a beam of light composed mostly of red and yellow (crimson). This latter phenomenon is particularly pronounced on a day when fine dust or smoke particles are present, or at sunset.

sional analysis as a logical procedure [*Phil. Mag.* **34**, 59–70].

In scientific stature, he is ranked alongside **Stokes** and **Kelvin**. The special feature of his work is its extreme accuracy and definiteness, combining highest mathematical acumen with refinement of experimental skill.

Possessing an immense range of knowledge, he has filled up lacunae in nearly every part of classical physics, and although he made no discovery which captured the popular imagination, he added analytic refinement to many branches of physics. His papers are often difficult to read but never diffuse or tedious, and his mathematical treatment is never needlessly abstruse, for when his analysis is complicated it is only because the subject-matter is so.

Rayleigh was born in Essex, the son of the second baron of a barony created in 1821 at George IV's coronation. He went to Trinity College, graduated as senior wrangler in 1865, and obtained the first Smith's prize of the year. He married in 1871 and from 1879 to 1884 was a Cavendish professor of experimental physics at the University of Cambridge, in succession to Clerk Maxwell. In 1887 he became a professor of natural philosophy at the Royal Institution of Great Britain. In 1908 he became the chancellor of Cambridge University.

Classical Thermal Physics

Thermal physics[492] *unites the disciplines of heat thermodynamics and statistical mechanics. Heat is a form of energy, and the science of heat deals with the changes in the properties of matter accompanying the transfer of energy through the mechanisms of work and the heat flow. It is an experimental science and the data obtained is represented by empirical laws, many of which can be justified a posteriori by theory.*

The name thermodynamics (from the Greek $\theta\varepsilon\rho\mu o s$ = *hot,* $\delta\upsilon\nu\alpha\mu is$ = *force) is given to that branch of physics which deals with the relations between*

[492] For further reading, see:

- Kittel, C., *Thermal Physics*, John Wiley & Sons: New York, 1969, 418 pp.

thermal and mechanical energy — the transformations of heat into work and vice versa.

Thermodynamics is an axiomatic science, and a purely mathematical discipline. The laws governing the transformation of energy through work and heat are derived from a few basic postulates, and important relations are obtained between the properties of systems in thermal equilibrium.

Thermodynamics – both its equilibrium and near-equilibrium branches – theory contributes to the understanding of matter and the physical world: it provides quantitative values for various properties of matter*, it gives information about the* possibility and impossibility of processes *and shows the direction of* evolution *of a macroscopic system. It also provides methods for testing the* stability *of a given state of the system.*

Thermodynamics is with us on a daily basis. With its help, we can understand how our car functions, predict why water boils, and understand the formation of clouds, rain, or snow.

The power of thermodynamics lies in the fact that it describes and correlates directly observable properties of diverse substances. This is done without using any detailed *knowledge of the internal structure of the bulk matter. With relatively few laws and variables, an impressive number of remarkable conclusions can be drawn for complex systems containing a great number of individual molecules. The specific nature of substances enters into the theory via a few parameters, such as heat capacities or molar volumes.*

The science of thermodynamics introduces the new concept of temperature; *it is absent from classical mechanics, as well as from the theory of electricity and magnetism and from atomic physics*[493]*. This concept is introduced through the Zeroth Law of thermodynamics:* There exists a property — temperature — such that the equality of temperature is a condition for thermal equilibrium between two systems or between two parts of the same system.

Most empirical laws, however fall outside the scope of thermodynamics, and the irreversibility of thermal processes seems to violate the more basic laws of mechanics. The fact is that thermodynamics would be an empty discipline, with no application in nature, were not matter composed of a myriad of molecules, or at least a large number of degrees of freedom.

The basis for both the empirical laws of heat and the postulates of thermodynamics is found in statistical mechanics*, and the latter gives a concrete picture of the abstractions of thermodynamics, such as entropy, temperature, internal energy and other system variables.*

[493] Historically, the subject of thermodynamics first arose *before* the atomic nature of matter was understood.

Mechanics is founded on certain general principles, such as the conservation of energy and momentum, that are applicable to the motion of interacting particles. When Newton's second law is translated into mathematical language, the solution of mechanical problem requires in turn the solutions of systems of 2^d *order ordinary differential equations, in which difficulties are encountered already with 3 interacting masses. Properties of matter in bulk (called* macroscopic *properties), as we ordinarily observe them, are the result of a collective actions of a large number of atoms and molecules*[494].

It is not only practically impossible, but also unnecessary to take into account the motions of each of these molecules in detail in order to determine the bulk properties of the matter, such as its pressure and temperature[495]. *Thus, to describe processes involving a very large number of particles, special methods must be devised. These methods are, by necessity, of a statistical nature. An important concept is that of the* probability of distribution *of the particles among the different dynamical states in which they may be found.*

A short historical survey is adequate at this point; Fire has fascinated and terrorized the human race throughout its history, but by the time of the great Ice Ages, humans had learned to tame fire into a constructive source of useful heat.

Until the end of the 18^{th} *century, fire was mainly used for heating, cooking, melting, and as a source of light. In some civilizations of antiquity, fire was a subject of worship and an agent of purification. In ancient Persia fire symbolized Ahura Mazda, the god of Zoroastrianism. In Greek mythology, Prometheus saved the human race by bringing the celestial gift of fire from the sun. The Aztec, Norse, and Hindu pantheons also had their gods of fire.*

That fire generates power can be seen by anyone watching a covered boiling kettle of water. ***Heron of Alexandria*** *made use of hot vapor in the construction of the first aeolipile (early gas turbine), which was used as a miracle agent in temples.*

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[494] In one cubic cm of gas at STP there are about 3×10^{19} molecules. The collective behavior of such a gigantic number of particles is basically the result of their quantum-mechanical *electromagnetic interaction*, since gravitational interaction plays only a minor role and the strong and weak interactions affect mainly nuclear processes. Familiar processes, such as melting and vaporization, diffusion, viscosity, thermal and electrical conductivities, thermionic emission, heat capacity, latent heat, etc. fall in this category of collective properties.

[495] *The temperature of a system in thermal equilibrium is a quantity related to the average kinetic energy per particle of the system*, the relation depending on the structure of the system.

The new industrial society at the turn of the 18^{th} *century needed coal at an ever increasing rate. Rising water in coal mines had to be eliminated, and muscle power was too slow and inefficient to do this.* ***Denis Papin*** *(1690) conceived the first vacuum-producing steam pump. A few years later such pumps were operational in English mines thanks to the ingenuity of* ***Thomas Savery*** *(1698) and* ***Newcomen*** *(1705). In* 1765, ***James Watt*** *modified the extremely inefficient Newcomen pump into a more efficient device by using a thermodynamic property, the* adiabatic expansion, *and also by introducing an automatic control inside the engine. The pump was transformed by* ***Fulton*** *(1807) into the first* steam engine. *This revolutionary discovery opened up the era of heat engines or* machines. *Engines drastically changed the nature of human societies, turning them into industrial societies. The development of heat engines was followed by a theoretical approach to the interrelationship between heat and mechanical motion.*

The idea that heat is a form of energy was first suggested by the works of ***Count Rumford*** *(1798) and* ***Davy*** *(1799). It was then stated explicitly by* ***R.J. Mayer*** *(1842), but gained acceptance only after the careful experimental work of* ***Joule*** *(1843 to 1849). The first theoretical analysis of heat engines was given by* ***Sadi Carnot*** *(1824), who thus became the founder of the new branch of macroscopic science — thermodynamics.*

Thermodynamic theory was formulated in consistent form by ***R.J.E. Clausius*** *and* ***Lord Kelvin*** *around* 1850, *and was greatly improved by* ***J.W. Gibbs*** *in several fundamental papers (1876–1878).*

The atomic approach to macroscopic problems began with the study of the kinetic theory of dilute gases. This subject was developed through the pioneering work of ***Clausius***, ***J.C. Maxwell*** *and* ***L.E. Boltzmann***. *Maxwell discovered the distribution law of molecular velocities in* 1859, *while Boltzmann formulated his fundamental integro-differential* transport equation *in* 1872. *The kinetic theory of gases assumed its modern form when* ***S. Chapman*** *and* ***D. Enskog*** *(1916–1917) approached the subject by developing systematic methods for solving the Boltzmann equation.*

The more general discipline of statistical mechanics also grew out of the work of ***Boltzmann***, *who (1872) further succeeded in giving a fundamental microscopic analysis of irreversibility and the approach to equilibrium. He was first to give the probabilistic interpretation of entropy.*

The theory of statistical mechanics was then developed further by the contributions of ***J.W. Gibbs*** *(1902). Although the advent of quantum mechanics has brought many changes, the basic framework of the modern theory is still the one which Gibbs formulated.*

Beginning in the 1970*'s, physicists recognized the close mathematical affinity between the statistical mechanics of condensed matter and fluctuations of*

fields in the second-quantized vacuum, and began applying this connection to advance both disciplines.

1876–1902 CE **Josiah Willard Gibbs** (1839–1903, U.S.A.). Theoretical physicist and chemist. Among the most prominent scientists produced by the United States.

In a path-breaking paper: "On the Equilibrium of Heterogeneous Substances" (1876–1878) Gibbs applied the principles of thermodynamics to the determination of chemical equilibrium (of chemical reactions rates). In this he helped lay the foundations of chemical thermodynamics and modern physical chemistry. He actually converted large parts of the physical chemistry of his day from an empirical to a deductive science. The importance of this work was soon recognized by Maxwell. The new concepts of '*free energy*' and '*chemical potential*' were introduced by Gibbs. Although Gibbs performed few experiments, his theory led to such practical results as the production of ammonia, dyes, drugs and plastics.

In his work he preferred a laconic, mathematician's style, making sure to say what was necessary for the logical structure of his argument — and little more. His spare and abstract style, and unwillingness to include a variety of examples and applications to particular experimental situations, made his work very difficult for potential readers. As a consequence, the literature of the 19^{th} century contains many rediscoveries of results already published by Gibbs. Such major figures as **Helmholtz** and **Planck** independently developed their own thermodynamic methods for treating chemical problems, quite unaware of the treasures concealed in his 1876–1878 paper[496].

While for Clausius and his contemporaries, thermodynamics was the study of heat and work, Gibbs eliminated these concepts from the foundations of the subject in favor of *state functions* — energy and entropy — and thermodynamics became the theory of properties of matter at equilibrium. Among other innovations, he gave an explicit derivation to Liouville's equation.

[496] In 1892 Rayleigh wrote to Gibbs urging him to expand on his ideas, saying that the original memoir was "*too condensed and too difficult for most, I might say all, readers*". Gibbs answered that he thought that his paper did seem "*too long*"....

In 1884 Gibbs coined the name '*statistical mechanics*', but he had not built molecular concepts into his papers on thermodynamics because he "*had no need for that hypothesis*", to paraphrase Laplace.

Gibbs is the father of our present-day '*vector analysis*'[497] (1881). He abstracted the vector and scalar concepts from the framework of Hamilton and Grassmann (thus disentangling them from the quaternion idea), and put them within a structure convenient to geometry and physics[498]. Similarly, he constructed the algebra and calculus of second-rank tensors (known as dyadics) on the basis of Grassmann's 'gap' products. From the point of view of physics, the Gibbs' vector calculus was a major simplification and improvement of Hamilton's quaternions. When attacked by the quaternionophil P.G. Tait, he replied: "*The world is too large, and the current of modern thought is too broad, to be confined to the* '*ipse dixit*' *even of a Hamilton*".

In 1886 he emphasized the superior generality of Grassmann's indeterminate product in dyadic and matrix algebra, over the unique product insisted upon by Hamilton. The vectors of Gibbs gradually displaced quaternions as a practical applied algebra. In the end, however, quaternions returned to physics under the guise of *Pauli matrices*, representing the action of the angular-momentum operators on *quantum-mechanical spinors*.

Gibbs was born in New Haven, Connecticut. His father was a professor of sacred literature in Yale Divinity School. He entered Yale College in 1854, graduated in 1858, and received his doctorate of engineering in 1863. He taught Latin and natural philosophy until 1866, when he went to Europe, studying in Paris (1866–1867), Berlin (1867), and Heidelberg (1868). Returning to New Haven in 1869, he was appointed professor of mathematical physics at Yale College in 1871[499], a position he held until his death. His interest in thermodynamics arose while endeavoring to improve the governor of James Watt's steam engine. In analyzing its equilibrium, he began to develop methods by which equilibriums of chemical processes could be calculated.

Gibbs remained a bachelor, living in the household of his surviving sister. He was a man of few words. Once, at a gathering of scientists, he was asked to give a talk on the subject "The role of mathematics in the physical sciences". He rose and issued just four words: "*Mathematics is a language*".

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[497] For further reading, see:

- Gibbs, J.W., *Vector Analysis*, Dover, 1960, 436 pp.

[498] Gibbs introduced the *notation* $\nabla \cdot \boldsymbol{a}, \quad \nabla \times \boldsymbol{a}$ for the divergence and curl of a vector field $\boldsymbol{a}$, respectively.

[499] Yale University refused for seven years to pay a salary to Willard Gibbs, already famous in Europe, on the ground that his studies were "non relevant".

Origins of Classical Statistical Physics[500] *(1850–1902)*

In 1865, ***Joseph Loschmidt***[501] *(1821–1895) gave the first estimate of Avogadro's number N (number of molecules in 22.4 liters of gas at standard temperature and pressure). Calculating from the newly developed kinetic theory of gases, he obtained the approximate value of* $N = 6 \times 10^{23}$*. Thus, under conditions prevailing (say) in a living room, the number of molecules in 1 cm*^{3} *of nitrogen gas is of the order of* 10^{20}*. This astronomical size of N is the major reason why attempts to arrive at thermophysical results from a purely mechanical point of view are foredoomed to fail without the use of statistics.*

Although a number of statistical problems (e.g., the explanation of some properties of gases on the basis of the notion of molecular motions), were considered by ***Newton***, ***D. Bernoulli****, and a number of other scientists back in the* 18^{th} *century, the appearance of statistical physics as an independent branch of physics dates to the second half of the* 19^{th} *century.*

In 1857, ***Clausius*** *clearly indicated that heat energy is the kinetic energy of random motions of molecules.*

In 1859 he introduced the useful concept of the mean free path and gave a correct molecular-kinetic explanation of the phenomena of thermal conductivity and viscosity. Also in 1859, ***Maxwell*** *fused together statistical ideas with those of mechanics in a kinetic theory that resulted in the law of distribution of the velocities of gas molecules, that now bears his name. In his kinetic theory he analyzed events involving single molecules, making special assumptions about the nature of inter-particle forces, while assigning to probability notions a mere subsidiary role.*

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500 To dig deeper, see:

- Feynman, R.P., *Statistical Mechanics*, Perseus Books, 1998, 354 pp.
- Ruhla, C., *The Physics of Chance*, Oxford University Press, 1992, 222 pp.
- Harris, S., *An Introduction to the Theory of the Boltzmann Equation*, Dover, 2004, 221 pp.
- Brown, A.F., *Statistical Physics*, Edinburgh University Press: Edinburgh, 1968, 307 pp.
- Zeldovich, Ya.B. et al., *The Almighty Chance*, World Scientific, 1990, 316 pp.

501 In the same year Loschmidt also obtained estimates of *molecular diameters* from measurements of liquid density and gaseous viscosity.

A further fundamental development was due to ***Boltzmann*** *(1877). Instead of scrutinizing separate microscopic events, he supplemented the mechanical laws of general validity by far-reaching* probability hypotheses *of comparable importance, thus subjecting the atomic particles themselves to statistical analysis. This choice was made by* ***Maxwell*** *as well in his kinetic theory, but his efforts were confined to an examination of the distribution of particles w.r.t. their velocity components only.*

Boltzmann generalized this procedure to encompass the distribution of particles in relation to position coordinates as well. ***Gibbs*** *(1884–1902) crowned the achievements of Clausius, Maxwell, and Boltzmann with decisive researches of his own. In his works, statistical physics obtained a fundamental substantiation suitable for arbitrary systems, and not only gaseous ones.*

The Gibbs ensemble *is treated at present as a fundamental principle whose role in statistical physics can be compared with that played by Newton's equations in classical mechanics or by Maxwell's equations in electrodynamics. Gibbs' book* Elementary Principles in Statistical Mechanics *(1902) played the same role in statistical physics as Maxwell's* Treatise *did in electrodynamics; the molecular-kinetic substantiation of the phenomenological science of classical thermodynamics, commenced by Boltzmann, was completed by Gibbs.*

Thus, a new discipline was established that succeeded in deriving the facts of phenomenological thermodynamics from postulates more fundamental *than the laws of thermodynamics. Moreover, it also provided numerical values for individual macroscopic properties, and finally alerted us to the possibility of rarely occurring fluctuation effects — a direct result of the conceptual researches initiated mainly by* ***Maxwell***, ***Boltzmann***, ***Gibbs*** *and* ***Einstein***. *This branch of physics, called* statistical thermodynamics, *seeks to deduce the thermodynamic properties of matter and energy from the laws of governing the behavior of its microscopic, or atomic, constituents.*

Let us highlight two fundamental ideas of this approach, beginning with Gibbs' derivation of Liouville's theorem, *using concepts of* Hamiltonian dynamics.

Consider a closed system of gas (molecules, electrons, stars) described by N generalized coordinate vectors $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N$, and generalized momenta $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N$. The $6N$-dimensional space spanned by the vectors $(\boldsymbol{p}_1 \cdots \boldsymbol{p}_N;\ \boldsymbol{q}_1 \cdots \boldsymbol{q}_N)$ shall be called the Γ-space or the phase space of the system. A point *in Γ-space, having $6N$ scalar coordinates, represents a state of the entire gas at a specific moment, and is known as a* representative point *or the* phase point *of the system. In terms of this phase point in Γ-space, the temporal development of the mechanical system, whose state is represented by it, can be surveyed geometrically.*

The position of a phase point at a given time t_0, corresponding to some initial state of the system, might be a matter for arbitrary decision. Once this choice has been made, Hamilton's equations of motion, viz., $\dot{q}_i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$ $(i = 1, 2, \ldots, 3N)$, uniquely determines the position of the phase point at any other (earlier or later) time t. The phase point therefore describes in the course of time a curve in Γ-space, known as a phase orbit *or* trajectory; *due to the uniqueness of the solutions of the above equations, each point in Γ-space is traversed by only one trajectory — that is, trajectories cannot intersect themselves or one another. Each position of the phase point at any given time describes a* microstate *of the system. Since the gas molecules are continually in motion, every conceivable microstate (compatible with constraints and conservation laws) will be approached arbitrarily closely sooner or later if the allowed Γ-space region is finite in volume.*

*A complete specification of the system's state (phase point) requires knowledge of the motions and positions of all microscopic particles at some time. In practice, however, we do not have (and are not interested in) the detailed information that is required to specify a particular microstate. We are usually observing the "*average*" behavior of a system with a given set of* macroscopic *properties (such as density distribution, velocity profile, pressure, temperature, etc.) which constitute a* macrostate *of the system.*

It is obvious that a very large number of microstates all correspond to a given macroscopic condition of the gas. Through macroscopic measurements we would not be able to distinguish between two different microstates that satisfy the same macroscopic condition. Thus when we speak of gas under certain macroscopic condition, we are in fact referring to a practically infinite number of microstates. In other words, we refer to a collection of systems, identical in composition and macroscopic conditions but existing in different microstates. Gibbs called such a collection of systems an ensemble.

Each member of the ensemble is a virtual *copy of the* real *system. The virtual systems are, of course, not identical in all respects. Indeed, the similarity extends only as far as the Hamiltonian functions of the systems, and the virtual systems may differ vastly among themselves and from the real system with respect to the configuration of the* velocities and positions of their particles. *Thus each member of the ensemble is an independent system* with its own phase point. *So we turn our attention from the individual phase point representing the real system to the assembly of phase points in Γ-space representing the totality of every system of the ensemble.*

Because these systems differ in their microscopic states, their corresponding phase points will correspondingly occupy different positions, so that the assembly of points will spread out in a "cloud" over a finite region in Γ-space.

This diversity reflects our ignorance about the microstate of the real system that serves as a prototype for the construction of the virtual systems.

One of the main goals of statistical thermodynamics is to find the correct statistical distribution of phase points under given macroscopical physical conditions (mechanical, chemical and thermal forces, initial and boundary conditions, etc.). To achieve this goal, one must first inquire how an assembly of phase points, initially arranged in phase space in an arbitrary manner, will evolve with the passage of time.

The situation may be conveniently described by a density function $\rho(p,q,t)$, *where* (p,q) *is an abbreviation for* $(\boldsymbol{p}_1\cdots\boldsymbol{p}_N;\ \boldsymbol{q}_1\cdots\boldsymbol{q}_N)$, *so defined that* $\rho(p,q,t)d^{3N}pd^{3N}q$ *is the expected fraction of representative points which at time* t *are contained in an infinitesimal volume element* $d^{3N}p\,d^{3N}q = dq_1dq_2\cdots dq_{3N}dp_1dp_2\cdots dp_{3N}$ *of* Γ*-space centered about the point* (p,q).

An ensemble is completely specified by $\rho(p,q,t)$*: Given* $\rho(p,q,t)$ *at any given time, the evolution of the random phase point with time is governed by the Hamiltonian* $H(p_1\cdots p_{3N};\ q_1\cdots q_{3N})$ *through to the equations of motion*

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i} \quad (i = 1,\ldots,3N).$$

Since H *does not depend on time derivatives of* p *and* q*, these Hamilton equations guarantee that the locus of a phase point is either a simple closed curve or a curve that never intersects itself. Moreover, since the total number of microstates in an ensemble is conserved, the number of phase points leaving an arbitrary volume* ω *in* Γ*-space with surface* S *must be equal to the rate of decrease of the number of phase points in the same volume. Hence*

$$\frac{d}{dt}\int_\omega \rho d\omega = \int_S dS(\boldsymbol{n}\cdot\boldsymbol{V}\rho),$$

where $\boldsymbol{V}$ *is a* $6N$*-dimensional vector with components*

$$\boldsymbol{V} = (\dot{p}_1,\dot{p}_2,\ldots,\dot{p}_{3N};\ \ \dot{q}_1,\dot{q}_2,\ldots,\dot{q}_{3N}),$$

and $\boldsymbol{n}$ *is the vector locally normal to the surface* S*. Using the divergence theorem in* $6N$*-dimensional space, we obtain the continuity equation for the phase-space density function:*

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{V}) = 0,$$

where ∇ *is the* $6N$*-dimensional gradient operator*

$$\nabla \equiv \left(\frac{\partial}{\partial p_1},\frac{\partial}{\partial p_2},\ldots,\frac{\partial}{\partial p_{3N}};\ \ \frac{\partial}{\partial q_1},\frac{\partial}{\partial q_2},\ldots,\frac{\partial}{\partial q_{3N}}\right).$$

Performing the indicated differentiations and using the equations of motion in the form

$$\frac{\partial \dot{p}_i}{\partial p_i} + \frac{\partial \dot{q}_i}{\partial q_i} = 0,$$

there emerges the mathematical formulation of Liouville's theorem:

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \sum_{i=1}^{3N} \left(\frac{\partial \rho}{\partial p_i}\dot{p}_i + \frac{\partial \rho}{\partial q_i}\dot{q}_i \right) = 0.$$

Geometrical interpretation: if we 'ride' on the trajectory of a phase-point in Γ-space, we will at all times measure the same density of representative points in its neighborhood. Hence the distribution of phase-points moves in Γ-space like an incompressible fluid. *Also, the co-moving (Euclidean) spatial volume element $d^{3N}pd^{3N}q$ changes its* shape, *but retains its* volume *throughout.*

If the virtual systems are required to be closed *and* conservative[502] *[so that H does not depend explicitly on time, and can be put equal to a constant E, the energy of the system], and if $\rho = \rho(H)$ it then follows that $\frac{\partial \rho}{\partial t} \equiv 0$ or $\rho = \rho(p, q)$. This means that the density in phase space does not vary with time and depends only on energy.*[503]*The ensembles defined in this way is called a* stationary ensemble, *and is said to be in* statistical equilibrium. *The equilibrium situation is thus guaranteed at all times if the phase points at an arbitrary instant t_0 are distributed in Γ-space with a density $\rho(p, q, t_0) = \rho(H)$.*

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[502] In Gibbs' large number of similar simultaneous systems (ensemble), all the gases are composed of the same number of molecules as the gas in the real system, and they are placed in vessels having the same shape. The energies of the different systems, however, are allowed to extend from E to $E + dE$, where dE is very small. This spread of energy is essential to the proof of Liouville's theorem, but once the incompressibility has been established, we may restrict our attention to the systems of energy E.

[503] This means that a sufficiently long time has elapsed so that macroscopic equilibrium is achieved, and it assumes there are no other conservation laws but energy, or that if there are, the system constraints prevent the system from exchanging the other conserved quantities with its environment. It is also assumed that *the phase point spends on average equal amounts of time* in all microstates compatible with the conservation laws (*ergodic hypothesis*). If the system is allowed to exchange with its environment other conserved quantities (molecules of various species, volume, electric charge etc.), then ρ will in general depend on *all* such conserved quantities, not just H.

Note that $\rho(p,q)$ *is actually the probability per unit volume that a phase point be found in an infinitesimal volume of the* Γ*-space. Being an* N*-particle distribution function we change its notation to* f*, where*

$$\int f(\boldsymbol{W}_1, \boldsymbol{W}_2, \ldots, \boldsymbol{W}_{_N}, t) d^6\boldsymbol{W}_1 \cdots d^6\boldsymbol{W}_{_N} = 1$$

and $\boldsymbol{W}_i \equiv (\boldsymbol{q}_i, \boldsymbol{p}_i)$ *denote the location of an individual particle in its 6-dimensional phase subspace. If the forces are conservative*

$$d\boldsymbol{p}_i/dt = -\partial\Phi_i/\partial\boldsymbol{q}_i,$$

where Φ_i *is the potential at particle* i *due to the other particles. Liouville's theorem then assumes the form*

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{N}\left[\boldsymbol{p}_i \cdot \frac{\partial f}{\partial \boldsymbol{q}_i} - \frac{\partial \Phi_i}{\partial \boldsymbol{q}_i} \cdot \frac{\partial f}{\partial \boldsymbol{p}_i}\right] = 0.$$

A special case of Liouville's theorem is known as the collisionless Boltzmann equation. Boltzmann, unlike Gibbs, used a 6-dimensional (not $6N$*-dimensional) phase space of a single representative molecule, which* ***Paul*** *and* ***Tatyana Ehrenfest*** *(1912) later termed the* μ*-space. A point in this space is* $\boldsymbol{W} = (\boldsymbol{q}, \boldsymbol{p})$*, and the velocity of its flow is given by the 6-vector* $\dot{\boldsymbol{W}} = (\boldsymbol{p}, -\nabla\Phi)$*.*

With this notation, the above Liouville's theorem is simplified to

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \boldsymbol{p} \cdot \nabla f - \nabla\Phi \cdot \frac{\partial f}{\partial \boldsymbol{p}} = 0.$$

Given Φ*, it is a differential equation for the unknown distribution function* $f(\boldsymbol{r}, \boldsymbol{p}, t)$*. Here,* $\frac{Df}{Dt}$ *represents the rate of change of the density of phase points as seen by an observer who moves through phase space with a molecule at velocity* $\dot{\boldsymbol{W}}$*, and* $\frac{Df}{Dt} = 0$ *implies that the phase-space density* f *around the moving phase point of a given molecule remains the same.*

Jeans *(1919) first applied the Boltzmann collisionless equation to stellar dynamics, where the role of gas molecules is played by non-colliding stars. Integrating the equation over all possible velocities, he assumed that:*

(1) the range of velocities over which we are integrating does not depend on time,

(2) $\boldsymbol{\Phi}$ *does not depend on* $\boldsymbol{p}$*,*

(3) $f(\boldsymbol{r}, \boldsymbol{p}, t) = 0$ *for sufficiently large* $|\boldsymbol{p}|$ *(there are no stars that move infinitely fast).*

This led him directly to the continuity equation (obtained by integrating the Boltzmann equation over all $\boldsymbol{p}$*)*

$$\frac{\partial \nu}{\partial t} + \operatorname{div}(\nu \boldsymbol{u}) = 0$$

where

$$\nu = \int f d^3\boldsymbol{p}, \qquad \boldsymbol{u} = \frac{1}{\nu}\int f\boldsymbol{p} d^3\boldsymbol{p}.$$

Multiplying the Boltzmann equation by $\boldsymbol{p}$ *and integrating again over all momenta (subject to the former assumptions), one arrives at the analog of Euler's equation of fluid flow:*

$$\nu\frac{\partial \boldsymbol{u}}{\partial t} + \nu\boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nu\nabla\Phi - \operatorname{div}\left[\nu(\mathfrak{Q} - \boldsymbol{u}\boldsymbol{u})\right],$$

where $\mathfrak{Q} = \frac{1}{\nu}\int \boldsymbol{u}\boldsymbol{u} f d^3\boldsymbol{p}$ *is a symmetric stress tensor.*[504]

When encounters of molecules (stars) are taken into account, the phase-space density of individual molecules changes with time and we may write $\frac{Df}{Dt} = M(f)$*, where the collision term* M *denotes the co-moving rate of change of* f *due to encounters (collisions). Boltzmann derived the explicit form of* M *under the assumptions:*

(1) only binary elastic collisions are taken into account (dilute gas);

(2) the walls of container are ignored;

(3) the effect of the external forces on the collision cross-section is ignored;

(4) the velocity of a molecule is uncorrelated with its position (molecular chaos, valid for sufficiently low gas densities).

When these assumptions are translated into mathematics, and the physics of elastic binary collision is applied, the end result is the Boltzmann transport equation:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}_{\boldsymbol{r}} + \frac{\boldsymbol{F}}{m}\cdot\boldsymbol{\nabla}_{\boldsymbol{v}}\right) f = \frac{s^2}{2}\int |\boldsymbol{U}\cdot\boldsymbol{e}|(f'f_1' - ff_1)\, d\omega\, d^3\boldsymbol{v}_1,$$

where $\boldsymbol{U} = \boldsymbol{v}_1 - \boldsymbol{v}$, $\quad -\nabla\Phi = \frac{\boldsymbol{F}}{m}$, $\quad d\omega =$ *solid angle element about the vector* $\boldsymbol{v}_1$*. Here* s *is the cross-section radius of a molecule,* $\boldsymbol{e}$ *is an arbitrary*

504 Under some further, often reasonable approximations, macroscopic continuum-mechanics PDE's such as the Navier-Stokes (NS) equation of fluid dynamics, can be derived via momentum-averaging techniques such as described above. The extra term in the NS equation (absent in Euler's equation) manifests viscosity, which is a molecular-collision effect.

unit-vector [that may be taken as $\boldsymbol{e} = (0,0,1)$ *without loss of generality], and*

$$f = f(\boldsymbol{r}, \boldsymbol{v}, t); \qquad f_1 = f(\boldsymbol{r}, \boldsymbol{v}_1, t);$$
$$f' = f\,[\boldsymbol{r}, \boldsymbol{v} + (\boldsymbol{U}\cdot\boldsymbol{e})\boldsymbol{e}, t];$$
$$f_1' = f\,[\boldsymbol{r}, \boldsymbol{v}_1 - (\boldsymbol{U}\cdot\boldsymbol{e})\boldsymbol{e}, t].$$

The mathematical problem of the kinetic theory of gases, now consists of the solution of the preceding non-linear integro-differential equation.

Returning to Gibbs' $6N$*-dimensional* Γ*-space, and having discussed the law of time-evolution of phase points in an ensemble, the next logical step is the establishment of a correspondence between a given* real system *and a suitable virtual ensemble. In other words, we must specify mathematically what is meant by the* average *behavior of a macroscopic system.*

There are two types of averages that are of interest. The first of these is the ordinary average at a given time *over all systems of the ensemble, known as the* ensemble average.

The second average of interest is the average of an observable entity for a given system of the ensemble *over some very large time interval.*

The ergodic[505] *hypothesis, first advanced by* **Boltzmann** *(1887), states that for* stationary *time processes (in which there is no preferred origin in time for the statistical description of observable entities, i.e., the ensemble is invariant under a time shift) the two averages are the same.*

The ergodic hypothesis *expresses the central assumption of classical as well as quantum statistical thermodynamics.*[506] *Attempts to prove it have given rise to the renowned* ergodic problem, *which has bedeviled eminent physicists and mathematicians for the past century.*

So far this hypothesis has been proved for a time average over an infinitely long time (both in classical and quantum mechanics) under certain assumptions that are too abstract to be easily stated and that remain to be justified. In physical experiments, however, we do not average over an infinite time, but over a finite time that is very short by macroscopic standards. It is plausible that this time interval can be effectively considered infinite because it is to be compared with characteristic molecular times, e.g., molecular collision mean-free time.

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505 The term *ergodic* is a combination of the Greek words for *energy* and *path*.

506 In quantum statistical thermodynamics – of which the classical version is a mere approximation – the *phase-space-density* function f is replaced with a *Hilbert space* density operator $\hat{\rho}$, and the Boltzmann transport equation is replaced by the *Fokker-Planck quantum Master-equation*.

***Von Neumann** (1931–1932) was able to formulate a single necessary and sufficient condition for the validity of the ergodic hypothesis. At present, therefore, the ergodic problem appears to have been reduced to the problem of demonstrating that the von-Neumann condition is fulfilled.*

The Rise of the 'New World', III
The Coming of American Technology (1876–1966)

*One hundred years after its inception, the United States of America began the marathon race for world supremacy in technology with the inventions of the telephone (**Bell**, 1876), the phonograph (**Edison**, 1877), the light bulb (**Edison**, 1884), the movie camera (**Edison**, 1889), the vacuum tube (**de Forest**, 1907) and finally the advent of the mass-produced automobile (**Ford**, 1908).*

*The Michelson-Morley experiment (1887), the establishment of the astronomical observatories at Lick (1888) and Yerkes (1900) and the experiments of **Millikan** (1910) have put the United States in the first league of the world's efforts in physics and astronomy. During the second half of the 19^{th} century, the U.S.A. continued to share in the leading trends of Europe. American scholars were trained at foreign universities. However, toward the end of the century, cultural exchange between Europe and the United States became less one-sided.*

By the early 1960's America had reluctantly come to realize that it possessed, as a nation, the most potent scientific complex in the history of the world. Eighty per cent of all scientific discoveries in the preceding three decades had been made by Americans. The United States had 75 per cent of the world's computers, and 90 per cent of the world's lasers. The United States had three and a half times as many scientists as the Soviet Union and spent three and a half times as much money on research; the U.S. had four times as many scientists as the European Economic Community and spent seven times as much on research. Most of this money came, directly or indirectly, from Congress, and Congress felt a great need for men to advise them on how to spend it.

1876 CE **Alexander Graham Bell** (1847–1922, U.S.A.). Scientist, educator and inventor. Bell was born in Edinburgh, Scotland and educated at the University of Edinburgh and the University of London. He moved with his father to Canada in 1870. In 1872 he became a professor of vocal physiology in Boston University. In 1876 he exhibited an apparatus embodying the results of his studies in the transmission of sound by electricity[507], and this invention, with improvements and modifications, constitutes the modern commercial telephone.

The telegraph had been invented before Bell's time. Noises, music and signals had been sent over electrified wires. But human *speech* had never been effectively sent by wire[508]. Many inventors were working to accomplish this, but Bell was the first to succeed. His great invention was the result of many years of scientific training. Bell exhibited his telephone at the Centennial Exposition in Philadelphia in June 1876.

The first telephone company, The Bell Telephone Company, came into existence in July 1877. Bell was frequently called upon to testify in lawsuits brought by men claiming they had invented the telephone earlier. Several of these suits reached the Supreme Court of the United States. Bell spent most of his later life at his estate on Cape Breton Island, Nova Scotia. He disliked the telephone because it interrupted his experiments in his laboratory. He died at his Nova Scotia home.

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[507] Bell's primitive telephone transmitter (1876) consisted of a membrane capable of moving in the field of a permanent magnet and an electromagnet that was fed by the dc current of a battery. The undulation of pressure on the membrane, caused by sound, generated an induced current signal which activated another membrane at the receiving end. The apparatus at each end acted both as a receiver and a transmitter. Bell uttered the first telephone message on March 10, 1876. He had spilled some acid on his cloths and was calling to his assistant, Thomas A. Watson, for help: "*Mr. Watson, come here. I want you*". In October 1876 Bell and Watson held the first two-way long distance telephone conversation between Boston and Cambridge, Mass., a distance of 3 kilometers. An American inventor, **Elisha Gray** (1835–1901), disputed Bell's claims as the inventor of the telephone. He filed a claim with the United States Patent Office only two hours after Bell filed a claim for a workable telephone (1876). Western Union supported Gray's claim in a bitter suit, but the claims were disallowed (Gray, however, made a fortune on other devices, such as simultaneous transmission of messages).

[508] With perhaps one exception: In 1860, **Jacob Reis** exhibited the first device which could transmit speech over a 100 meter wire. Bell acknowledged that he drew upon Reis' ideas in the construction of his telephone.

Efforts to improve the telephone transmitter led to the development of the *microphone* — a general device for changing sound waves into electrical signals (a telephone is essentially a simple type of a microphone). Other independent microphone inventors and improvers were: **Emile Berliner** (1877), **David Edward Hughes** (1878; also invented the *teleprinter*), **Thomas Edison**, **Francis Blake**, **Henry Humings**, **Charles Cuttris**, **Jerome Redding** and **E.W. Siemens**.

Note that Bell tried to develop a telegraph capable of sending multiple messages and accidentally discovered the principle of the telephone.

1876 CE *Flooding* in the Bay of Bengal region. Sea-waves 15 meter high, poured into the Ganges River delta and flooded an area of 380 km^2. Loss of life estimated at 215,000. It may have been a tsunami of seismic origin in the Andaman Islands. Again, on Nov. 12, 1970, cyclonic and tidal waves devastate the same region (now *Bangladesh*); 300,000 to 500,000 perished.

1876–1877 CE Great crop failure in India led to outbreak of *cholera* through which ca 3 million perished.

1876–1880 CE **Samuel (Siegfried Karl von) Basch** (1837–1919, Germany). Physician. Laid the foundation for the diagnosis of high blood pressure (hypertension). Invented the first simple and reliable apparatus for taking a person's blood pressure.

Basch was born in the ghetto of the Old City of Prague and obtained his medical degree in Vienna (1862). Became the personal physician of the Emperor Maximilian of Mexico until the latter's tragic end (1867).

In appreciation for Basch's loyal services, Kaiser Franz Joseph knighted him, an honor very rarely bestowed upon a Jew at that time. Two years later Basch was appointed the first lecturer in Experimental Pathology at the University of Vienna, but it took many years before he was given a modest laboratory of his own. Appointed (1877) professor at Vienna. From 1876 he published numerous papers on blood pressure and its measurement. All non-electronic instruments, so-called *sphygmomanometers*, are based on the original one invented by Basch[509].

1876–1883 CE **Paul Emile Appell** (1855–1930, France). Mathematician. Contributed significantly to the fields of analytical mechanics (non-holonomic systems), differential equations and special functions (*Appell's polynomials*). Although his work lacks central themes, seminal ideas and dramatic results,

[509] Like many other new inventions, the instrument of Basch was at first subjected to scorn and ridicule.

he was a technician who used the classical methods of this time to answer open questions, work out details and make natural extensions in the mainstream of the late 19^{th} century.

Appell was born in Strasbourg. He was educated at the École Normale and became a professor at the University of Paris (1903–1925). He married a niece of Hermite (1881).

1876–1892 CE **Isaac** (**Eduard**) **Schnitzer** (**Emin Pasha**, 1840–1892, Austria). Explorer of central Africa, ornithologist, meteorologist and physician. The southern inlet of Lake Victoria bears his name.

Isaac was born in Oppeln, Upper Schlesia to Jewish parents, and baptized after their death (1846). He later (1870) converted to Islam and took the name Emin Pasha. He was appointed governor of the Equatorial Province of Egypt by General C.G. Gordon (1878) and was murdered (1892) by slave traders whose activities he opposed.

1876–1897 CE **Robert** (**Heinrich Hermann**) **Koch** (1843–1910, Germany). Physician. The father of modern *bacteriology*, which he established as a separate science. He developed new techniques for straining, incubating, and growing bacteria, which remain the basis of the bacteriological study of infections.

Koch discovered and isolated the bacilli of *anthrax* (1876), *tuberculosis* (1882) (the first definite discovery of a specific microbe causing specific human disease), *cholera* (1883), and *bubonic plague* (1897). He won the 1905 Nobel prize for physiology or medicine for his work on tuberculosis.

Koch was born at Klausthal, Hanover and studied medicine at Göttingen. In 1885 he was appointed a professor at the University of Berlin.

The origins of Microbiology (1676–1900)

Stimulated by development of light microscopy, scientists began the study of microscopic organism (microorganisms). These organisms (most of which cannot be seen without a microscope) include: algae, bacteria molds, protozoans, fungi and viruses. They are sometimes called microbes. *Biologists*

specialize in the study of various kinds of microorganisms. For example, bacteriologists *work with bacteria,* mycologists *are concerned with fungi, and* virologists *with viruses.*

Nearly all microorganisms measure less than 0.1 mm (100 micron) across, and many must be studied with microscopes that magnify objects at least 1,000 times. Most viruses are so tiny that they can be seen only with electron microscopes that magnify many thousands of times.

Viruses are called acellular *microorganisms because they do not have true cell structures. All other microorganisms are* cellular*. They have cell membranes, cytoplasm, and a nuclear body. Bacteria are the smallest single-celled organisms. The smallest bacteria may measure only ca* $\frac{4}{10}$ *of a* micron*.*

About 10,000 small viruses could be packed into a cell the size of one of these bacteria. Over a billion such cells could be packed into one of the largest microbial *cells — the cells of a certain algae.*

Microbiology as a discipline is defined by the techniques *employed:*

- *Microscopy and Strains*
- *Sterilization*
- *Getting a pure culture*
- *Composition of culture media*
- *Anaerobe, aerobe, microaerophile*

Bacteria are one-celled organisms, first seen by **Leeuwenhoek** *(1676). Originally confused with protozoa, bacteria were variously called* animalcules *or microbes.*

During the 18th century bacteria contributed to the 'spontaneous generation controversy' as **Spallanzani** *(1767–1768) refuted Needham's assertion that microbes appeared in sealed flasks of boiled broth. Bacterial studies outside medicine remained superficial until 1872 when* **F. J. Cohn** *(1828–1898) defined and named bacteria, distinguishing four groups on the basis of external form and specific fermentative activity. He recognized bacteria that take nitrogen from simple ammonia compounds, elucidated their life-cycles, identified spores and suggested that bacteria were motile cells devoid of walls. Determining bacterial temperature limits,* **Cohn**, **Pasteur** *and* **Tyndall** *effectively ended the spontaneous generation controversy with their studies on sterilization.*

The link between Leeuwenhoek's microorganisms (1676) and the induction of infectious diseases by bacteria was not recognized for another 200 years![510]*Indeed, proof that microbes cause diseases in humans was first given by* ***Bassi*** *(1835), anticipating Pasteur and Koch.* ***Semmelweiss*** *(1847) proved that puerperal fever is contagious.*

Casimir Davain *(1812–1882, France), physician, was first to produce experimental infection in animals with blood containing the anthrax bacillus and first to suggest that the bacillus caused the disease (1850–1863).* ***Joseph Lister****, influenced by the discoveries of* **Pasteur***, introduced carbolic acid as an antiseptic in surgery (1867).* ***Robert Koch*** *(1876) confirmed Davain's suggestions. Koch also developed techniques for handling bacteria, improving upon* ***Carl Weigert****'s (1845–1904) original use of methyl violet to stain them, introducing solid nutrient media (agar-agar) to grow pure cultures, and devising methods for fixing bacteria.*

Viruses are sub-microscopic, obligate intracellular parasites: they are produced from the assembly of pre-formed components, whereas other agents 'grow' from an increase in the integrated sum of their components & reproduce by division. Virus particles (virions) themselves do not 'grow' or undergo division. Viruses lack the genetic information which encodes apparatus necessary for the generation of metabolic energy or for protein synthesis (ribosomes). No known virus has the biochemical or genetic potential to generate the energy necessary for driving all biological processes, e.g. macromolecular synthesis. They are therefore absolutely dependent on the host cell for this function.

- *Viroids are small (200–400 nm), circular RNA molecules with a rod-like secondary structure which possess no capsid or envelope.*
- *Virusoids are satellite, viroid-like molecules, somewhat larger than viroids (e.g. approximately 1000 nm).*
- *Prions are infectious agents believed to consist of a single type of abnormally-folded protein molecule with no nucleic acid component. They are believed to infect other, normal proteins by somehow inducing in them their own folding abnormality.*

Viruses infect all types of living cells — animals, plants & bacteria.

[510] **Girolamo Fracastoro** made the first scientific statement (1546) on the true nature of contagion and transmission of diseases by germs, but he had no physical idea about these agents.

Virology began with ***Edward Jenner's*** *vaccination against smallpox (1796).* ***Pasteur*** *(1881) made the first artificially produced virus vaccine (rabies).* ***Dimitri Iwanowski*** *(1864–1920, Russia) explained (1892) the infectiousness of tobacco mosaic disease by showing that it can be transmitted via cell-free filtration from leaves of diseased plants to leaves of healthy plants. During the 1890s increased knowledge of soil and water bacteria was responsible for completing the explication of the nitrogen, sulphur and carbon cycles.*

Nodule-forming bacteria living in the roots of leguminous plants were found to fix atmospheric nitrogen. As a result of ***Winogradski's*** *(1856–1953) and* ***M. Beijerinck's*** *(1851–1931) work on anaerobic bacteria, knowledge of a whole world of organisms able to live on elementary nitrogen, iron or sulphur has emerged.*

In 1898, ***Friedrich Loeffler*** *(1852–1915, Germany) and* ***Paul Frosch*** *(1860–1928, Germany) discovered that a virus is responsible for foot-and-mouth disease. In 1900,* ***Walter Reed*** *(1851–1902) proved that yellow fever was caused by a virus spread by mosquitoes.* ***K. Landsteiner*** *(1900) demonstrated that poliomyelitis is caused by a virus.*

1876–1909 CE **Otto Wallach** (1847–1931, Germany). Organic chemist. Pioneered in the field of alicyclic compounds which formed the basis for the *perfume industry*. Awarded the 1910 Nobel prize in chemistry.

Wallach was born to Jewish parents. Studied under **Kekule**. Professor at Bonn (1876) and Göttingen (1889–1915).

1877–1881 CE **Wilhelm Friedrich Philipp Pfeffer** (1845–1920, Germany). Physiological botanist. First to measure *osmotic pressure*[511] and determine through it molecular weights of proteins. Made pioneering studies of *respiration*, *transpiration*, *photosynthesis*, *metabolism*, *transport in plants*

[511] He used a membrane of $Cu_2[Fe(CN)_6]$ (discovered in 1864 by **Moritz Traube** (1826–1894)), to make accurate measurements of osmotic pressures. The semipermeable membrane container was filled with a sugar solution and immersed in a vessel of water. He then connected a mercury-filled manometer to the top of the semipermeable container, and was able to show that the pressure was directly proportional to *concentration* (and hence inversely to *volume*), and also directly to the absolute *temperature*, i.e. $PV = kT$. Van't Hoff used Pfeffer's measurements (1886) to derive the law of osmotic pressure.

and *mycorhiza* (a fungus entering into symbiotic partnership with roots of trees). His work on *osmosis* was of fundamental importance in the study of cells, because *semipermeable membranes* surround all cells and play a large part in controlling their internal environment.

Pfeffer was born in Grebenstein, near Kassel, and was trained as a pharmacist at Göttingen (Doctorate, 1865), Marburg and Bonn. He became a professor at Bonn (1873) and at Leipzig (1887–1920).

1877–1887 CE **Emile Berliner** (1851–1926, U.S.A.). Electrical engineer and inventor. Invented the variable-resistance *microphone* (1877, ahead of Edison), and the *gramophone record* (1887).

Berliner was born in Hanover, Germany to a Jewish family, and was educated in his native place and Wolfenbüttel, where he graduated in 1865. He emigrated to the United States in 1870 and settled in Washington D.C. There he worked as a clerk, salesman, and assistant in a chemical laboratory. He studied electrical engineering, and in 1876 began experimenting with Bell's newly invented telephone. In 1877 Berliner succeeded in refining it with his invention of the loose-contact telephone transmitter[512]. The Bell Telephone

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[512] In contradistinction to Bell's telephone transmitter, the electrical resistance at the *contact of two conductors* is made to vary with the sound pressure on the membrane. This variable resistance generates, in turn, a variable electric signal that flows to the receiver and creates there the inverse effect. Berliner's patent was issued on April, 4, 1877. Almost simultaneously with Berliner, on July 21, 1877, Edison improved on this idea by replacing the two conductors in contact by a small cell of *carbon powder*.

The overall operation of the modern telephone transmitter (mouth piece) is as follows: behind the mouth piece of the phone lies a thin metal disc called a *diaphragm*. When a person talks into the telephone, the diaphragm vibrates in accordance with the sound pressure waves. Behind the diaphragm lies a small cup filled with tiny grains of carbon. A low-voltage electric current, activated by batteries (at the telephone company), travels through the grains. The electric resistance of the powder depends on its packing (loose of tight), which in turn is determined by the sound pressure on the diaphragm. Thus, the electric current through the powder grains replicates the pattern of the sound waves.

The *receiver* (ear piece) consists of a permanent magnet and an electromagnet located at the edge of a diaphragm and causing it to vibrate. The permanent magnet holds the diaphragm close to it permanently. The electromagnet controls the vibrations of the diaphragm. The electric current from the transmitter flows through the coils of the electromagnet, which pulls the diaphragm away from the permanent magnet. This causes the diaphragm to vibrate and set up sound waves.

Company immediately purchased the rights to his invention, which *for the first time made the telephone practical for long-distant use*. In 1887 he improved Edison's phonograph by introducing the *gramophone record* — a laterally cut shallow grooved disc. He also invented a way to press duplicate records from one master disc (by making a wax disc from which a 'negative' metal matrix was made for producing endless 'positives'). The patent was acquired by the Victor Talking Machine Company and served as a basis for the modern gramophone.

In his later years Berliner engaged in aviation experiments: between 1919 and 1926 he built three helicopters, which he tested in flight himself.

1877–1889 CE **Thomas Alva Edison** (1847–1931, U.S.A.). Distinguished inventor. Invented the *phonograph* (1877), the carbon-powder *microphone* transmitter (1877), the *incandescent electric light* (1879); contributed to the development of motion pictures (1889) and the memeograph machine (1887) and patented 1093 inventions in his lifetime. He had only 3 months of formal schooling in his whole life(!).

Edison was born at Milan, Erie county, Ohio, of mixed Dutch and Scottish descent. His parents moved to Port Huron, Michigan, when he was 7 years old. At the age of 12 he became a train news-boy on the railway to Detroit. At 15 he became a telegraph operator, and was employed in many cities in the United States and Canada, but frequently neglected his duties in order to carry on studies and experiments in electrical science. In 1869 Edison came to New York City, and soon afterwards became connected with the Gold & Stock Company. He invented an improved printing telegraph for stock quotations, for which he received $40,000. He then established a laboratory and factory for further experiments and for the manufacture of his inventions. On Oct. 19, 1879, after many failures, Edison finally succeeded in placing a filament of carbonized thread in a bulb[513]. On Dec. 21, 1879 the news of Edison's

[513] Edison can hardly claim to be the bulb's sole inventor. He was neither the first to come up with the incandescent light bulb idea. Contrary to popular opinion, the key ingredient he used for his light bulb — carbon — was certainly not unique. Carbon had been an ingredient of experimental light bulbs 50 years before Edison. At least 3 or 4 serious inventors, in England, France and the United States, were working on the incandescent lamp in the 1870s. They had the right ingredients and had functioning light bulbs. **Joseph Wilson Swan** (1828–1914, England) had lit residences in 1879 with his British bulb. **Hiram Stevens Maxim** (1840–1916, England and USA) had filed for incandescent light patents in 1878 and 1879 and had carbon incandescent lamps burning for twenty four hours at a time. **Hippolyte Fontaine** (1833–1917, France) displayed his version of a carbon-vacuum bulb in 1876. **William Eve Sawyer**

invention of the electric incandescent light-bulb astounded the world. Nations and individuals honored the modern Prometheus as probably no other person has been honored while alive. He died in West Orange, N.J.

Edison was a poor businessman and his inventions had never made him the money he thought he was entitled to[514].

(England) issued a patent (1878) on a carbon-nitrogen light bulb. It is still debatable whether he or Edison perfected the first light bulb. [He died in prison (1883) while serving a sentence for murder.] However, these pioneers succeeded in producing workable bulbs only on a *small scale*. Edison, from the start, designed his lamp to be part of a total electrical system the size of a city, complete with electric dynamos to produce the electricity and wires and fuses to distribute and control it. Only Edison discerned that the lamp and the system had to work as a unit and had to match. In addition, it was Edison's enormous wealth, influence, and power that allowed him to create the entire system from scratch in his New Jersey laboratories, set up a power station to light New York City with his new bulbs, and influence an eager press and public into believing his bulb to be the superior to all others.

514 He blamed it all on the Jews. As a close associate of the motor-car tycoon Henry Ford, he also propagated anti-Semitic views, habitually grousing about "*Jewish conspiracies*".

History of Sound Reproduction — from antiquity to Berliner

The ancient Greeks knew that sound as heard by the ear consisted of vibrations of air which, at certain frequencies, could even cause objects to vibrate. Records indicate that resonating panels were commonly used to improve acoustics of Greek theater. Thus, the origins of recorded sounds can be traced as far back as the ancient Greeks.

The colossal "vocal" statue of Memnon at Thebes was built about 1500 BCE with the ability to make the sound of the harpstring every day to greet Memnon's mother, the Goddess of the Dawn. The secret of this sound was lost when the original statue was destroyed in 27 CE by an earthquake. Back in the year 18 BCE, the Romans installed metal vases in their amphitheaters, specially tuned to certain frequencies.

The wheel was the first mechanism used to record sound, with pegs positioned to strike chimes as the wheel was rotated by hand. In the Middle Ages, music was reproduced by cylinders with attached pins that would strike certain keys or bells when rotated. Automatic carillons were built in the 14th century and the oldest surviving barrel organ dates from 1502. Renaissance Europe was fascinated with automata, or automatic music boxes that used elaborate clockwork gears to produce motions and sounds.

The most famous automata was a mechanical duck by ***Jacques de Vaucanson*** *in 1745 that flapped its wings, raised up on its legs, stretched its neck and moved its intestines that were visible from the outside. Influenced by Vaucanson and by the flood of new inventions from the Industrial Revolution, the French silk-weaver* ***Joseph Marie Jacquard*** *was awarded a medal at the Paris Exhibition of 1801 for an automatic loom that used punched cards to "record" a complex pattern for textiles woven by a loom with controls connected to spring-loaded keys. The long punched-card strips held down the keys until a hole allowed a key to open and start a loom operation.*

The weavers of Lyons burned Jacquard's loom in 1808 (where his statue is now located in Lyons) but Napoleon recognized its significance and awarded Jacquard a pension and royalties. The punched card was adapted by music instrument designers such as ***Charles Dawson*** *who exhibited a "jacquard organ" at the London Exhibition of 1851 that used cardboard strips to control the air bellows. From this would come the player piano of the 1880s and* ***Herman Hollerith****'s punch card tabulator for the 1890 census, a precursor of the modern computer.*

In 1855, the first successful sound recording device was developed by ***Edouard-Leon Scott de Martinville*** *(1817–1879, France). He called his invention the 'phonautograph'. It used a mouthpiece horn and membrane fixed to a stylus that recorded sound waves on a rotating cylinder wrapped with smoke-blackened paper. This device did not record the sound itself, only a graphical image of the sound. There was no way at the time to play the sound back*[515]*.*

In 1877 ***Thomas Edison*** *(1847–1931, USA) designed his "tinfoil phonograph"*[516]*. The device consisted of a cylindrical drum wrapped in tinfoil and*

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515 In March 2008, a group of American audio historians unearthed in an archive in Paris a recording of the human voice made on April 9, 1860 by **Scott**. It is a 10-seconds recordings of a singer crooning the folk song "Au Clair de la Lune". This phonautogram was made playable (converted from squiggles on paper to sound) by scientists at the Lawrence Berkeley National Laboratory in Berkeley, California. The Berkeley scientists used optical imaging and a "virtual stylus" on high-resolution scans of the phonautogram, deploying modern technology to extract sound from patterns inscribed on the soot-blackened paper almost a century and a half ago he recording was played in public on March 28, 2008 at Stanford University.

Scott's 1860 phonautogram was made 17 years before Edison received a patent for the phonograph and 28 years before an Edison associate captured a snippet of a Handel Oratorio on a wax cylinder.

Scott is in many ways an unlikely hero of recorded sound. He was a man of letters, not a scientist, who worked in the printing trade and as a librarian. He published a book on the history of shorthand, and evidently viewed sound recording as an extension of stenography. In a self-published memoir in 1878, he railed against Edison for "appropriating" his methods and misconstruing the purpose of recording technology. In his memoir, Scott scorned his American rival and made brazen appeals to French nationalism: "What are the rights of the discoverer versus the improver? Come, Parisians, don't let them take our prize." Thus, Scott went to his grave convinced that credit for his breakthrough had been improperly bestowed on Edison.

516 The invention of this first "*talking machine*" is most commonly attributed to **Edison**, in part because of the publicity that attended his celebrity and the theatrical power of his demonstrations, and in part because previous invention had earned him the means to have the device built. However, the first to *build* a phonograph was his top laboratory mechanic **John Kruesi**. The first to conceive of a workable design was the Frenchman **Charles Cros** who delivered viable plans for a machine that would use *disks* to the French Academy of Sciences in April 1877, several months before Edison happened on his idea.

Note that Edison was seeking to improve the *telephone* in 1877, when he discovered the recording device known as the phonograph.

mounted on a threaded axle. A mouthpiece attached to a diaphragm was connected to a stylus that etched vibrational patterns from a sound source on the rotating foil. For playback, the mouthpiece was replaced by a "reproducer". When the stylus was made to travel over the grooves, it made the membrane vibrate in response to the depressions in the grooves. Hence the motion of the stylus could reproduce the original sound.

So far, electricity was not directly involved in this mechanical sound-recording devices. But after the wheel, electricity would soon become another method of recording sound. To see the evolution of this process, one must go back in time to 1832, the year in which ***Samuel Morse*** *began the design of the telegraph. The electric telegraph was the stimulus for inventors to search for better methods of sending* and recording *all kinds of messages, including voice and music.*

David Edward Hughes*, a Professor of Music at St. Joseph's College in Kentucky, invented in 1855 a keyboard telegraph with rotating type-wheel printer that grew into the modern telex industry. In Germany, telegraph printers were patented as early as 1848 and* ***Philip Reis*** *invented an acoustic transmitter in 1861 that used a diaphragm to open and close an electrical circuit. He called it a "telephone" hoping to use it to reproduce speech and music but was unsuccessful.* ***Elisha Gray*** *and his Western Electric Company in Chicago had also invented an improved telegraph receiver, calling it a "telephone" after 1874 because it produced a wide range of sounds, but failed to make a similar transmitter.*

Herman Helmholtz *made an electric tuning-fork "sounder" device that used an electromagnetic coil, tuning fork and cardboard tube "resonator" to amplify the sound. His purpose was the scientific study of sound that was published in the influential 1862 book "Sensation of Tone".*

Berliner's gramophone (1887) was based on Scott's phonautograph and Cros's disc. But in spite of its superiority over Edison's cylinder machines, the Berliner gramophone was slow to attract attention. However, a new wax engraving process improved recording quality dramatically, and by 1901 the Gramophone company recorded four stars of the Russian Imperial Opera. A few years later, it recorded the voice of ***Enrico Caruso****, and a worldwide recording odyssey began.*

The invention of the phonograph and other sound reproduction machines began a new way of producing historical archives. Expressions of the human voice were no longer limited to their abstraction as words on the page, and the artistry and passion of a musical performance could be presented outside human memory. People could bring the sounds of the world into their homes,

and a global culture began to arise out of the mixture of influences that a broad diversity of recordings could provide. Before radio and sound motion pictures, the phonograph and other "talking machines" reigned for several decades as the great modern innovation in audio culture and entertainment.

Evolution of Celestial Mechanics[517]

Astronomy is the oldest science, and in a certain sense the parent of all sciences. The relatively simple and regularly recurring celestial phenomena first taught men, in the days of ancient Greece, that Nature is systematic and orderly.

For a long time progress was painfully slow. Centuries of observations and attempts at theories for explaining them were necessary before it was finally possible for **Kepler** *(1571–1630) to derive his laws, which are first approximations to the description of the way in which planets move. The wonder is that, in spite of the distractions of the constant struggles incident to an unstable social order, there should have been so many men who found their greatest pleasure in patiently making the laborious observations which were necessary to establish the laws of celestial motions.*

The work of Kepler closed the preliminary epoch of 2000 *years or more, and the discoveries of* **Newton** *(1642–1727) opened another. The invention of the calculus furnished for the first time a mathematical machinery, which was suitable for grappling with such difficult problems as the disturbing effect of the sun on the motion of the moon, or the mutual perturbations of the planets. It was fortunate that the telescope was invented at about the same time; for without its use, it would not have been possible to acquire the*

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517 For further reading, see:

- Moulton, F.R., *An Introduction to Celestial Mechanics*, Dover: New York, 1970, 436 pp.
- Sterne, T.E., *An Introduction to Celestial Mechanics*, Interscience Publishers, 1960, 206 pp.

accurate observations which furnished the grist of data for the mill of the new mathematical theory.

The history of celestial mechanics during the 18^{th} *century is one of continual triumphs. The analytical foundations laid by* ***Clairaut*** *(1713–1765),* ***d'Alembert*** *(1717–1747) and* ***Euler*** *(1707–1783) formed the basis for the achievements of* ***Lagrange*** *(1736–1813) and* ***Laplace*** *(1749–1827).*

Their successors in the 19^{th} *century further developed, largely by the same methods, the theory of motions of moon and planets. They advanced the theoretical calculations to higher levels of precision, and compared them with more and better observations. In this connection, the names of* ***LeVerrier*** *(1811–1877),* ***Delaunay*** *(1816–1872),* ***Hansen*** *(1795–1874) and* ***Newcomb*** *(1835–1909) are especially noteworthy.*

Near the close of the 19^{th} *century, a third epoch was entered. It is distinguished by new points of view and new methods which, in power and mathematical rigor, surpassed all that went before. It was inaugurated by* ***Hill*** *(1838–1914) in his 'Research on the Lunar Theory', but owes most to the contributions of* ***Poincaré*** *(1854–1912) to the problem of three bodies.*

Celestial mechanics should be regarded as one of the splendid achievements of the human mind. No other science is based on so many observations extending over so long a time. No other scientific theory except Quantum Electrodynamics, has been empirically vindicated to such an outstanding accuracy.

1877 CE **Asaph Hall** (1829–1907, U.S.A.). Astronomer. Discovered two satellites of Mars with diameters 11 km and 6 km respectively, which he named *Phobos* (fear) and *Deimos* (terror). Curiously enough, these satellites were predicted by **Jonathan Swift** (1667–1745) already in 1726, in his "*Gulliver's Travels*"!

1877 CE **George William Hill** (1838–1914, U.S.A.). Mathematical astronomer. Contributed significantly to the theory of lunar motion. Also developed a theory of the motions of Jupiter and Saturn. Hill was first to use infinite determinants to analyze the motion of the moon's perigee. [This work was published in 1877 under the arcane title: "*On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and the moon*".] In his work he came across a class of homogeneous, linear, 2^{nd} order differential equation with real, periodic coefficients. Such an equation is now

known as "*Hill's equation*"[518]. It has numerous applications to problems in engineering, physics and astronomy. It includes as special cases the equations of **Mathieu**, **Lamé**, **Whittaker-Hill**, **Hermite** and **Picard**.

Hill was born in New York and educated at Rutgers College. In 1861 he joined the staff of scientists working in Cambridge, Massachusetts, on the *American Ephemeris and Nautical Almanac*. There he was assigned the task of calculating the American ephemeris, work he was later authorized to continue at his rural home in West Nyack, NY.

During 1881–1892 Hill resided in Washington D.C., working for the Navy Department in the Nautical Almanac Office. In 1898 he accepted the chair of astronomy at Columbia University. Since few students were qualified to comprehend graduate-level work in celestial mechanics, Hill objected to receiving pay, and finally resigned in 1901. He remained a recluse in West Nyack, devoted to his researches and to his large scientific library, which he bequeathed to Columbia University. Illness during his last years reduced his physical activity, and a failing heart brought his career to a close.

[518] Its standard form is $y'' + [\lambda + Q(x)]\, y = 0,$ where λ is a parameter and $Q(x)$ is a real periodic function of x with period π. The fundamental importance of Hill's equation for stability problems was established by **Lyapunov** in 1907.

Lunar Theory — Part II (1687–1878)

The mathematical theory of the orbital motion of the moon is known as lunar theory. It is one of the most complex and difficult problems of dynamical astronomy. Its solution required the combined efforts of the greatest mathematicians since Newton, and its history extends back twenty centuries or more.

Before Newton, the problem was that of devising empirical curves for anomalies in the motion of the moon around the earth. After the establishment of universal gravitation as the primary law of celestial motions, the problem was reduced to that of integrating the differential equations of the moon's motion, and testing the results by comparison with observations.

Modern research developed naturally from the results of the ancients. In the hands of ***Hipparchos*** *(ca 135 BCE), observations were brought to a degree of precision which is truly marvelous in comparison with the level of other branches of physical science in that age.*

Hipparchos *discovered the 'annual equation'* and ***Ptolemy*** *(150 CE) discovered 'evection'. The 'variation' was discovered by* ***Tycho Brahe*** *in about* 1600. *The inclination of the moon's orbit and the regression of the nodes were discovered in* 1670 *by* ***John Flamsteed*** *(1646–1719,* England*).*

The modern lunar theory began with **Newton***, and consisted in determining the motion of the moon from the universal theory of gravitation. Newton tried to explain the rotation of the line of apsides in his 'Principia', but his predictions accounted for only about half the observed apsidal rotation. In* 1749, ***Clairaut*** *found that Newton had neglected some small terms in the equations, and he brought theory and fact into agreement by taking such terms into account. However, more than a century later, in* 1872, *the correct calculations were also discovered among Newton's unpublished papers: he had detected his own error but had never bothered to correct it in print!*

Since the days of Newton, the methods of analysis have succeeded those of geometry. In the 18^{th} *century the development of lunar theory was almost entirely the work of five men:* **Euler** *(1707–1783),* **Clairaut** *(1713–1765),* **d'Alembert** *(1717–1783),* **Lagrange** *(1736–1813)* and **Laplace** *(1749–1827).*

The first complete explanation of the irregularities in the motion of the moon was given by **Newton** *both in his published and unpublished manuscripts. Newton regarded lunar theory as being very difficult and he confided*

to Halley in despair that it "made his head ache and kept him awake so often that he would think of it no more".

In the 18^{th} *century lunar theory was developed analytically by* ***Euler****,* ***Clairaut****,* ***d'Alembert****,* ***Lagrange*** *and* ***Laplace****. This intensive work was motivated by the general demand, in the* 18^{th} *century, for accurate lunar tables for the use of navigators in determining their position at sea. This, together with the fact that the motions of the moon presented the best test of the Newtonian Theory, induced the English Government and a number of scientific societies to offer very substantial prizes for lunar tables agreeing with observations within certain narrow limits.*

Euler *published some imperfect lunar tables in* 1746*. In* 1747*,* ***Clairaut*** *and* ***d'Alembert*** *presented to the Paris Academy, on the same day, memoirs on the lunar theory. Each had trouble in explaining the motion of the perigee. In* 1749*, Clairaut found the source of the difficulty (also discovered by Euler and d'Alembert a little later). In* 1787*,* ***Laplace*** *explained the cause of the secular acceleration of the moon's mean motion.*

The immediate successors of Laplace, ***M.C. Damoiseau*** *(1768–1846) and* ***Plana*** *(1781–1864), carried out his method to a high degree of approximation. They integrated the equations of motions by expressing the time in terms of the moon's true longitude. Then, by inverting the series, the longitude was expressed in terms of the time.*

A second method was followed by ***Hansen*** *(1795–1874),* ***Lubbock*** *(1803–1865),* **de Pontécoulant** *(1795–1874) and* **Delaunay** *(1816–1872) during the years* 1832–1867*. According to this school, the moon's coordinates are obtained in terms of the time by a direct integration of the differential equations of motion. The expressions for the longitude, latitude and parallax appear as infinite trigonometric series, in which the coefficients of the sines and cosines are themselves infinite power series in small entities [eccentricities of moon and earth orbits, sine of half the moon's inclination etc.]. However, by this method, the series converge slowly and the final expressions of the moon's longitude are overlong and complicated.*

An entirely different approach, based on a method suggested by ***Euler****, was taken up by* ***George William Hill*** *(1838–1914) and continued by* ***John Couch Adams*** *(1819–1892) and* ***Ernest William Brown*** *(1866–1938).*

Euler conceived the idea of an iterative scheme, starting with a zeroth order solution of the problem in which the orbit of the moon is supposed to lie in the ecliptic and have no eccentricity, while that of the earth is taken to be circular. The additional terms were then found, which were multiplied by the first powers of the eccentricities and of the inclination. Then the terms of the second order were found, and so on to any desired order. This method is

superior by far to the method of Laplace, since the convergence is faster and high precision is achieved even after a small number of iterations.

Hill improved on Euler's method, and worked it out with greater rigor and detail.

Differential Equations and Special Functions[519] *(1694–1879)*

Most of the special functions of mathematical physics and their corresponding differential equations were discovered during the 18^{th} *and* 19^{th} *centuries. The differential equations were usually encountered in the solutions of*

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519 To dig deeper, see:

- Andrews, G.E., R. Askey and R. Roy, *Special Functions*, Cambridge University Press, 2000, 661 pp.
- Vilenkin, N.J., *Special Functions and the Theory of Group Representations*, American Mathematical Society, 1968, 613 pp.
- Bell, W.W., *Special Functions for Scientists and Engineers*, Van Nostrand, 1968, 247 pp.
- Erdélyi, A. (Editor), *Higher Transcendental Functions*, 3 Volumes, McGraw-Hill Book Company: New York, 1953–1955.
- Magnus, W., F. Oberhettinger and R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer-Verlag: Berlin, 1966, 508 pp.
- Kamke, E., *Gewöhnliche Differentialgleichungen*, Chelsea Publishing Company: New York, 1959, 666 pp.
- Zwillinger, D., *Handbook of Differential Equations*, Academic Press: Boston, 1989, 673 pp.
- Inch, E.L., *Ordinary Differential Equations*, Dover Publications: New York, 1956, 558 pp.
- Havil, J., *Gamma*, Princeton University Press, 2003, 266 pp.

geometrical, astronomical or physical problems. The pioneers in this field were the illustrious ***Bernoullis*** *and* ***Euler****. They were later followed by* ***Legendre****,* ***Gauss****,* ***Jacobi*** *and others. In principle, the common special functions of mathematical physics evolved in a systematic way from the solutions of the wave-equation, heat-equation and the Laplace equation in the various orthogonal curvilinear coordinate systems. Historically, however, many of the well-known special functions were discovered independently, through solutions of problems in the various branches of science and engineering.*

The historical pattern is as follows:

A. The Gamma function

The first non-elementary function to be discovered after the scientific revolution. Its notation $\Gamma(z)$ *was introduced by Legendre in* 1814*. Euler's formula for this function was given in* 1729 *and most of its properties*[520] *were discovered by* ***Gauss****,* ***Legendre****,* ***Neumann*** *(1848),* ***Weierstrass*** *(1856) [who also showed that the Gamma-function does not satisfy any differential equation with rational coefficients] and* ***Hankel*** *[expression of* $\Gamma(z)$ *as a contour integral, 1864]. The associated Beta-function was introduced by Euler (1772).*

- Watson, G.N., *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 1966, 804 pp.
- Robin, Louis, *Functions Spheriques De Legendre et Functions Spheroidales*, Gauthier Villars: Paris, 1957–1959, Vols I-III (201 pp., 384 pp., 289 pp.)
- Goursat, E., *Differential Equations*, Ginn and Company, 1945, 300 pp.
- Titchmarsh, E.C., *The Theory of the Riemann Zeta-Function*, Oxford University Press: Oxford, 1951, 346 pp.
- Edwards, H.M., *Riemann's Zeta Function*, Academic Press: New York, 1974, 315 pp.
- Vallée, O. and M. Soares, *Airy Functions and Applications to Physics*, Imperial College Press, 2004, 194 pp.

[520] When $\Re\{z\} > 0$, $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$ is known as the Eulerian integral of the second kind. It leads to the basic recursion relation satisfied by the Gamma-function $\Gamma(z+1) = z\Gamma(z)$. Euler's formula is $z\Gamma(z) = \prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n}\right)^{-1}$.

B. Bessel functions (1694–1824)

In 1694 ***Johann Bernoulli*** *discovered the so-called Riccati's equation,* $y' = y^2 + x^2$, *in a paper on curves, but did not solve it. In* 1703, ***James Bernoulli*** *communicated to Leibniz a solution of the above equation in the form of infinite power series. In* 1738 ***Daniel Bernoulli*** *was engaged in the problem of the lateral oscillation of a heavy uniform chain. His solution was given in terms of a series, now described as a Bessel function of order zero.*

Euler*, returned to this problem in* 1781*, deriving the equation of motion for the chain's horizontal displacement. By an extremely ingenious analysis he found the three gravest eigenperiods of the chain. Earlier, in* 1764*, Euler had investigated the vibrations of a stretched membrane. He wrote the partial differential equation of its transverse displacement in polar coordinates and then proceeded to obtain the ordinary differential equation of the motion's amplitude, known today as the 'Bessel equation'. He also gave its explicit solution in terms of an infinite series. This investigation of Euler contains the earliest appearance of a Bessel function of general integral order.*

In 1770*, Lagrange encountered the Bessel-function in the astronomical problem of the Kepler equation*[521]*, and gave an expansion of the radius vector*

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[521] The planetary elliptic orbit is given parametrically by means of the *eccentric anomaly* angle E, via the relations

$$r = a(1 - e\cos E), \qquad \cos\theta = \frac{\cos E - e}{1 - e\cos E},$$

where e is the eccentricity, θ is the *true anomaly* and r is the radius-vector. The auxiliary angle E is a solution of *Kepler's transcendental equation*

$$E - e\sin E = \frac{2\pi}{p}(t - T) = M,$$

where p is the orbital period and $t - T$ is the time, measured from the perihelion passage T. The quantity M, known as the *mean anomaly*, is the angle which the radius-vector would have described if it had been moving uniformly with average rate $\frac{2\pi}{p}$. Once the *Kepler equation* $E - e\sin E = M$ is solved for given M, the orbit $\{r, \theta\}$ is known for all times.

It turns out that the Fourier-series expansions of the radius-vector and of E are:

$$r = a\left[1 + \frac{1}{2}e^2 + \sum_{n=1}^{\infty} B_n \cos(nM)\right],$$

$$E = M + \sum_{n=1}^{\infty} A_n \sin(nM),$$

of the orbit in terms of coefficients that are infinite-series representations of Bessel functions. ***Fourier*** *(1822) and* ***Poisson*** *(1823) obtained similar series in problems of heat diffusion in solid circular cylinders and spheres. Finally,* ***Bessel*** *(1824) made a systematic study of the functions that now bear his name, in connection with the Kepler's problem.*

C. Legendre functions (1784–1884)

In 1784, ***Legendre*** *studied the gravitational attraction of spheroids. In the course of his work he expanded* $(1-2hz+h^2)^{-1/2}$ *in a power series of* $z=\cos\theta$. *The coefficients of* h^n *in this expansion were named later after him. These coefficients,* $P_n(z)$, *are of frequent use not only in potential theory, but in other branches of analysis as well and are called today Legendre polynomials.*

The algebraic and analytic properties of these polynomials were investigated by Legendre himself and by his followers during the next 100 *years [****Rodrigues*** *(1814),* ***Murphy*** *(1833),* ***Dirichlet*** *(1837),* ***Neumann*** *(1848–1862),* ***Heine*** *(1851–1878),* ***Christoffel*** *(1858),* ***Frobenius*** *(1871),* ***Mehler*** *(1872),* ***Schläfli*** *(1881),* ***Ferrer*** *(1877),* ***Hobson*** *(1891) and many others].*

A more extended class of Legendre functions is the Legendre associated polynomials and functions. A function connected with the associated Legendre function $P_n^m(z)$ *is the function* $C_n^\nu(z)$, *which for integral values of* n *is defined to be the coefficient of* h^n *in the expansion of* $(1-2hz+h^2)^{-\nu}$ *in ascending powers of* h. *It has been studied by* ***Gegenbauer*** *(1874–1893).*

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with

$$A_n = \frac{2}{n}J_n(ne),$$

$$B_n = -\frac{e}{\pi n}\int_0^{2\pi}\sin u\sin(nu-ne\sin u)du = -2\left(\frac{e}{n}\right)J_n'(ne),$$

where J_n is the Bessel function of order n and J_n' is its derivative w.r.t. the argument.

D. The hypergeometric functions (1755–1879)

This general function includes as special cases most of the familiar functions of elementary analysis [e.g. Chebyshev polynomials]. The function was known to **Euler** *(1755), who discovered a number of its properties, but it was studied systematically by* **Gauss** *(1811–1812), who gave the earliest satisfactory treatment of the convergence of an infinite series. Gauss' work initiated far-reaching development in many branches of analysis, in infinite series, general theories of linear differential equations and functions of complex variables.*

Another classical differential equation of considerable importance is the confluent hypergeometric equation[522]*, otherwise known as the Kummer equation (***Kummer***, 1836). Special cases and associated functions of this class are: Parabolic cylinder functions (***Weber***, 1869), Hermite polynomials (***Hermite***, 1864), Laguerre polynomials*[523] *and functions (***Laguerre***, 1879), Error functions*[524]*, Gamma functions*[525] *(***Euler***, 1729;* **Legendre***, 1876;* **Schlömilch**,

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[522] Gauss' hypergeometric equation is

$$x(1-x)y'' + [c - (a+b+1)x]\, y' - aby = 0.$$

A solution which is regular at $x = 0$ is given by the hypergeometric series

$$y = {}_2F_1(a, b; c; x) = 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \dots.$$

If the independent variable is changed from x to $u = bx$, the former equation becomes

$$u\left(1 - \frac{u}{b}\right) y'' + \left[(c-u) - \frac{(a+1)}{b}u\right] y' - ay = 0.$$

If we let $b \to \infty$ it becomes

$$uy'' + (c-u)y' - ay = 0,$$

which is the *confluent equation*. A solution of this equation is given by the series

$$u = {}_1F_1(a; c; x) = 1 + \frac{a}{c}x + \frac{a(a+1)}{2!c(c+1)}x^2 + \frac{a(a+1)(a+2)}{3!c(c+1)(c+2)}x^3 + \dots.$$

[523] Laguerre and Hermite polynomials have important applications in quantum mechanics of the hydrogen atom and the linear harmonic oscillator, respectively.

[524] $Erfc(x) = \int_x^\infty e^{-t^2}dt$ and $Erf(x) = \int_0^x e^{-t^2}dt$ occur in connection with the theories of probability, observation errors and heat conduction.

[525] $\gamma(n, x) = \int_0^x t^{n-1}e^{-t}dt$.

1871) and the Logarithmic integral[526] *(**Euler**, 1755). **Thomas Clausen** (1801–1885) introduced the generalized hypergeometric function (1828)*

$$ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)=\sum_{k=0}^{\infty}\frac{\Pi_{i=1}^{p}(a_i)_k z^k}{\Pi_{i=1}^{q}(b_i)_k k!}, $$

where the notation $(a)_k = a(a+1)\cdots(a+k-1)$ *is known as the Pochhammer symbol (**Leo Pochhammer**, 1841–1920, Germany).*

Various mathematical constants, all elementary functions, and many special functions can be expressed in the hypergeometric notation; for example:

$$ \cos(z) = {}_0F_1\left(\frac{1}{2};\frac{-z^2}{4}\right) $$

$$ \log(z+1) = z\ {}_2F_1(1,1;2;-z) $$

$$ \mathrm{erf}(z) = \frac{2z}{\sqrt{\pi}}\ {}_1F_1\left(\frac{1}{2};\frac{3}{2};-z^2\right)^2 $$

$$ \pi = 4-\frac{8}{9}\ {}_3F_2\left(\frac{1}{2},\frac{1}{2},1;\frac{5}{2},\frac{5}{2};-1\right) $$

$$ \pi = \frac{426880\sqrt{10005}}{13591409}\Bigg/\left({}_3F_2\left(\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1;\frac{-1}{151931373056000}\right)\right. $$
$$ \left. -\frac{30285563}{1651969144908540723200}\ {}_3F_2\left(\frac{7}{6},\frac{3}{2},\frac{11}{6};2,2;\frac{-1}{151931373056000}\right)\right) $$

E. Elliptic functions and integrals[527] (1655–1920)

The study of the theory of elliptic and associated functions is of great importance for its close relation to the development of the general theory of functions of complex variable and for its important applications in various branches of mathematics, physics and engineering. Extensive and detailed elaboration of the subject has provided a testing ground for discovery and improvement of the general theorems of complex variable: the theorems of

[526] $\ell_i(x) = \int_0^x \frac{dt}{\log t}$.

[527] The terminology for elliptic integrals and functions has changed during their investigation. What were originally called *elliptic functions* are now called *elliptic integrals* and the term elliptic functions is reserved for a different idea. We will use modern terminology throughout this section to avoid confusion.

Liouville[528] *(1847) and* ***Picard*** *(1879), the theorems of multiple periodicity and many other important results of the theory of functions of complex variable.*

*The theory of elliptic and associated functions has a very wide field of applications in the analytical theory of numbers [****Gauss****,* ***Jacobi****,* ***Hermite****,* ***Hardy****,* ***Ramanujan****] and in the theory of equations [****Hermite****,* ***Kronecker****, 1858], where the general solution of a quintic equation was obtained in terms of elliptic functions.*

*In the field of geometry, elliptic functions are of great use in studying the properties of certain classes of curves. In mechanics, the earliest applications were to the problems of the simple pendulum with a finite amplitude (****Euler****), the spherical pendulum (****Lagrange****,* ***Richelot****, 1852) and the motion of a rigid body about a point (****Legendre****). Further applications appear in the theories of potential, elasticity, electrostatics and heat conduction.*

*The first encounter with elliptic integrals resulted from attempts to harness the calculus for the rectification of the ellipse (****Wallis****, 1655). What is now known as an* elliptic integral*, occurs in the researches of* ***Jakob Bernoulli*** *on the* Elastica *(1694). He was first to notice that these integrals cannot be expressed in terms of elementary functions*[529]*.*

Giulio Carlo Fagnano dei Toschi *(1682–1766, Italy, 1715) made extensive research of elliptic integrals and proved that the difference of any two elliptic arcs is algebraic.* ***Euler*** *was acquainted with the results of Fagnano in* 1751 *and obtained from it suggestions for his proof of the addition theorem of elliptic integrals (1761). He systematically studied the geometri-*

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[528] If $f(z)$ is analytic for all values of z and if $|f(z)| < K$ for all z, where K is a constant [so that $|f(z)|$ is bounded as $z \to \infty$], then $f(z)$ is constant. This theorem furnishes short and convenient proofs of some of the most important results in analysis, e.g. that an elliptic function $f(z)$ with no poles in a cell is merely a constant.

[529] An ellipse has the parametric equations $x = a \sin \theta$, $y = a \cos \theta$, where $a > b$ and the eccentric angle θ measured from the minor axis. If s is the arc length parameter measured clockwise around the curve from the end B of the minor axis, then

$$ds^2 = \sqrt{(dx^2 + dy^2)} = \sqrt{(a^2 \cos^2 \theta + b^2 \sin^2 \theta)}\, d\theta = a\sqrt{(1 - e^2 \sin^2 \theta)}\, d\theta$$

where $e = \sqrt{1 - \frac{a^2}{b^2}}$ is the eccentricity. Thus, the length arc from B to any point P were $\theta = \varphi$ is given by $s = a \int_0^{\varphi} \sqrt{1 - e^2 \sin^2 \theta}\, d\theta = aE(u, e)$, where E is the elliptic integral of the second kind.

cal applications of elliptic integrals, and proposed that they be recognized as primitive new transcendentals to be investigated on their own merits. ***Legendre*** *worked more than* 40 *years on the systematic development of this vast field.*

Abel *(1827),* ***Jacobi*** *(1827) and* ***Gauss*** *(1797, unpublished) revolutionized this subject, and opened the floodgates to* 19^{th} *century analysis, with the simple idea of inverting the elliptic integral*[530]. *Abel's first important discovery in this connection was the double periodicity of this inverse function, known as an elliptic function, which thus started the study of elliptic functions proper. Abel also generalized the elliptic integrals, and considered the integrals (and their inverse functions) later named as Abelian integrals and Abelian functions.*

Jacobi *(1827–1829) was the principal and most accomplished investigator of elliptic and theta functions*[531]. *He obtained their properties by purely algebraic methods. His analysis is so complete that practically most of the results known today are to be found in his works.*

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[530] Instead of directly investigating the elliptic integral

$$y = \int^x \left[(1 - c^2x^2)(1 + e^2x^2)\right]^{-1/2} dx,$$

they proposed to consider the inverse function $x = F(y)$. This procedure is similar to that of defining the inverse circular function and the logarithmic functions by integrals $\sin^{-1} x = \int^x \frac{dx}{\sqrt{1-x^2}}$, $\log x = \int^x \frac{dx}{x}$ respectively, and then establishing the properties of the circle and exponential functions from the corresponding inverse functions. The advantage of this artifice is that instead of having to consider, say, the ∞-ly *multiple-valued* function $y = \sin^{-1} x$, restricted to the range $-1 < x < 1$ (for real values of x), we may, by writing the same equation as $x = \sin y$, treat x as *single valued* function of y, which is much easier to deal with.

[531] The four functions

$$\theta_1(z, q) = 2\sum_{n=0}^{\infty}(-)^n q^{(n+\frac{1}{2})^2} \sin(2n + 1)z,$$

$$\theta_2(z, q) = 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} \cos(2n + 1)z,$$

$$\theta_3(z, q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nz,$$

$$\theta_4(z, q) = 1 + 2\sum_{n=1}^{\infty}(-)^n q^{n^2} \cos 2nz,$$

The next epoch in the development of the theory of elliptic functions started with the works of ***Liouville*** *(1844–1851) and* ***Weierstrass*** *(1854). In their approach, the elliptic functions are not introduced as inverse functions of primitives of elliptic integrals — instead they are suitably defined by some important properties, such as double periodicity*[532]*, meromorphism etc.*

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are known as the *theta functions*. They solve the *heat-conduction equation*

$$k\frac{\partial^2\theta}{\partial z^2} = \frac{\partial\theta}{\partial t},$$

where k is the *diffusivity* and $q = e^{-4kt}$. For most applications it is convenient to pass to complex time τ such that $q = e^{\pi i\tau}$, $\operatorname{Im}\tau > 0$, $|q| < 1$, to secure convergence. Clearly θ_1, θ_2 are periodic with period 2π and θ_3, θ_4 are periodic with period π. Interestingly enough, the series are *also pseudoperiodic* in z with period $\pi\tau$, namely $\theta_1(z+\pi\tau, q) = -N\theta_1(z,q)$ where $N = q^{-1}e^{-2iz}$, etc. The theta function can also be represented as a product of *partition functions*, e.g.

$$\theta_4(z,q) = \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1}e^{2iz})(1-q^{2n-1}e^{-2iz}).$$

[Euler was first to study, in 1748, the partition function $\prod_{m=1}^{\infty}(1-q^m x)$.]
The solution of

$$\left(\frac{dy}{du}\right)^2 = (1-y^2)/(1-k^2y^2)$$

is $y = sn(u,k)$, known as the *Jacobian elliptic function of* u. It can be shown that

$$y = \frac{\theta_3(0,q)}{\theta_2(0,q)}\frac{\theta_1(x,q)}{\theta_4(x,q)},$$

where $x = \frac{u}{[\theta_3(0,q)]^2}$ with $k^2 = \left[\frac{\theta_2(0,q)}{\theta_3(0,q)}\right]^4$.

532 In analogy to the relation $\frac{1}{(\sin z)^2} = \sum_{m=-\infty}^{\infty}\frac{1}{(z-m\pi)^2}$, Weierstrass defined a new function

$$\wp(z) = z^{-2} + \sum_{m,n}{}' \left\{\frac{1}{(z-\Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2}\right\}$$

where $\Omega_{m,n} = 2m\omega_1 + 2n\omega_2$, and $\sum' = \sum_{m,n}$, $(m,n) \neq (0,0)$.
It turns out that $\wp(z)$ is doubly periodic, with no singularities but poles. Hence it is an elliptic function, and satisfies the differential equation:

$$\left[\wp'(z)\right]^2 = 4\wp^2(z) - g_2\wp(z) - g_3$$

with

$$g_2 = 60\sum_{m,n}{}'\Omega_{m,n}^{-4}, \quad g_4 = 140\sum_{m,n}{}'\Omega_{m,n}^{-6}$$

The final stage in the development of elliptic functions, involving their mysterious relations to number theory, is marked by the contributions of ***Hermite****,* ***Klein****,* ***Kronecker****,* ***Dedekind****,* ***Poincaré*** and ***Ramanujan****. Their approach was based on two novel ideas: one is the concept of* automorphic function *introduced by Poincaré, and the other is the concept of* modular function*, which arises from the behavior of the* modulus *k^2 as a function $f(\tau)$ in the complex τ-plane, where $f(\tau) = \frac{\theta_2^4(0,\tau)}{\theta_3^4(0,\tau)}$.*

F. Differential equations with periodic coefficients (1868–1940)

In 1868, **Émile Léonard Mathieu** *(1835–1890,* France*) determined the vibrational modes of a stretched membrane having an elliptical boundary. Solving the two-dimensional wave equation in elliptical coordinates by the method of separation of variables, a second order equation, known today as the Mathieu equation*

$$[y'' + (a - 2q\cos 2z)y = 0]$$

is obtained. Its solutions are the Mathieu functions. Other problems of mathematical physics which lead to the Mathieu equation are:

(1) Tidal waves in a cylindrical vessel with an elliptical boundary.

(2) Certain forms of steady vortex motions in an elliptical cylinder.

(3) Diffraction of sound and electromagnetic waves by a right elliptical cylinder.

(4) Propagation of waves in elliptical wave guides.

(5) Heat conduction in an elliptical cylinder.

(6) Amplitude distortion in a moving-coil loudspeaker.

H.G. Hill *(1887) investigated the mean motion of the lunar perigee by means of a generalized form of Mathieu equation*

$$[y'' + \{a - 2q\psi(2z)\}y = 0\,,$$

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and

$$z = \int_{\wp}^{\infty} [4t^3 - g_2 t - g_3]^{-1/2} dt.$$

where

$$\psi(2z) = -\sum_{r=1}^{\infty} \theta_{2r} \cos 2rz \Bigg].$$

In 1883, ***G. Floquet*** *proved an important theorem concerning the general nature of the solutions of Hill's equation. Since then, an extensive literature has accumulated on the exact, asymptotic and numerical solutions of Mathieu functions [**Lindemann** (1883), **Stieltjes** (1884), **Maclaurin** (1899), **Hilbert** (1904), **Sieger** (1908), **Whittaker** (1914), **Ince** (1924–1939), **Erdélyi** (1934–1936), **Bickley** (1940)].*

1877–1902 CE **Benjamin Baker** (1840–1907, England). Engineer and bridge constructor. Involved in the construction of Metropolitan railways and in designing the cylindrical vessel in which *Cleopatra's needle* was brought over from Egypt to England (1877–1878). Designed and erected the *Forth Bridge* in Scotland (with John Fowler) in 1890. Directed the construction of the Assam dam (1902). Pioneered in the construction of intra-urban railways in deep tubular tunnels built up of cast iron segments. Author of many papers on engineering subjects.

Baker was born near Bath and received his early training in a South Wales ironworks. He afterwards became associated with John Fowler in London.

1878–1886 CE **Francois Marie Raoult** (1830–1901, France). Physicist and chemist. Discovered (1878) that the depression of the *freezing points*[533] of liquids due to the presence of a substances dissolved in them is proportional to the solute's *molarity (moles per unit solvent weight), and number of dissociated ions per molecule of solute with a coefficient depending only upon the solvent (cryoscopic coefficient).* Introduced (1886) the law named after him, stating that in a dilute solution, the lowering of the *vapor pressure* of the solvent is proportional to the *molecular weight* of the substance dissolved, unless the solvent is an electrolyte. Both phenomena afforded new methods of determining the molecular weight of substances.

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533 The bare fact that the presence of dissolved substances in water lowers its freezing-point was already known to the English physician **Charles Blagden** (1748–1820). ["Experiments on the cooling of water below its freezing point", *Phil. Trans.* **78**, 120–130, 1788; and "Experiments on the effect of various substances in lowering the point of congelation in water", *Phil. Trans.* **78**, 277–312, 1788.]

W. Ostwald and **J.H. van't Hoff** used Raoult's laws to support the hypothesis of electrolytic dissociation.

Raoult was born at Fournes en-Weppes Nord and taught at Grenoble (1867–1901).

1878–1913 CE **Ferdinand de Saussure** (1857–1913, Switzerland). Linguist. The father of modern linguistics. His book *Memoir on the primitive system of vowels in Indo-European languages* (1878) was a major breakthrough in comparative philology. His most famous work *Course in General Linguistic* was published posthumously by his students (1915).

de Saussure led the *structural movement* against the *comparative*[534] method and formulated many basic principles of structural linguistics (1906) which apply to *all* languages. He urged the development of a general science of signs aided by mathematics — suggestions which were intensively followed by modern philologists.

de Saussure was born in Geneva. He studied Indo-European languages at the Leipzig and Berlin Universities. After several years of teaching in Paris, he returned to his native Geneva (1891) and remained there until his death.

1879 CE **Charles Émile Picard** (1856–1941, France). Mathematician. Advanced research in the fields of analysis, algebraic geometry and mechanics.

Developed an existence theorem for differential equations based on the method of *successive approximations.*

Picard worked on quadratic forms, Abelian functions and the allied theories of discontinuous and continuous groups of transformations. His work led to a study of the algebraic manifold, now known as the *Picard variety*, which play a fundamental role in algebraic geometry.

In 1879 Picard proved the theorem named after him. This theorem became the starting point for many important studies in the theory of complex functions[535].

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[534] The early *structuralists* believed that the comparativists overemphasized languages as written in the past, and ignored languages as spoken today. The structuralists also disagreed with the traditional method of describing languages by *paradigms* (patterns) of conjugations and declensions. They studied many non-Indo-European languages and found that some do not have conjugations and declensions. Thus, these languages could not be described by the traditional method.

[535] In the neighborhood of an isolated essential singularity, a one-valued function takes every value, with one possible exception, an infinite number of times.

During 1878–1899, Picard held various posts with the universities of Paris, Toulouse and the École Normale Supériere. In 1898 he was appointed professor at the University of Paris.

1879 CE **Edwin Herbert Hall** (1855–1938, U.S.A.). Physicist. Discovered the "Hall effect" in which a voltage is produced across a current-carrying conductor in a magnetic field. The Hall voltage is perpendicular to both the direction of the current and the direction of the magnetic field, and proportional to the current and the magnetic field. Since different material produce different Hall voltages, scientists can use the Hall effect as a probe of the electronic structure of various materials[536].

Hall was born at Great Falls, ME. He was a professor at Harvard University (1888–1921).

1879 CE **Joseph Stefan** (1835–1893, Austria). Physicist. Empirically discovered the law that the total energy radiated by a blackbody per unit area-time (u) is proportional to the 4^{th} power of the temperature T: $u = \sigma T^4$. In 1884, **L. Boltzmann** derived the law theoretically[537], and since then it is known as the "*Stefan-Boltzmann law*" and σ as the "*Stefan-Boltzmann constant*". At any rate, Stefan's contribution was the first important step toward the understanding of black-body radiation, from which sprang the idea of the quantum of radiation.

Stefan was born at St. Peter, Austria and did his major work at the University of Vienna [lecturer, 1858; full professor, 1863; director of the Physical Institute, 1866].

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536 **Hall** designed a series of thought experiments which demonstrated that the magnetic force should deflect the charge carriers and cause them to collect on one side of the conductor. Charge carries in most metals bear a negative charge, (*electrons* were discovered by **J.J. Thomson** in 1897).

The *Hall effect* is easily understood in terms of the simple free-electron model of **Drude**, but for metals with valence > 1 (e.g. Be, Mg, In and Al), the explanation of experimental results require a quantum treatment of the effect.

In 1980, **Klaus von Klitzing** discovered the *quantized Hall effect* (Nobel prize, 1985), in which changes in resistance in a plate kept in a magnetic field at temperatures near absolute zero occur in discrete steps instead of continuously; it is an example of quantum behavior that is directly observable macroscopically.

537 The laws of Stefan-Boltzmann and Wien follow purely as a result of the general laws of thermodynamics and the electromagnetic nature of radiation. Indeed, Boltzmann was able to provide their theoretical basis *without* the use of Planck's radiation formula.

1879–1906 CE **Henri Jules Poincaré** (1854–1912, France). Outstanding and versatile mathematician, theoretical astronomer, and philosopher of science. He is known as 'the last of the universalists' since he was the last man to have had a creative command of the whole of mathematics as existed in his day, including: algebra, geometry, arithmetic and analysis — as well as the entire gamut of mathematical physics (celestial mechanics, general analytical mechanics, optics, elasticity, thermodynamics, potential theory, electromagnetism). He will probably be the last man who will ever be in this position.

Poincaré did not dwell on any particular field long enough to round his work. A contemporary said of him: "*He was a conqueror, not a colonist*". He had an unusually retentive memory for everything he read, and could also visualize what he heard. Throughout his life he was able to perform complex mathematical calculations in his head, and could quickly write a paper without extensive revisions. He produced more than 30 books and 500 technical papers. His major achievements are:

- Virtually founded the theory of automorphic functions[538]. He found that these functions are associated with transformations arising in non-Euclidean geometry (1879–1887). Such functions are generalizations of trigonometric functions $(a = d = 1,\ c = 0,\ b = 2k\pi)$ and elliptic functions. **Hermite** had studied such transformations for the restricted case in which the coefficients a, b, c, d are integers, and satisfy $ad - bc = 1$, and had discovered a class of elliptic modular functions invariant under the restricted transformations. But Poincaré's generalization uncovered a broader category of functions, known as *zeta-Fuchsian functions*, which could be used to solve 2^{nd} order linear differential equations with algebraic coefficients.

  Poincaré found that two automorphic functions, invariant under the same group, are connected by an algebraic equation. Conversely, the coordinates of a point on any algebraic curve can be expressed in terms of automorphic functions, and hence by uniform functions of a single parameter [e.g. $x^2 + y^2 = a^2$ is parametrically represented by $x = a\cos t$, $y = a\sin t$].
- Developed the theory of asymptotic series representation of functions (1886).
- Father of *algebraic topology* (Poincaré Conjecture, 1904).

[538] An automorphic function $f(z)$ of the complex variable z is one which is analytic, except for poles, in a domain D and which is invariant under a denumerably infinite group of linear fractional transformations $z' = \frac{az+b}{cz+d}$, i.e. $f\left(\frac{az+b}{cz+d}\right) = f(z)$.

- Established combinatorial topology, i.e. the study of intrinsic qualitative aspects of spatial configurations that remain invariant under continuous 1–1 transformations[539].

  Among other things, he generalized *Euler's formula* $V - E + F = 2$ to n dimensional space. Instead of vertices (V), edges (E) and faces (F), we then have $0-, 1-, 2-, \dots, (n-1)$-dimensional entities. If the numbers of these entities is $N_0, N_1, \dots, N_{n-1}$ respectively, then the equation $N_0 - N_1 + N_2 - \cdots = 1 - (-1)^n$ applies to the manifolds corresponding to the simple polyhedra. For $n = 3$ this reduces to *Euler's formula.*

Poincaré's generalization furnished a determination of the *regular* polytopes in higher dimensional spaces [in 3 dimensions there are only 5 regular polyhedra: *Tetrahedron* ($V = 4$, $E = 6$, $F = 4$); *Cube* ($V = 8$, $E = 12$, $F = 6$); *Octahedron* ($V = 6$, $E = 12$, $F = 8$); *Dodecahedron* ($V = 20$, $E = 30$, $F = 12$); *Icosahedron* ($V = 12$, $E = 30$, $F = 20$)].

- Advanced the qualitative study of *nonlinear differential equations* by the introduction of *topological* arguments. To describe the nature of a singular point he introduced the notion of an *index*[540].

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[539] Transformations which are continuous and which have a continuous inverse transformation. Such transformations are known as *homeomorphisms.* Two homeomorphic manifolds are said to be *topologically equivalent*.

[540] Consider a singular point P_0 and a simple closed curve C surrounding it. At each intersection of C with the solutions of

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)} ,$$

there is a direction angle of the trajectory, which we shall denote by ϕ and which can have any value from 0 to 2π radians. If a point now moves in a counterclockwise direction around C, the angle ϕ will vary; and after completion of the circuit around C, ϕ will have the value $2\pi I$ where I is an integer or zero (since the direction angle of the trajectories has returned to its original value). The quantity I is the index of the curve. It can be proved that the index of a closed curve that contains several singularities is the algebraic sum of their indices. The index of a closed trajectory (and of no other simple closed curve) is $+1$.
The nature of the trajectories can be determined by the characteristic equation, and so the index I of a curve should be determinable by knowing just the differential equation. One can prove that

$$I = \frac{1}{2\pi}\int_C d\left(\arctan\frac{P}{Q}\right) = \frac{1}{2\pi}\int_C \frac{Q\,dP - P\,dQ}{P^2 + Q^2} ,$$

where the path of integration is the closed curve C.

- Proved the *recurrence theorem*, stating that a non-dissipative dynamical system having a finite energy and confined to a finite volume, will, after sufficiently long time, return to an arbitrary small neighborhood of almost any initial state.

- Contributed[541] substantially to the classical 'n-body problem' in celestial mechanics: given the present masses, velocities, and mutual positions of n bodies, how long will they remain stable in their present orbits? In his solution, Poincaré initiated the qualitative theory of non-linear differential equations. He developed new mathematical techniques and made fundamental discoveries on the behavior of the integral curves of differential equations near singularities (1889–1895).

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[541] One of Isaac Newton's most important discoveries was that two bodies moving under the influence of each other's gravitational fields, both follow ellipses (or hyperbolas or parabolas). The question was: how do three bodies move under Newtonian gravitational forces? The 'two-body problems' is "integrable" — the laws of conservation of energy and momentum restrict solutions to such an extent that they are forced to take simple mathematical form. The suspicion that three bodies can move *chaotically* (chaos — apparently random motion with purely deterministic causes — too complicated to occur in an integrable system) predates the recognition of chaos in mathematics; indeed, it was one of the key historical steps in its discovery.

In 1887, King Oscar II of Sweden was worried about the stability of the solar system. Will it persist forever, behaving such as it does today, or will a planet escape or crash into the sun?

A prize of 2500 crowns, offered by the king to anyone who could solve the problem, was won by the leading mathematician of the day, Henri Poincaré — even though he did not answer the question. However, what Poincaré achieved was more important — the introduction of qualitative geometrical methods into dynamics. It led him to discover some curious behavior that we now recognize as *chaos*.

He found it in the '*restricted three-body problem*' (an idealization in which one body is assumed to have such a small ("test") mass that the other two are not affected by it). This test body does, of course, respond to the gravitational fields of the two more massive bodies. The question is: does the chaos persist if we make the model more realistic by including the very small but non-zero gravitational effects of the almost massless test body in our calculations?

In 1994, **Zhihong Xia** of the Georgia Institute of Technology, U.S.A., proved that a system of three bodies is not integrable (i.e., it has no conserved quantity other than energy and linear and angular momentum) and that the full three-body problem is chaotic.

- Contributed to the theory of numbers by demonstrating how the concept of binary quadratic forms (developed by Gauss) could be cast in a geometric form (1904).
- Made important contributions to the theory of equilibrium of gravitating rotating fluid masses. In particular he described the conditions of stability of the pear-shaped figures that played a prominent part in evolution models of celestial bodies.

Many of the problems he tackled were seeds of new ways of thinking,which have since grown and flourished in 20^{th} century mathematics.

Poincaré was born in Nancy into a distinguished family [his first cousin, Raymond Poincaré (1860–1934) was president of the French republic during WWI]. He studied at the École Polytechnique (1872–1875), devoting himself to scientific mining, and took his doctorate in 1879. He was lecturer at Caen and then moved to the University of Paris in 1881, where he held several professorships in mathematics and science.

Poincaré died of embolism, a week after a successful prostate operation. He was in his 59^{th} year and at the height of his powers — '*the living brain of the rational sciences*', in the words of Painlevé.

The Poincaré Conjecture[542] *(1904–2003)*

"One of the early successes of topology was to show that just two topological invariants, the Euler characteristic *and* orientability*, are all you need to be able to distinguish any two closed surfaces. That is to say, if two surfaces have the same Euler characteristic and are either both orientable or both non-orientable, then they are in fact the same — even if one is unable to see how to continuously deform one into the other. This result is called the classification theorem for surfaces, since it says that one can classify all surfaces (topologically) by means of just these two attributes.*

542 Quotations are from Keith Devlin's book *The Millennium Problems*, Basic Books: New York, 2002.

Loosely speaking, the classification theorem for surfaces is proved by taking a sphere as the basic surface and measuring the degree to which any given surface differs from a sphere — what would one have to do to a sphere to turn it into that surface. This corresponds to our ordinary intuition that a sphere is the simplest, most basic, and, some might say, the most aesthetically perfect closed surface.

It should be pointed out that in this case, the operations to be performed on a sphere to turn it into some other surface go beyond the normal topological operations of continuous deformations. Indeed, if one changes a sphere by means of twisting, bending, stretching, or shrinking, the resulting object, topologically, will still be a sphere. To classify surfaces by seeing how they can be constructed from a sphere, one has to allow cutting and stitching together in addition to the usual twisting, stretching, etc. Topologists refer to this process as "surgery." The term is apt, since a typical surgical operation involves cutting one or more pieces from the sphere, twisting, turning, stretching, or shrinking each of those pieces, and then sewing those pieces back into the sphere again.

The classification theorem tells us that any orientable surface is topologically equivalent to a sphere with a certain number of "handles" sewn onto it. You get a handle by cutting two holes into the sphere and joining them together by means of a tube. Any non-orientable surface is equivalent to a sphere with a certain number of "crosscaps" sewn in. You get a crosscap by cutting a hole in the sphere and sewing a Möbius band to the boundary of the hole. As with the Klein bottle, in ordinary three-dimensional space one cannot do this without the Möbius band passing through itself; one needs four dimensions to do it properly.

In the early years of the twentieth century, Poincaré and other mathematicians set out to classify higher-dimensional analogues of surfaces — which they called "manifolds." Not surprisingly, they tried an approach similar to the one that had worked for two-dimensional surfaces. They sought to classify all three-dimensional manifolds (called "3-manifolds" for short) by taking a three-dimensional analogue of a sphere (called a "3-sphere") as basic and measuring the degree to which any 3-manifold differs from that 3-sphere.

One has to be careful here. A regular surface such as a sphere or a torus is a two-dimensional object. The figure the surface encloses is three-dimensional, of course, but the surface itself is two-dimensional. Apart from a plane, any surface can be constructed only in a space of three or more dimensions. Thus, any closed surface requires three or more dimensions. For instance, it takes three dimensions to construct a sphere or a torus, four dimensions to construct a Klein bottle. Yet a sphere, a torus, or a Klein bottle is a two-dimensional

object — a surface that has no thickness and can, in principle, be constructed from a flat, perfectly elastic sheet.

But just as a sphere can be regarded as a two-dimensional analogue (in three-dimensional space) of a circle (which is a one-dimensional object – a curved line – in two-dimensional space), so too we can imagine a three-dimensional analogue (in four-dimensional space) of a sphere. Well, actually, we can't imagine it. But we can write down equations that determine such an object, and study "it" mathematically. Indeed, physicists routinely study such imaginary objects, and use the results to help understand the universe we live in. The 3-manifolds, i.e., the three-dimensional analogues of surfaces (which exist in spaces of four or more dimensions), are sometimes called hypersurfaces, with the three-dimensional analogue of a sphere being called a hypersphere.

There is no mathematical reason to stop at three dimensions. One can write down equations that determine manifolds of 3, 4, 5, 6, or any number of dimensions. Once again, these considerations turn out to be more than idle speculation. The mathematical theories of matter that physicists are currently working on view the universe we live in as having 11 dimensions. According to these theories, we are directly aware of three of those dimensions, and the others manifest themselves as various physical features such as electromagnetic radiation and the forces that hold atoms together.

Poincaré attempted to classify manifolds of three and more dimensions by taking a "sphere" of the respective dimension as a base figure and then applying surgery, A natural first step in this endeavor was to look for a simple topological property that tells you when a given (two-dimensional) surface is topologically equivalent to a sphere. (Even in the simple case of regular two-dimensional surfaces, a surface might appear extremely complicated and yet turn out to be continuously deformable to a sphere.)

In the case of two-dimensional surfaces, there is such a property. Suppose you were to take a pencil and draw a simple closed loop on the surface of a sphere. Now imagine the loop shrinking in size, sliding over the surface as it does so. Is there a limit to how small the loop can shrink? Obviously not. One can shrink the loop until it becomes indistinguishable from a point. Mathematically, one cane shrink it until it actually becomes a point.

The same thing is not necessarily true if one starts with a loop drawn on a torus. One can draw loops on a torus that cannot be shrunk down to a point. No loop that goes right around the ring of the torus can be shrunk down indefinitely, nor can any loop that encircles the torus like a belt.

The shrinkability to a point of any loop drawn in a surface is a topological property of the surface that is unique to spheres. That is to say, if one has

a surface on which every loop (the "every" is important here) can be shrunk down to a point without leaving the surface, then that surface is topologically equivalent to a sphere.

Is the same true for a three-dimensional hypersphere? This is the question Poincaré asked in the early 1900s, hoping that a speedy positive answer would be the first step on the road to a classification theorem for three-dimensional hypersurfaces. He developed a systematic method — called homotopy theory *— for studying (using methods of algebra) what happens to loops when they are moved around a manifold and deformed.*

Actually, that's not quite what happened. At first, Poincaré tacitly assumed *that the loop-shrinking property for 3-manifolds did characterize the 3-sphere. After a while, however, he realized that his assumption might not be valid, and in 1904 he put his doubts into print, writing (in French): "Consider a compact three-dimensional manifold V without boundary. Is it possible that the fundamental group of V could be trivial, even though V is not homeomorphic to the three-dimensional sphere?" Stripping away the technical terms, what Poincaré asked was, "Is it possible that a 3-manifold can have the loop-shrinking property and not be equivalent to a 3-sphere?" That was the birth of the Poincaré conjecture.*

As it turned out, his question did not get a speedy answer. Nor, indeed, a slow answer, despite the best efforts of a number of leading topologists. As a result, finding a proof (or a disproof) of the Poincaré conjecture rose to become one of the most sought-after prizes in mathematics."

Thus, in 1904, Poincaré conjectured that every simply connected, closed, orientable 3-dimensional manifold[543] is homeomorphic to the sphere of that dimension[544] (the surface of a 4-dimensional solid sphere). This conjecture has been generalized to read:

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[543] A 3-dimensional manifold is a space such that every point has an open neighborhood homeomorphic to a 3-dimensional Euclidean space. From the point of view of STR, we are living on a 3-dimensional sphere.

[544] Earlier, Poincaré asserted that any two closed manifolds that have the same *Betty numbers* and torsion coefficients are homeomorphic. But he soon gave an example of a 3-dimensional manifold that has the Betti numbers and torsion coefficients of the 3-dimensional sphere but is not connected. Hence he added simple connectedness as a condition.

He then showed that there are 3-dimensional manifolds with the same Betti numbers and torsion coefficients but which have different fundamental groups and so are no homeomorphic. However, **James W. Alexander** showed (1919) that two 3-dimensional manifolds may have the same Betti numbers, torsion coefficients, and fundamental group and yet not be homeomorphic.

> *"Every simply connected, closed, n-dimensional manifold that has Betti numbers and torsion coefficients of the n-dimensional sphere, is homeomorphic to it."*

This generalized conjecture has been proved by ***Stephen Smale*** *(1960) for $n \geq 5$ and for $n = 4$ by* ***Michael Freedman*** *(1982). The case $n = 2$ is classical (and was known even to 19th century mathematicians), and the case $n = 1$ is trivial.*

Thus, by 1982, the only unsettled case of the Poincaré conjecture was the one originally posed by Poincaré, in three dimensions. This happened because 2-dimensional space is too small to have room for any serious complexity, and 4- or higher dimensional space is so big that the complexities can be arranged nicely. In 3 dimensions there is a creative tension: big enough to be complicated; too cramped to be easily simplified. What was needed was a line of attack that exploited the special properties of 3-dimensional manifolds. This feat was finally achieved in 2003 by ***Grigori Perelman****, of the Steklov Institute of Mathematics, of the Russian Academy of Sciences in St. Petersburg. It carried with it a prize of one million dollars, given by the Clay Mathematical Institute. (For an account of recent development in this field see "The Poincare Conjecture" by D. Oshea, Walker and Co. New York 2007, 293 pp.)*

Worldview XXV: Poincare

* * *

"Mathematics is the art of giving the same name to different things. [As opposed to the quotation: Poetry is the art of giving different names to the same thing]."

* * *

"Later generations will regard Mengenlehre (set theory) as a disease from which one has recovered."

* * *

"What is it indeed that gives us the feeling of elegance in a solution, in a demonstration? It is the harmony of the diverse parts, their symmetry, their happy balance; in a word it is all that introduces order, all that gives unity, that permits us to see clearly at once both the ensemble and the details."

* * *

"Mathematicians do not study objects, but relations between objects. Thus, they are free to replace some objects by others so long as the relations remain unchanged. Content to them is irrelevant: they are interested in form only."

* * *

"The mind uses its faculty for creativity only when experience forces it to do so."

* * *

"Mathematical discoveries, small or great, are never born of spontaneous generation. They always presuppose a soil seeded with preliminary knowledge and well prepared by labor, both conscious and subconscious."

* * *

"Absolute space, that is to say, the mark to which it would be necessary to refer the earth to know whether it really moves, has no objective existence... The two propositions: "The earth turns round" and "it is more convenient to suppose the earth turns round" have the same meaning; there is nothing more in the one than in the other."

* * *

"...by natural selection our mind has adapted itself to the conditions of the external world. It has adopted the geometry most advantageous to the species or, in other words, the most convenient. Geometry is not true, it is advantageous."

* * *

"The problem is not what is the ANSWER, the problem is in what is the QUESTION."

* * *

"THOUGHT is the lightening between two infinities of blackness. But it is the lightening which matters."

1879–1918 CE **John Moses Browning** (1855–1926, USA). Inventor and designer of firearms. Designed a series of pistols, rifles and shotguns. The United States Army adopted his machine-gun (1890) and his automatic rifle (1918). He became internationally famous for designing and inventing automatic arms, including the Browning automatic rifle. He was born in Ogden, Utah.

1880 CE **Charles Louis Alphonse Laveran** (1845–1922, France). Physician and parasitologist. Discovered the blood parasite causing *malaria*[545] (Nobel prize for physiology or medicine, 1907).

[545] The British physician-bacteriologist **Ronald Ross** (1857–1932) discovered (1898) in India that mosquitoes can transmit malaria to birds. For this he was awarded (1902) the Nobel prize. The Italian zoologist **Giovanni Battista Grassi** (1854–1925), building on the work of Ross, determined that malaria is spread to humans by the *Anopheles* mosquito (1899). In that year, the complete life-cycle of the parasite became known.

Laveran worked as a military physician (1870–1896). While in Algeria to study malarial fever (1878–1883) he found microscopic parasites in red blood cells of human victims of the disease. He continued his research with the Pasteur Institute in Paris (1896–1922). Author of *Traité des fievres palustres* (1884) and *Traité d'hygienne militaire* (1896).

1880 CE **Aurel Edmund Voss** (1845–1931, Germany). Mathematician. Was first to derive the *contracted* Bianchi identity between the covariant derivatives of the components of the Riemann tensor.

It was discovered independently by **Ricci** in 1889, and then again in 1902 by **Bianchi**. Voss also derived a generalization of Gauss' formula. He was a professor at the University of Münich during 1902–1923.

1880 CE *Piezoelectricity* discovered by **Pierre and Jacques Curie** (certain crystals develop an electric charge on the surface when stretched or compressed along an axis).

1880–1885 CE **Charles Sumner Tainter** (1854–1940, England and USA). Engineer and inventor. Constructed the first system that utilized a photocell to convert sound into light (1880). With **A.G. Bell** developed a working prototype of the '*gramophone*' which used a *wax cylinder* rather than **Edison**'s tinfoil cylinder (1880–1881). Also, with Bell, he invented the '*photophone*' (1881) — an apparatus that transformed sound into light signals which in turn activated a photocell[546].

The name malaria was coined in the 17^{th} century by Dr. **Francisco Torti** by combining the Italian names for "bad" and "air", and it has been called the shakes, the fevers, the ague, and many other things, none affectionate. **Hippocrates** reported several clinical types of malaria. Untreated malaria may kill about one percent of those infected. The survivors, prone to relapse, may suffer from anemia, weakness, sexual impotence, chronic abortion, or secondary infections — all of which lower the value of the individual to self, family, and community.

Throughout men's history, few diseases have played so tragic a role as malaria. It has killed or incapacitated more people than all plagues, wars, and automobiles. Endemic malaria contributed to the downfall of Greece (after 400 BCE). **Alexander the Great** died of it in June 323 BCE, and it was again the malaria that taxed heavily the vitality of Rome in its declining years. **Oliver Cromwell** died of malaria in 1658. Troops in many wars during history were inactivated by malaria, e.g. over 10 percent of the U.S. overseas armies in 1943 had malaria.

546 According to his futuristic idea, the recording and reproduction of sound utilized photocells and a magnetic induction sensing device. However, in lack of elec-

Tainter developed the *dictaphone* (1885), a machine that could record dictations.

1880–1885 CE **James Alfred Ewing** (1855–1935, Japan and Scotland). Physicist and engineer. Helped **John Milne** to construct in Japan (1880) the first useful *seismograph system* for recording local earthquakes. Investigated magnetic properties of iron, steel etc. Observed and named the phenomenon of *hysteresis*[547] (1885).

Ewing was professor at Tokyo (1878–1883), Dundee (1883–1890), Cambridge (1890–1903) and the University of Edinburgh (1916–1929).

1880–1890 CE **Wilhelm Killing** (1847–1923, Germany). Mathematician. An original and profound mathematical thinker. His work was neglected, but his ideas, results, and methods served as a basis to the later works of **Élie Cartan**, **Hermann Weyl**, **Noether**, **Wedderburn**, **Coxeter** and many others. In his epoch making paper: "*Die Zusammenensetzung der stetigen, endlichen Transformationsgruppen*" [Mathematische Ann. 1888–1890; four parts] he originated such key notions as the rank of an algebra, semi-simple algebra, Cartan algebra, root systems and Cartan integers. Weyl's theory of the representation of semi-simple Lie groups would have been impossible without

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tronic amplification (in 1886), the typical output of a single cell was not enough to be 'audible'. [In fact, it was not enough even to light a small flashlight bulb.]

[547] Greek origin meaning: lagging behind. A remarkable aspect of *ferromagnetism*: the tendency of ferromagnetic materials to retain initial magnetization, explained by the fact that the magnetic domains offer *resistance to orientation*. The magnetization of weakly magnetized substances varies linearly with the field strength. However, the magnetization of ferromagnetics (substances capable of having magnetization in the absence of an external magnetic field) depends on $\boldsymbol{H}$ in an intricate way: μ depends on $\boldsymbol{H}$, and consequently $\boldsymbol{B} = \mu\boldsymbol{H}$ depends *nonlinearly* on $\boldsymbol{H}$. If a sample is initially magnetized, we obtain the first portion of the curve in which $\boldsymbol{B}$ increases with $\boldsymbol{H}$ until it begins to flatten off due to *saturation*. On decreasing the external field, the curve does not follow the same path and shows a positive value of $\boldsymbol{B}$ when $\boldsymbol{H} = 0$. This is known as *residual magnetization* in the sample. When $\boldsymbol{H}$ is reversed, it is found that $\boldsymbol{B}$ finally becomes zero at some negative value of $\boldsymbol{H}$, known as the *coercive force*. The other half of the *hysteresis loop* is then obtained by making $\boldsymbol{H}$ still more negative until reverse saturation is reached, and then returning $\boldsymbol{H}$ to the original positive saturation value. From an engineering point of view, these substances are of immense importance and have very important technological consequences.

the results of the above paper. Roughly one third of the extraordinary work of Cartan was based on Killing's paper[548].

Killing's theorem enumerating all possible structures for finite dimensional Lie algebra (which he invented in 1880 independently of **Sophus Lie**) over the complex numbers, was used by Cartan and **Molien** as a paradigm for the development of the structure theory of finite dimensional linear associative algebras.

It was Killing who discovered the exceptional Lie algebra E_8, which today figures prominently in superstring theory. Killing introduced the notion that a vector field on a manifold represents a flow, which induces a continuous change of coordinates on the manifold. A *Killing vector* of a metric is a flow that leaves the metric tensor invariant. It manifests a symmetry (isometry) in the metric. Otherwise stated, the Lie-derivative of the metric tensor along the *Killing vector* vanishes on the manifold.

Under a given coordinate transformation $x \to x'$, the metric tensor $g_{\mu\nu}(x)$ is transformed according to the relation

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x),$$

or equivalently

$$g_{\mu\nu}(x) = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} g'_{\rho\sigma}(x').$$

If we require that the transformed metric be the *same* function of its argument x'^μ as the original metric $g_{\mu\nu}(x)$ was of its argument x^μ (*form-invariance*), we can write the last equation as

$$g_{\mu\nu}(x) = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} g_{\rho\nu}(x').$$

In general, this equation is a very complicated restriction on the function $x'^\mu(x)$. It can be greatly simplified by descending to the special case of an *infinitesimal coordinate transformation* $x'^\mu = x^\mu + \epsilon\xi^\mu(x)$ with $|\epsilon| \ll 1$. Then, to first order in ϵ,

$$0 = \frac{\partial \xi^\mu(x)}{\partial x^\rho} g_{\mu\sigma}(x) + \frac{\partial \xi^\nu(x)}{\partial x^\sigma} g_{\rho\nu}(x) + \xi^\mu(x)\frac{\partial g_{\rho\nu}(x)}{\partial x^\mu}.$$

Note that the r.h.s. of this equation is exactly the *Lie derivative* of the metric tensor w.r.t. the vector field $\xi^\mu(x)$.

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[548] John A. Coleman, *The greatest mathematical paper of all time*. The Mathematical Intelligencer **11**, 29–38, 1989. Cartan was meticulous in noting his indebtedness in 63 references to Killing in his 1894 thesis.

This can be written in terms of derivatives of the covariant components $\xi_\sigma = g_{\mu\sigma}\xi^\mu$:

$$\begin{aligned}0 &= \frac{\partial \xi_\sigma}{\partial x^\rho} + \frac{\partial \xi_\rho}{\partial x^\sigma} + \xi^\mu\left[\frac{\partial g_{\rho\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\sigma}}{\partial x^\rho} - \frac{\partial g_{\rho\mu}}{\partial x^\sigma}\right] \\ &= \left[\frac{\partial \xi_\sigma}{\partial x^\rho} - \Gamma^\mu_{\sigma\rho}\xi_\mu\right] + \left[\frac{\partial \xi_\rho}{\partial x^\sigma} - \Gamma^\mu_{\rho\sigma}\xi_\mu\right] = \xi_{\sigma;\rho} + \xi_{\rho;\sigma},\end{aligned}$$

where $\xi_{\sigma;\rho}$ is the *covariant derivative* of $\xi_\sigma(x)$.

Any four-vector field $\xi_\sigma(x)$ that satisfy the last equation will be said to form a *Killing vector of the metric* $g_{\mu\nu}(x)$.

It can be shown that Killing vectors have two other useful properties:

- If A_i is a Killing vector, then $A_i\frac{dx^i}{ds}$ is constant along any *geodetic.*
- A necessary condition for the metric $g_{\mu\nu}$ to have a *hidden continuous symmetry (isometry)* is that it admits one Killing vector field per parameter of the symmetric (Lie) group.

Killing was born in Burbach, Westphalia. He began university studies in Münster (1865) but quickly moved to Berlin and came under the influence of **Kummer** and **Weierstrass**. In 1872 he completed his dissertation under Weierstrass. From 1868 to 1882 he was teaching at the gymnasium level in Berlin. On the recommendation of Weierstrass, Killing was appointed professor of mathematics at the Lyzeum Hosianum in Braunsberg, East Prussia (now Braniewo in the Ulsztyn region of Poland). The main object of the college was the training of Roman Catholic clergy, so Killing had to teach a wide range of topics — including the reconciliation of faith and science.

Although he was isolated mathematically during his ten years in Braunsberg, this was the most creative period of his mathematical life. He produced his brilliant work despite worries about the health of his wife and seven children, demanding administrative duties as rector of the college and as a member and chairman of the city council, and his active role in the church of St. Catherine.

In 1892 he was called back to his native Westphalia as professor of mathematics at the University of Münster, and he stayed there for the rest of his life.

1880–1894 CE **John Milne** (1850–1913, England and Japan). One of the founders of the science of seismology. Constructed the first seismograph

suitable for world-wide use, and set up seismological observatories to measure ground movements on a global basis (1892).

Milne was born in Liverpool. After working in Labrador and Newfoundland as a mining engineer (1873) and serving as a geologist in an expedition to the Sinai desert (1874), he accepted (1875) a position of professor of geology and mining at the Imperial College of Engineering, Tokyo.

An earthquake in 1880 near Yokohama prompted him to create the Seismological Society of Japan, the first of its kind. In 1881 he married Tone Horikawa and stayed in Japan until 1895. During his stay there he traveled all over the islands and set up a network of seismological stations equipped with his seismographs. Upon his return to England in 1894, he settled on the Isle of Wight and established a private seismological station there. His activity in his later years centered around the establishment of a global network of seismic stations. He died in Shide, Isle of Wight.

In 1974, the University of Tokyo donated a number of cherry tree saplings to be planted at Shide and at the Isle of Wight College of Arts and Technology as 'a living memorial' to the pioneer seismologist.

1880–1908 CE **Moritz Benedict Cantor** (1829–1920, Germany). Distinguished historian of mathematics. A professor of mathematics, who devoted only his final academic years exclusively to the history of his field. His monumental work *Vorlesungen über Geshichte der Mathematic* (1880–1908), which carried the study down to 1799, dwarfed all previous endeavors and is still unsurpassed.

Moritz Cantor, a relative of Georg Cantor, was born in Mannheim to a Jewish family; he was appointed a private docent in Heidelberg (1853) and the rest of his active life was spent in the service of the university (1863–1913).

His treatise on the history of mathematics remains the most elaborate ever produced, without any equal of all histories of science.

Science Progress Report No. 11 The Long Arm of Orthodoxy, or did Moses write the Torah?

The Pentateuch, the first five books of the Bible, has always held a special place in both Christianity and Judaism. Islam also owes much to these books. As the Torah (the "five books of Moses"), the books are especially venerated by Jews; they were the first part of the Bible to be admitted to the Hebrew canon.

Christianity accepts the entire Hebrew Old Testament as found in the Masoretic text, including the Pentateuch, as canonical, although there is some variation in organization. Much of early Christianity art depicts events from the Pentateuch. Almost everyone in the Western world is early exposed to stories of Noah and the ark, the passage across the Red Sea, and the story of Abraham and Isaac.

A passage in the book of Deuteronomy (31, 9), relating that Moses wrote the Torah, gave rise to the doctrine of the Mosaic authorship of the whole Pentateuch.

Serious doubts about the traditional Mosaic authorship began to be heard already in the Middle Ages; ***Isaac Ibn Yashush*** *(**Ibn Kastar**; 982–1068, Spain), grammarian, biblical commentator and personal physician to Muwaffak Mudshaid al Amiri in Toledo, Spain, was first to demonstrate that Moses could not have been the author of the Pentateuch in his book "Sefer hatserufim". Consequently, he was ferociously excoriated. His opinion were later shared by the Jewish scholar* ***Joseph Bonfils*** *(fl. 1370, Damascus) and by the Bishop of Avilla,* ***Alonso Tostado*** *(fl. 1450, Spain), at the cost of their being persecuted by their respective religious establishments.*

Even in the 17th century, it would have taken a brave person to deny that Moses personally wrote "his" five books. The philosophers ***Thomas Hobbes*** *(Leviathan, 1651),* ***Baruch Spinoza***[549] *(Tractacus Theologico-*

[549] Spinoza argued that those who believe Moses to be the sole literal author of the entire text must reconcile the following questions and facts:

- Could: "Now the man Moses *was* very meek, above all the men which were upon the face of the earth" [Numbers **12**, 3] been written by the meek man himself?

- The first of Edomite kings [*Genesis*, **36**] includes those who lived many years after Moses. Especially, how could Moses know [**36**, 31] about future Israeli kings? [See also: 2 *Samuel* **8**, 14].

Politicus, 1670) and the theologian ***Richard Simon*** *(1638–1712, France; 1678) pioneered the modern historical method of Biblical study and critique, citing many textural examples.*

- How could Moses write Deuteronomy **34**, 5–12? He certainly could not had known that "... there arose not a prophet like Moses...".

- In Deuteronomy **1**, 1 we read: "These are the words which Moses spoke unto all Israel beyond the Jordan..." This could have only been written by someone on the West bank of the river, but it could not have been Moses, since Moses never set foot on the Western side. [From **1**, 5 we know that "beyond" means the side of *Moab*, namely the Eastern side].

- In *Genesis*, **12**, 6 we read: "And Abraham passed through the land unto the place of Shechem... *and Cannanite was then in the land*". Clearly, this was written in a post-Mosian era when the Cannanite were not there anymore. The discrepancy was already noticed by **Avraham Ibn Ezra** (c. 1140 CE).

- Deuteronomy **3**, 11–15 says: "For only Og king of Bashan remained of the remnant of the Rephaim; behold, his bedstead was a bedstead of iron; is it not in Rabbah... to this day". The style and content of this passage could have been written only after Rabbah was taken by David (a few hundred years later).

- In Genesis **22**, 14 we are told that the mountain is called by Abraham *the mountain of God*, whereas the writer of the story called it mount *Moriah* [**22**, 2]. This last name, however, was chosen in the days of David for the site of the Solomon Temple, a fact unknown to Moses [2 *Chron* **3**, 1].

- In Genesis **14**, 14 we read: "... and pursued as far as Dan". However, the name of this city was given a long time after the death of Joshua [*Judges* **18**, 29].

- The Pentateuch contains many *terms* which Moses could not have unknown, description of *places* which Moses never visited and *linguistic forms* foreign to his time.

- Most of the laws in the Pentateuch and most of its narrative were *not* an integral part of the day to day living habits of the Hebrews during the time of Moses.

- Moses was so busy leading his people around the wilderness that he scarcely had time for extensive writing [*Exodus*, **18**, 14–18].

According to Jewish tradition, the original scrolls of the Pentateuch and additional biblical manuscripts were lost in the fire that destroyed the Solomon Temple (587 BCE). Ezra apparently reconstructed it from duplicate fragmentary documents. [*Hazon Ezra* **12**, 20–22, ca 100 CE].

For that, Spinoza was excommunicated from Judaism and exiled from Amsterdam. Simon's book Histoire critique du Vieux Testament was placed on the Catholic Index and he himself was expelled from his order just because he dared deny that Moses personally wrote "his" five books and because his superiors thought that his attack on Spinoza was insufficiently fierce. Both Catholics and Protestants joined forces to persecute him; from the 1300 copies of his book only six were saved from the stake.

The politician ***John Hampden the younger*** *(1656–1696, England), who translated Simon's book into English was imprisoned (1688), and later released from the Tower only after he publicly renounced his 'crime'. Depressed and humiliated, he eventually took his own life.*

During the 18th century there were more imprisonments and even assassination attempts against those who dare suggest that Moses was not the literal author. But as Victor Hugo once said, no army in the world can suppress an idea whose time has come. Biblical scholars n Europe, Christian and non Christian alike, began a modern scholarly textual investigation of sources of the Pentateuch.

This new trend began already in 1315 CE with ***Joseph Even Caspi****, [known also by his French name* ***Don Bonafous de L'argentera****] (1280–1340, France and Spain). In his book* Tirat ha-kesef *(1315) he discovered that in the book of* Genesis *there are instances where the* same *story is being told* twice *in entirely different terms, sometimes with* contradictions *between the two narratives. Moreover, in some versions God is called* Elohim *[e.g. Gen 1, 1 – 3, 23] while in the following chapters it is referred to as YHVH.*

Caspi, however, did not draw from it any conclusions concerning the layout and uniformity of the biblical narrative. Only 300 years later was this line continued by the German priest ***H.B. Witter****. He claimed (1711) that the book of Genesis was composed of two sources,*[550] *each describing the creation in a different way, a different style and using a different name of God.*

[550] *Examples*:

- *Gen 20*, Vs. **26**, 6–11: If these stories are not independent, they cannot be reconciled: for why should Abraham repeat his excuse ("She is my sister") with Abimelech, when it had already failed with Pharaoh (*Gen* **20**, 10–20). On the other hand why should Abimelech not learn from his bitter experience with Abraham and disbelieve Isaac that Rebekah was indeed his sister. The only logical way out of this dilemma is to assume that the Sarah-Pharaoh and Rebekah-Abimelech incidents belong to one source while the Sarah-Abimelech story belongs to another document.

- *Gen* **15** Vs Gen *21*, 9–21; two different stories on the escape of Hagar

- *Gen* **21**, 22–34 Vs **26**, 26–38; Abimelech makes covenants with both Abraham and Isaac?

The physician ***Jean Astruc***[551] *(1684–1766, France) began the modern scholarly textual investigation of sources of the Pentateuch. In his book: Conjectures sur les mémoires originaux dont il parait que Moyse s'est servi pour composer de livre de la Genèse (1753), he developed a method of separation of the sources in* Genesis *and* Exodus *into two parallel stories.*

Independently, the orientalist and historian ***Johann Gottfried Eich-***

- Gen **30**, 28–43 Vs **31**, 11–; two versions on how Jacob tricked Laban.
- Gen **31**, 44–54; two versions on the Jacob-Laban covenant.
- Gen **37**, 22–30; was Joseph saved by Reuben or Judah?
- Gen **42**, 37 Vs **43**, 9; who was surety for Benjamin, Reuben or Judah?
- Gen **42**, 27 Vs **42**, 35; where did the brothers discovered their money?
- The story of the *deluge* is a remarkable 'scissors and paste' work of two *inconsistent* sources [*Gen* **7**: 1–5, 7, 10, 12, 16–20 Vs *Gen* **7**: 8–9, 11, 13–16, where the editor put a line of his own in the intervals!]
- The *creation* account appears in two *inconsistent* sources: *Gen* **2**, 4–25 Vs **3**, 1–24, where the editor appears in **2**, 4. The reason both versions were included is that both were highly venerated and the redactor unwilling to discard either one.
- *Exodus* **20**, 1–17 Vs *Deuteronomy* **5** are two versions of the Decalogue. the Covenant Code and case law in *Exodus* **21** is at par with the primitive "law" of *Exodus* **34**, which does not agree with either of the Decalogue versions. The case law in *Numbers 5* and the religious law in *Numbers* **28–29** do not agree with case law of Exodus **21**.
- *Joshua* **1–12** and *Judges* **1**, tell two different stories on the conquest of Israel. The first tells us that the land fell in three swift and decisive military campaigns. The second describes it as a series of independent tribal actions that did not result in a complete occupation of the land – a slow and complex process, which is more in line with archaeological findings; *Judges* **2–16** tell of many setbacks and reverses in the process of consolidating control over Israel.
- Could Moses have written both *Gen:* **19**, 31–38 and *Lev:* **18**, 17; **20**, 14? Or else, could he have written both *Gen:* **38**, 18 and *Lev:* **18**, 15?

[551] M.D. Montpellier (1703); Prof. of anatomy at Toulouse (1710–1717); Prof. of Medicine: Montpellier (1717–1730), Paris (1731).

horn[552] *(1752–1827, Germany) founded the modern Old Testament criticism and pioneered scientific study of biblical literature in his book Einleitung in das Alte Testament (1780–1783). His conclusions:*

- *most of the writings of the Hebrews have passed through several hands.*
- *the so-called supernatural facts related to both Testaments were explicable on natural principles. It should be judged from the standpoint of the ancient world and accounted for by the superstitious beliefs which were then generally in vogue.*

The Protestant theologian ***Wilhelm Martin Leberecht de Wette***[553] *(1780–1849, Germany) first applied historical criticism to the Pentateuch in his book Beiträge zur Einleitung in das Alte Testament (1806–7). Showed that the book of Deuteronomy could not have been written earlier than 622 BCE, when it was found by a priest in the Solomon Temple. It may have been first written early in the reign of Josiah (ca 640–609 BCE) and edited later [the general idea was expounded already by* ***Eusebius Hieronymus*** *(c. 380 CE)].*

Following de Wette, about a century of biblical research has been devoted to the analysis of the process by which the books of the Bible emerged from a welter of traditions, oral and written, and the determination of the main stages of transmission until the present received text. The principal result has been the promulgation of the so-called "Documentary hypothesis", which is associated with the name of the German scholar ***Julius Wellhausen***[554] *(1844–1918) who gave it its classical formulation in his book Komposition des Hexateuch (1889).*

In his analysis Wellenhausen demonstrates that the Pentateuch is a composite work of many hands and periods, including priestly editors and professional scribes. It is the result of a long period of growth, compilation and transmission. Much of its narrative content derives from oral traditions which were subsequently reworked and expanded by revisers of various schools. This accounts for the fact that the five books of the Pentateuch contain so many

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552 Educated at Göttingen (1770–1774), Prof. Oriental Languages, Jena (1775), Prof. Exegesies of Testament and Political history, Göttingen (1778–1827).

553 Educated in Jena. Prof. of Theology, Heidelberg (1807–1810), Univ. of Berlin (1810–1819), Univ. of Basel (1822–1849). In 1819 de Wette was dismissed from Berlin University and banished from the Prussian Kingdom. His subsequent appointment at Basel was strongly opposed by the orthodox party for obvious reasons.

554 Professor at Halle (1882), Marburg (1845), Göttingen (1892).

duplications, inconsistencies, and even contradictions, not to speak of major stylistic differences, the result of the blending of diverse traditions and disparate points of view.

Wellhausen distinguished four sources in the Pentateuch, each of which is regarded as an independent 'document' which has been composed or compiled by a single author or editor. Various editors then put these sources together with necessary modifications, bridges, and adjustments to produce the connected whole. The dating of these documents is:

- *J, E: before the major prophets of the 9th–8th centuries. Covers the ancient period when festivities, cults, rituals, and ceremonies were connected to agriculture and nature (Exodus* ***23****, 24) and fertility stage of religion. The letter J stands for* Jehova *and the letter E for the* Elohim*, both the divine characteristics of God in the respective passages of the book of* Genesis*,* Judges *(in part),* Samuel *and* Kings*.*

- *D (Deuteronomist) in the century before the discovery of the document in the Temple (622 BCE). This was the final period of the Kingdom of Judah, where there was a tendency to make the religion feasts into historical national symbols (Deut* **16***) and turn the rituals into symbols that unite the nation under a single kingdom. It is spiritual and ethical stage of religion.*

- *P(priestly) during and after the exile (6th century BCE[555]). Feasts and rituals are independent of agriculture and secular policy. The corresponding biblical texts are the books of* Leviticus*,* Numbers *and parts from* Exodus *[*Especially: *Lev* **23***, Num* **28–29***]. It represents the* legal stage of religion*, principally concerned with rites, ceremonies, priestly duties, genealogy, and measurements.*

During the 19th century the results of the biblical sources research were met with vehement opposition by the religious establishment; the entire weight of Catholicism, Protestantism, and Judaism was arrayed against the scholars: not only were there four authors (name of which was Moses), but these men were actually suggesting that the accounts were written at very different times, centuries removed from each other!

[555] Professor **Yehezkel Kaufmann** (1889–1963, Israel) defended effectively (1937) an earlier date for P (making it roughly contemporary with D) and argued for the effective coalescence of J and E. Archaeological discoveries during the 20th century support his views that although the *books* of the Pentateuch obtained their final form in the days of **Ezra** (ca 445 BCE), the *sources* of the Torah are ancient, some dating as far back as the days of Moses.

One victim of this resistance was **William Robertson Smith** *(1846–1894), Scottish philologist, physicist, archaeologist, Biblical critic, and chief editor of the Britannica (from 1881). He was educated in the Universities of Aberdeen, Edinburgh (1866), Bonn and Göttingen and became Professor of Oriental Languages and Old Testament exegeses at Free Church College, Aberdeen. His articles in the 9th edition of the Britannica on the* Documentary hypothesis *aroused the anger of his dons and after a Church trial he was removed from his Chair (1881). He was later appointed Prof. of Arabic at Cambridge (1883). By the 20th century, the fury abated. Most mainline Protestants began to see that it really didn't matter who wrote the books; the content was the important part. The oppression of the Catholic Church melted when Pope Pius XII published "Divino Afflante Spiritu" (1943):*

"Let the interpreter then, with all care ... endeavor to determine ... the sources, written or oral, to which he had recourse and the forms of expression he employed".

Mark Twain (1835–1910), the most irreverent of writers, was actually a very religious man, but he did not subscribe to any orthodox set of beliefs, and he did not believe that the Bible was literally the word of God. He once said: "It ain't those parts of the Bible that I can't understand that bother me, it is the parts that I do understand".

1880–1899 CE **Adolf Johann Friedrich Wilhelm von Baeyer** (1835–1917, Germany). Among the major contributors to organic chemistry throughout the 19th century. Did pioneering work on organic dyes and the hydrochromatic compounds. Known especially for synthesis of *indigo* (1880) and formulation of its structure (1883), synthesis of Phthalein dyes, discovery of uric acid derivatives and investigation of polyacetone. He put forward a strain theory of carbon rings (1885) which explained the stability of ring compounds in terms of the distortion of their valence bonds from the normal angles of the tetrahedral carbon atom.

Baeyer's father was a Prussian general and his mother – an apostate Jewess. He studied under **Bunsen** and **Kekule** and held professional positions at Strasbourg, Berlin and Munich (1875–1915). Awarded the 1905 Nobel Prize for chemistry.

1881 CE **John Venn** (1834–1923, England). Logician. His book '*Symbolic Logic*' contained the '*Venn diagrams*', a system of overlapping circles (or ellipses, or other figures) for representing logical propositions.

Venn was born in Drypool, Hull, a descendant from a Devonshire family long distinguished for its erudition. Representing the 8^{th} generation of his family to study at Cambridge, he entered the university in 1853 and studied mathematics. Ordained a deacon in 1858 and priest in 1859, he served for a short period in parochial work. He resigned his orders in 1883 to devote himself entirely to the study and teaching of logic.

1881 CE **Charles Roy**. Made pioneering experiments demonstrating the nonlinear elastic behavior of the arteries.

Without the aid of any electronic devices, he constructed a gravity-driven apparatus that inflated isolated segments of blood vessel from human beings and other mammals, measured instantaneous pressure and volume, and plotted the results on a rotating drum called a *kymograph*.[556] He also tested strips of artery wall with an apparatus that plotted the fork-length curves for the tissue as it was stretched. With these data, Roy determined an artery wall's *nonlinear elasticity* and found that the distensibility of the human aorta decreases as a function of age. He also showed that the arteries distend considerably at resting blood pressure, which means that the *tensile stress is never zero*. Finally, he showed that the aorta is most compliant in the normal range of blood pressure.

In other experiments, he showed that heating makes the artery wall stiffer, so that an applied stress produces less strain. Thus Roy recognized that an artery wall had an elastic mechanism that was thermodynamically like that of natural latex *rubber* (caoutchouc) although the physics of this type of elastic material was not understood in Roy's time.

Modern research on synthetic rubber-like polymers, as well as on animal rubbers like *elastin*, has revealed that the elasticity of such polymer networks arises from changes in the entropy of the molecular chains; an imposed strain *increases order in the molecular network*, thereby decreasing its entropy. The *elastic force* arises from the tendency of the network to return to conformational states of higher entropy (or disorder), according to the laws of thermodynamics.

1881 CE **Clément Ader** (1841–1925, France). Engineer and inventor. Built and flew for 50 meters the *Eole*, a bat-winged, steam-powered airplane, 13 years ahead of the Wright brothers.

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[556] An instrument used to record temporal variations of any physiological or muscular process; it consists essentially of a revolving drum bearing a record sheet (usually of smoked paper) on which a stylus or pen-point travels to and fro at right angles to the motion of the cylinder. The drum is rotated by a mechanism at a uniform rate, or the rate is indicated by a time marker which registers on the sheet.

Discovered the *stereo effect* of sound by recording a live performance with the use of more than one microphone and subsequently delivered it via a number of phase-different signals to the listener's ears.

"The musical telephone" was a major attraction at the International Electrical Exhibition in Paris in 1881, where Ader demonstrated stereophonic transmission by telephone direct from the staged of the Paris Opera House and the Comédie Francaise. He used 12 stage microphones and laid the lines through the Paris sewers to the Exhibition Hall at the Palais de l'Industrie. Up to 48 listeners could hear the opera, using two receivers each, one for each ear. This first public broadcast entertainment was known as the *Theatrophone*.

In 1890, a commercial company, Compagnie du Theatrophone, was established in Paris, distributing music by telephone from various theaters to special coin-operated telephones installed in hotels, cafés etc., and to domestic subscribers.

The service continued until 1932. Elsewhere, trials of concerts by telephone were held, not only on a local basis but also with distribution over longer distances, e.g. from Paris to Brussels in 1887, and from Paris to London in 1891. A mixed service of news, telephone concerts and lectures was opened in Budapest in 1893.

1881–1886 CE **Lucien Gaulard** (1850–1888, France). Engineer and inventor.

Invented the first *power transformer* with annular core. With **John Dixon Gibbs** (England) succeeded in transmitting, for the first time (1883), a voltage of 2000 Volts over a distance of 40 km. This was the first effective device for long-distance transmission of AC power. In 1885, **Westinghouse** imported a number of Gaulard-Gibbs transformers and a Siemens AC generator to begin experimenting with AC networks in Pittsburgh, USA.

During his life, his work was not recognized in France. He fell into sever depression, was sheltered in a clinic and died there in 1888. A street in Paris now bears his name.

1881–1906 CE The great exodus of Jews from Russia. Organized government-incited massacres of the Jews in Russia [1871, 1881, 1903, 1905, 1906] was aimed at diverting public attention from serious interior problems and prevent the collapse of the crumbling Tzar's regime. The Nazis were to use exactly the same technique of violence-led legislation 52 years later.

In 1882 500,000 Jews living in the rural areas of the *Pale of Settlement*, were forced to leave their homes and live in towns and townlets (*shtetls*) in the Pale; 250,000 Jews living along the Western frontiers of Russia were also

moved into the Pale; 700,000 Jews living east of the Pale were driven into the Pale by 1881 [e.g. 2000 Jews of Moscow were expelled (1891) and 2000 Jews were deported from St. Petersburg (1891)]. By 1897 there were 5 million Jews living in the Pale and 320,000 outside it in Siberia, Baltic provinces, Caucasus, Russian Central Asia, Astrakhan and Terek regions.

Thus, from 1881 this vicious, mounting and cumulatively overwhelming pressure on Russian Jewry produced a panic flight from Russia westwards. This year was the most important year in Jewish history since the expulsion of the Jews from Spain (1492). It had wide and fundamental consequences in world history too. During 1881–1914, more than 2 million Jews from Russia, Poland and Romania moved to the United States. This was a completely new phenomenon, which in time changed the whole balance of Jewish influence in the world and had great future impact on American and world science.

1881–1912 CE **Paul Ehrlich** (1854–1915, Germany). Bacteriologist, physician and a distinguished 'microbe-hunter'. Revolutionized the whole aspect of the preventive and curative treatment of infections. Founded *chemotherapy* (1910), modern *hematology* (1885), and modern *immunology* (1897). Became known for discovering Salvarsan (arsphenamine)[557], the 'magic bullet' remedy for syphilis (1909). Salvarsan is also called "606" because it was the 606^{th} compound tested. Ehrlich shared the 1908 Nobel prize for physiology or medicine with Elie Metchnikov for their work on immunity. He dominated the first phase of the chemotherapeutic revolution which ended with his death and was not resumed until 20 years later with the discovery by **Domagk** of the antibacterial action of the dye *prontosil rubrum*.

Ehrlich was born in Strehlen, upper Silesia, near Wroclaw to Jewish parents. After graduating in medicine from the University of Leipzig (1878), he spent a period as medical assistant at the Charité Hospital in Berlin. He married the 19 year old Hedwig Pinkus (1884). In 1888 tubercle bacilli were found in his sputum and he was forced to spend the next two years in Egypt, until he was cured. In 1890 Ehrlich joined Koch in the latter's new Institute of Infectious Diseases. In 1896 an institute for serum research and control was created near Berlin under Ehrlich's direction. In 1898 he moved to Frankfurt am Main to head the State Institute for Experimental Therapy. Later the George Speyer House was built nearby, and from 1906 until his death Ehrlich directed both institutions.

Ehrlich was a forceful personality, often engaged in controversy yet inspiring great loyalty, smoking more than 25 cigars a day ["burning his life's candle at both ends" — as he used to joke], but drinking only mineral water. He used to mail himself postcards a few days before every family anniversary lest

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[557] *Dioxy-diamino-arseno-benzene-dihydrochloride.*

he forget them, and believed in the four big G's, the formula for successful work: *Geld* (money), *Geduld* (patience), *Geschick* (ability) and *Glück* (luck).

Ehrlich began his work by studying the capacity of aniline dyes for selective staining; and one can trace almost all his later work back to the specific interaction between chemical substances and particular biological structures that dyes reveal, whose full significance he was the first to see. An early discovery (1881) was the so-called 'mast-cell', a large cell with distinctive granules taking up basic dyes, now known to be rich is histamine and to mediate many allergic reactions. A year later (1882) he defined the 'eosinophil' cells that occur in blood; these cells are now known to be particularly involved in resistance to parasitic infections. Extending his work to bacteria, he was the first to stain tubercle bacilli. The same insight led him to recognize the 'blood-brain barrier' (1885), through which only drugs with sufficiently fat solubility can penetrate, but which otherwise prevents many substances dissolved in blood from reaching the brain.

Further discoveries in this period were that the dye *methylene blue* selectively stained nervous tissue *in vivo*; and that other dyes, which bleach on removal of oxygen, allowed the study of the varying oxygen demanded of different tissues. In 1890 he demonstrated that antibodies were transmitted in maternal milk to provide '*passive immunity*' to a newborn animal. He produced (1890) a diphteria antitoxin sufficiently concentrated for its first clinical use. Its success led to the necessity of a standardization of toxin and antitoxin, which Ehrlich solved by preparing a dried, evacuated reference sample of antitoxin as an international standard, with which the toxin was titrated. With this, Ehrlich established the field of immunology.

His '*side-chain theory*' (1897) envisaged that a toxin which could combine with (but not kill) a cell, would give rise to a proliferation of the cellular binding sites involved[558]. As this method of '*immunotherapy*' failed against diseases such a malaria and syphilis, Ehrlich turned again to *chemotherapy*, the use of chemicals – especially constructed 'magic bullets' – to find, bind to, and act on the parasite. A range of phenols proved to be inhibited by serum and too toxic. The dye '*trypan red*' was found effective but '*drug resistance*' developed — the first description of this phenomenon. Compound

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[558] **Ehrlich** postulated that the body's cells possess a great many "receptors" by which they combine with the food substances in the body fluids. He theorized that the metabolic products of certain bacteria combine with the receptors of some cells, thus injuring the cells. Ehrlich visualized receptors as unsatisfied chemical side chains. This is not far from the modern idea of receptors as domains in enzymes or other proteins, with which drugs of appropriate structure can combine. In fact, knowledge of cell receptors is now on the cutting edge of pharmacology and drug discovery.

No. 606 proved effective in human syphilis and, as *Salvarsan*, revolutionized its treatment. Adverse reactions, however, led to much controversy, until the safer No. 914, '*Neosalvarsan*' (1912) appeared.

The Ehrlichs has two daughters, the youngest of which, Marianne, was married to the mathematician **Edmund Landau**.

During the first year of World War I — the last of his life — Ehrlich's health deteriorated. Long years of heavy smoking of strong cigars (stimulants *needed* to withstand the enormous strain on his physical and intellectual constitution, and to stave off the effects of exhaustion, irregular meals, and improper food) took their toll and produced a disastrous effects on his system. He died of a stroke in Bad Homburg, Germany, and was buried in the Jewish Cemetery of Frankfurt.

Atop the two high columns at the entrance to the box-edged tomb were, visible from afar, the Star of David and the Snake of Aesculapius. At the head of the tomb, on a high stone carved from a natural block of marble, was a large vase of porphyritic rock containing trailing rose bushes with an abundance of blossoms. So his resting place was covered with the falling petals of these glowing flowers.

Ehrlich's whole life was one long fight for the promotion of medical science in the service of mankind. He had a deep-rooted and unwavering optimism, aiming always at perfection and ever more difficult targets, supported always by an unshakable faith in progress. He could have done much more for humanity had his life not been cut short by his premature death at the age of sixty-one. Striving for the health and happiness of the world he had overtaxed his own physical strength, and burnt the candle at both ends.

Ehrlich was a solitary thinker, inspired by humanitarian unselfish motives rather than by the struggle for power; single-minded, and able to elicit devotion from his followers. He was also a fearless rule-breaker, challenged on all sides by narrow-minded bureaucrats, and ill-treated by skeptical colleagues, stubbornly pursuing his research into a disease regarded with shocked abhorrence in his day[559].

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[559] Ehrlich's secretary, **Martha Marquardt**, worked for him during 1902–1915. Her first book 'Paul Ehrlich als Mench und Arbeiter', was published in memory of his 70^{th} birthday anniversary (1924). Most of the copies of this book were burnt by the Nazis on May 10, 1933. An extension of the previous work was therefore written and finished by her in 1940 in Paris, but remained unpublished owing to the ongoing war. In December 1946 Marquardt was able to revisit Frankfurt for a few weeks, and found a number of Ehrlich's old staff still at their posts in the Institutes in the Paul Ehrlich Strasse.

Edward G. Robinson (1893–1973; born Emanuel Goldenberg in Bucharest,

1881–1916 CE **Edward Emerson Barnard** (1857–1923, U.S.A.). Astronomer. Discovered the 5^{th} satellite of Jupiter[560], *Amalthea* (1892) and *Barnard's star*[561] (1916). Barnard was a pioneer in celestial photography: he made the first photographic discovery of a comet (1881). Barnard was also the first person to report seeing craters on Mars (however, he did not publish these observations for fear of ridicule). Around 1900, Barnard discovered the first of the *dark nebulae*, known today as *Barnard objects*[562].

Romania) played the character of Ehrlich in the movie *Dr. Ehrlich's Magic Bullet* (1940).

560 The first four had been detected by **Galileo Galilei** (1610). *Amalthea* is the last planetary satellite to be discovered without the aid of photography or spaceprobes.

561 The star with the largest known *proper motion* (10.3 arcseconds per year). It is due to the star's motion perpendicular to the line of sight. The largest proper motion of any naked-eye star is that of 61 cygni ($5''$ per year).

Barnard's star is at *distance* of 6.0 light-years from earth, its *absolute magnitude* is 13.22, its *apparent magnitude* is 9.54, *color* red, *surface temperature* 3250 K, *radius* about 0.2 solar radii. The question of whether it has planets, has not been answered.

562 For many years, astronomers have suspected that stars are born in cold, dark clouds of interstellar gas. If an interstellar cloud is warm, its atoms are moving about so rapidly that there is no chance for a protostar to condense from the agitated gases. If the temperature is low, however, then the atoms are moving slowly enough to allow denser portions of the cloud to contract gravitationally into clumps that collapse to form new stars. Many of these cold clouds are scattered across the Milky Way. In some cases they appear as dark regions silhouetted against glowing background nebulosity, such as the famous Horsehead Nebula. In other cases they appear as dark blobs that obscure background stars. These are *Barnard's objects*.

The Ether Hypothesis[563]

Aristotle *(ca 350 BCE) supported the theory that all of space is filled with the four elements: fire, water, earth, air and a fifth element — the ether. The ether was supposed to serve as a medium through which the stars moved in their daily courses around the earth. During the following* 2200 *years, the term 'ether' and 'vacuum' became largely synonymous. With the takeover of the Copernican world-view in the* 16^{th} *century the ether concept, in its Greek sense, disappeared from science. It was given, however, a new 'task': to serve as a medium for the transmission of light and gravitation. This view, known as the 'ether hypothesis', was mainly due to* **Descartes** *(1637) who specified the physical properties of the ether as an elastic, weightless material.*

Huygens *(1678), in his wave theory of light, needed the ether not only as a medium for the wave motion but also to explain its finite velocity and its refraction. The later discoveries of stellar aberration (1728) and the Doppler effect (1842), in which the velocities of the light and of its sources or detectors were combined, added support to the existence of the ether. The years* 1821–1838 *witnessed the development of the elastic ether theory. In* 1821, *the engineer* ***C.L.M.H. Navier*** *established the theory of elasticity of solid bodies, discerning that matter consists of countless particles (mass points, atoms) exerting on each other forces along the lines joining them. [****A.L. Cauchy*** *(1828) derived the equations of elasticity by means of the continuum concept.]*

Further development of this theory was due to **S.D. Poisson** *(1828),* ***G. Green*** *(1838),* **J. Maccullagh** *(1837) and* **Franz Ernst Neumann** *(1798–1895, Germany, 1835). At this point the ether was assumed to be a luminiferous (light-carrying) elastic solid, capable of accommodating transverse wave-motion. However, all efforts to reach a consensus regarding a* mechanical *model of the ether, met with total failure.*

Following the advent of the electromagnetic field concept [introduced by ***Faraday*** *(1846) and* **Maxwell** *(1865)], the discovery of the electromagnetic nature of light and the recognition that the internal forces in material media*

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563 For further reading, see:

- Whittaker, E.T., *A History of the Theories of Aether and Electricity*: *From the Age of Descartes to the Close of the Nineteenth Century*, Longmans, Green and Company: London, 1910, 475 pp.
- Whittaker, E.T., *A History of the Theories of Aether and Electricity*: *The Modern Theories 1900–1926*, Philosophical Library: New York, 1954, 319 pp.

are of electrical and magnetic origin — the ether concept underwent further modification: It was stripped of all its material attributes and left with only the properties of imbuing all space outside and inside *matter, and of carrying electric and magnetic fields.*

The state of the ether prior to 1881 *was this: The space of* mechanics *was regarded as empty wherever material bodies were not present. The space of* optics *was filled with ether, which had a certain mass, density and elasticity. The universe no longer consisted of isolated masses separated by empty space, but was completely filled with a thin rigid, elastic medium of ether in which the masses were floating. Ether and matter acted upon each other with mechanical forces and moved according to Newtonian laws.*

This doctrine was assisted by an additional working hypothesis, stated as follows: The ether in astronomical space, far removed from material bodies, is at rest in an inertial frame. [If this were not the case, parts of the ether would be accelerated which in turn would bring about changes in density and elasticity, detectable through analysis of star light reaching us from those regions.] Hence, absolute space is at rest relative to the ether.

If the ether indeed defined a system of reference which was absolutely at rest, then the motion of the earth (for example) relative to it, should in principle be detectable.

Now, the detection of this motion by means of light waves *depended on whether or not the Galilean principle of relativity remains valid for optical phenomena. The ether theory gives the following answer to this question: the optical Doppler effect is indeed determined by the relative motion of the source of light and of the observer in accordance with Galilean relativity — but only if quantities of second-order are neglected*[564]*. Hence, the classical principle of relativity holds only approximately for optical wave phenomena.*

This furnishes us with a means of establishing motions relative to the ether. Indeed, ***Maxwell*** *(1879) called attention to the fact that by observing the eclipses of Jovian moons, it should be possible to ascertain a motion of the whole solar system relative to the ether, by measuring the variation in the* ***Roemer*** *apparent* 6*-month delay of these eclipses, over a* 6*-year interval (half*

[564] Let a light source, at rest relative to the ether, be monitored by two observers at rest in respective frame S (ether frame) and S' that move with relative uniform velocity $\boldsymbol{v}$. The origins of the frames, O and O', coincide at $t' = t = 0$. Let the train of waves be associated with an electric field $\boldsymbol{E} = \boldsymbol{E}_0 \sin(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)$, $k^2 = \frac{\omega^2}{c^2}$. Since the phase is invariant (representing as it does the number of wave-crests) we have: $\boldsymbol{k}' \cdot \boldsymbol{x}' - \omega' t' = \boldsymbol{k} \cdot \boldsymbol{x} - \omega t$, as seen by the two observers [$\boldsymbol{k}$ = wave-number vector, $\boldsymbol{x}$ = spatial coordinates, ω = angular frequency, t = time]. Assuming the *Galilean transformation* $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}t$, $t = t'$, one obtains

the Jovian year). However, the experimental errors in such a measurement are too large to obtain a meaningful result.

The experiments of ***Michelson*** *(1881) and Michelson and Morley (1887) have shown that the velocity of light in terrestrial measurements is not influenced by the motion of the earth* even to the extent involving quantities of the second order.

The immediate result of this experiment was that the ether lost its last material properties, namely — its ability to move and its specific location: For if we must assume that light propagates with a fixed velocity relative to all observers, independent of the velocity of its source and irrespective of the *velocities of its observers* [which may move with different velocities relative to each other], *then this ether is totally superfluous*[565].

$$\boldsymbol{k}' = \boldsymbol{k}, \quad \omega' = \omega\left(1 - \frac{\boldsymbol{v}\cdot\widehat{\boldsymbol{k}}}{c}\right), \quad c' = c - \boldsymbol{v}\cdot\widehat{\boldsymbol{k}}, \quad \omega = 2\pi\nu, \quad \omega' = 2\pi\nu',$$

where $\widehat{\boldsymbol{k}}$ is a unit vector normal to the wave-front, c' is the *phase velocity* in frame S', and ν, ν' are the frequencies (in cycles per second) in the respective frames. In the simple case where $\boldsymbol{v}$ is parallel to $\widehat{k}$ (along the x-axis, say),

$$c' = c - v, \quad \nu' = \nu\left(1 - \frac{v}{c}\right).$$

Here c is the velocity of light measured by an observer at rest relative to the ether.

If, on the other hand, an observer at rest in the ether measures the frequency of a *moving source* with intrinsic frequency ν_0 and velocity v_0, the observed frequency is $\nu = \frac{\nu_0}{1-\frac{v_0}{c}}$. For cases where $\frac{v_0}{c} \ll 1$, we find

$$\nu \sim \nu_0\left(1 + \frac{v_0}{c}\right).$$

If we assume a simultaneous motion of the source of light (v_0) and the observer (v), the observed frequency ν' is

$$\nu' = \nu\left(1 - \frac{v}{c}\right) \sim \nu_0\left(1 + \frac{v_0 - v}{c}\right).$$

So, *to first order*, the Galilean relativity principle holds for the optical Doppler effect.

565 In a sense, though, the $3\,^\circ$K cosmic microwave background radiation (CMBR), permeating the universe, has replaced the ether in modern cosmology; at least, the velocity of the solar system w.r.t. it can, and has, been measured through the terrestrially-observed 'Doppler shift' in that cosmic temperature — once

Sic transit gloria ether. Indeed, after **Einstein** *had expounded the special theory of relativity (1905), the ether was banished from classical physics. After* 1915, *when the general theory of relativity interpreted gravitation as the intrinsic geometry of space-time, besides the CMBR, one might say that another, modern version of the ether entered physics – namely, dynamical spacetime itself!*

This "ether", though, is locally invariant under non-accelerating changes of reference frame. Space (the vacuum) is dynamical not just by virtue of GTR (where undulations in it are called gravitational waves), but also in Quantum Field Theory.

1881–1933 CE **Albert Abraham Michelson** (1852–1933, U.S.A.). Experimental physicist. Established that the velocity of light is not influenced by the motion of the earth thereby showing that the ether hypothesis must be abandoned.[566]

Michelson spent 50 years in improving his measurements of the velocity of light, ending in 1933 with a value of 299,744 km/sec [2 km/sec higher than the value accepted in the 1970's]. Michelson was the first American scientist to win the Nobel prize in physics (1907).

Michelson was born to Jewish parents in Strelno, Prussia. He came to the United States with his parents in 1854. At 17 he entered the U.S. Naval Academy, from which he graduated in 1873, serving as a science instructor there until 1879. In 1880 he traveled to Europe and during 1880–1881 built an interferometer by means of which he was able to demonstrate in Berlin (1881) that there was no motion of the earth relative to the ether. He repeated this experiment with **Edward William Morley** (1838–1923, U.S.A.) in 1887,

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the shift due to the earth's motion relative to the sun is subtracted. (The result is a few hundred km/sec.)

566 The experiment's "null" result quite obviously dominated the work of **Lorentz** and many others. But it was *not* the road along which special relativity evolved. **Einstein** said (1945) that at the time he wrote his basic paper on relativity (1905) he had never heard of the experiment. Einstein elaboration on STR began with his rejection of the "luminiferous ether", and in that sense Michelson's experiment was not decisive. Einstein's reasoning is sufficiently simple and logical, and there is every reason to use it in expounding the special theory of relativity.

with an improved interferometer which he built with a grant of 200 dollars obtained from Alexander Graham Bell. In 1883 he accepted a position of professor of physics at the Case School of Applied Science in Cleveland. At 1892 he was appointed professor and head of the department of physics at the University of Chicago, a position held until his retirement in 1929. Michelson's other achievements in physics are:

(1) Measured the earth's mean rigidity[567].

(2) Measured the diameter of the star Betelguese[568] (Alpha Orionis) by means of the partial coherence of light arriving from its opposite edges (1920).

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567 He used an interferometer to measure the bodily tide in the solid earth by the disturbance of the water level in two vertical tubes with a long horizontal connection underground [*Astrophys. J.* **39**, 105–138, 1914; *Astrophys. J.* **50**, 330–345, 1919].

568 *Michelson's stellar interferometer* (1890) measures the small angular dimensions of remote astronomical objects; a star is presumed to be a circular distribution of partially coherent point sources such that it has a uniform brilliance. If one were to perform an interference experiment with this source, in which a double-slit aperture was used, as in Young's experiment, then the distance between the slits would have to be less than the lateral coherence width in order to obtain distinct interference fringes. In practice, a telescope objective, diaphragmed by two equal small apertures, is used to view the starlight, of effective wavelength $\overline{\lambda}$ (mean wavelength of a narrow spectral band) and angular source diameter θ. The star's *visibility* is shown to be

$$\varphi = |\gamma_{12}(0)| = 2\left|\frac{J_1(\pi h\theta/\overline{\lambda})}{\pi h\theta/\overline{\lambda}}\right|,$$

where h is the smallest value of the slit separation for which the visibility of the fringes is minimum. The first zero of φ occur when

$$\pi h\theta/\overline{\lambda} = 3.83, \quad \text{or} \quad h = 1.22\frac{\overline{\lambda}}{\theta}.$$

Once h and $\overline{\lambda}$ are known, θ is calculable. Michelson employed *mirrors* to increase the effective distance between the slits. This enabled him to measure very small angular diameters (even for nearby stars, angular diameters are of the order of hundredths of a second of arc, with corresponding lateral coherence width of the order of several meters). Pointing the 100 inch reflector telescope of the Mount Wilson Observatory toward *Betelguese* (α Orionis), the fringes formed by the interferometer were made to vanish at $h = 121$ inches, and with $\overline{\lambda} = 5800$ Å, $\theta = 0.047$ seconds of arc. Using its known distance (determined from parallax measurements), the star's diameter turned out to be about 280 times that of the sun! (it is a red giant). In the 1990's, Betelguese's disc was optically resolved

(3) Suggested the use of the wave-length of the red line of Cadmium as the basis for a new standard of length (1893). This suggestion was accepted in 1960.

Science and Economy

In 1881, Albert Michelson, using a $ 200 grant from Alexander Graham Bell, had built an instrument called the interferometer, with which he disproved the existence of the mysterious ether that was supposed to fill all space. Today's atom smashers, in dramatic contrast, cost hundreds of millions of dollars to build and operate – and there is no guarantee that anything as momentous as Michelson's discovery will result.

By the mid 1970's, the honeymoon between science and the US Government had ended. Pressures on the Federal Government multiplied, and Congress was becoming tougher about spending. Scientists were told that "... the American people cannot afford to finance science as a hobby horse — science for the fun of it. They envision practical science as a workhorse for the people — research that produces a better quality of life... Congress wants scientific research that gets results".

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and imaged (in the usual, ray-optics manner) by the orbiting Hubble space telescope.

The Operational Calculus[569]

__Leibniz__' differential notation (1672) made it possible to consider the differential operator as an algebraic quantity independent of the function operated upon. Several mathematicians, among them __Lagrange__, __Laplace__ (1812) and __Cauchy__ (1827) employed this idea, so fundamental for the operational calculus. An explanation of the success of the algebraic treatment of the differential operators was sought in other fields of mathematics. Laplace, for example, explained the operational methods by means of the Laplace transform, whereas Cauchy used the Fourier theorem.

__Servois__ (1814) thought that the reason why algebraic treatment was applicable to differential operators was that the latter obeyed the commutative and distributive laws. __Boole__ (1859) created his own version of an abstract algebraic approach to differential operators.

However, all the above contributions consisted only in the introduction of operational methods into analysis. It was __Oliver Heaviside__ (1893), whose work stimulated a systematic use of operational methods in physical and technical problems. It was he who presented an abundance of mathematical and physical methods and results. In fact, most of his methods stemmed from his need to solve practical problems associated with work as an operator of the great Northern Telegraph Company. However, Heaviside developed a formal calculus, suited for his own purpose.

The pure mathematicians of his time would not deal with this nonrigorous theory, but in the 20^{th} *century several attempts were made to rigorize Heaviside's operational calculus. These attempts can be grouped into two classes: the one leading to a representation of the operational calculus in*

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569 For further reading, see:

- Van Der Pol, B., *Operational Calculus*, Cambridge University Press, 1959, 415 pp.
- Scott, E.J., *Transform Calculus*, Harper and Brothers: New York, 1955, 330 pp.
- McLachlan, N.W., *Modern Operational Calculus*, Dover: New York, 1962, 218 pp.
- Carslaw, H.S. and J.C. Jaeger, *Operational Methods in Applied Mathematics*, Oxford University Press, 1953, 359 pp.

*terms of integral transforms [**Bromwich**[570] (1916), **Carson** (1917), **van der Pol** (1929)] and the other leading to an abstract algebraic formulation [**P. Levy** (1926), **Mikusinski** (1949)]. Also, **Schwarz**'s creation of the theory of distributions (1945) was very much inspired by problems in the operational calculus of Heaviside.*

It is remarkable that the theory of linear operators of Hilbert spaces and Banach spaces and the theory of von Neumann algebras, which was developed in the period 1900–1940, *did not interact at first with the development of the operational calculus. There are three reasons for this:*

(1) The operational calculus had its source in practical applications of differential equations whereas operators in Banach spaces developed from theoretical interest in integral equations.

(2) The operational calculus was successful in practice but lacked in rigorous interpretation, whereas the theory of integral equations had clear concepts but no effective methods of solution.

(3) Operational calculus developed at the fringe of the mainstream of mathematics and was scarcely used by practitioners, while the theory of operators in Banach space occupied a central position in the mathematics of the 20^{th} *century and was created by pure mathematicians.*

The development of the operational calculus provides an illustrative example of how a practical problem — long distance telegraphy — can influence mathematical theory. Thus, computation techniques that arose in engineering inspired an essential field of mathematics, namely the theory of distributions.

The Maxwellians (1879–1894)

After Maxwell's death (1879), a tightly knit group of British physicists, the self-styled 'Maxwellians' transformed the rich but confusing raw material of James Clerks Maxwell's Treatise *into a solid, concise and well-confirmed theory. Only with that transformation in the two decades after Maxwell's death*

[570] The Bromwich inversion integral $h(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\frac{f(p)}{p}dp$ of the Laplace transform $f(p) = p\int_0^\infty e^{-pt}h(t)dt$, was known to **Riemann** as early as 1859.

did 'Maxwell's equations' emerge in the form they have since retained, and the accompanying technological developments begin. Thus, the field-theoretical ideas of Faraday and Maxwell were clarified, consolidated, completed and reformulated into the core of thinking and research in physics.

In 1873, Maxwell published a two-volume Treatise on Electricity and Magnetism *that was destined to change the orthodox picture of physical reality. This treatise did for electromagnetism what Newton's* Principia *had done from classical mechanics. It not only provided the mathematical tools for the investigation and representation of the whole electromagnetic theory, but it altered the very framework of both theoretical and experimental physics. It was this work that finally displayed action-at-a-distance physics and substituted the physics of the field.*

Like Newton's Principia, *Maxwell's* Treatise *did not immediately convince the scientific community. The concepts in it were strange and the mathematics was clumsy and involved. Most of the experimental basis was drawn from the researches of Michael Faraday and one of Maxwell's purposes in writing his treatise was to put Faraday's ideas into the language of mathematical physics — precisely so that orthodox physicists would be persuaded in their importance.*

Maxwell died in 1879, midway through preparing a second edition of the treatise. At that time, he had convinced only a very few of his fellow countrymen and none of his continental colleagues. That task fell to his disciples.

During the twenty years that followed Maxwell's ideas were picked up in Great Britain, modified, organized and reworked mathematically so that the Treatise *as a whole and Maxwell's concepts were clarified and made palatable and irresistible to the physicists of the late 19th century.*

James Clerks Maxwell's theory of the electromagnetic field is generally acknowledged as one of the outstanding intellectual achievements of the 19th century. By the mid-1890s the 4 "Maxwell's equations" were recognized as the foundation of one of the most successful theories in all of physics, taking their place as companions to Newton's laws of mechanics. The equations were by then also being put to practical use, most dramatically in the emerging new technology of radio communication, telegraph, telephone, and electric power industries.

Surprisingly enough, Maxwell's Treatise *(1873) doe not contain the 4 famous* Maxwell's equations, *nor does it even hint at how electromagnetic waves might be produced or detected. These and many other aspects of the theory were thoroughly hidden in the version of it given by Maxwell himself.*

The task of digging out the "latent" aspects of his theory and of exploring its wider implications was thus left to a group of younger physicists: Between

1879 and 1894 these "Maxwellians"[571]*, led by* ***George Francis FitzGerald*** *(1851–1901),* ***Oliver Lodge*** *(1851–1940) and* ***Oliver Heaviside*** *(1850–1925), with a key contribution from* ***Heinrich Hertz*** *(1857–1894), transformed the rich but confusing raw material of the* Treatise *into a solid, concise, and well-confirmed theory (at least for free space) — the "*Maxwell's theory*" we know today. It was they who first explored the possibility of generating electromagnetic waves and then actually demonstrated their existence; it was they, along with* ***J.H. Poynting*** *(1852–1914), who first delineated the paths of energy flow in the electromagnetic field and then followed out the far-reaching implications of this discovery; it was they who recast the long list of equations Maxwell had given in his* Treatise *into the compact set now universally known as "Maxwell's"; and it was they who began to apply this revised theory to problems of electrical communications, with results that have transformed modern life.*

Scientific theories rarely sprig fully formed from the mind of one person; a theory is likely to be so refined and reinterpreted by later thinkers that by the time it is codified and passes into general circulation, it often bears little resemblance to the form in which it was first propounded.

1881–1912 CE **Oliver Heaviside**[572] (1850–1925, England). Eminent mathematical physicist. Originated modern operational calculus[573] (1893), laid the foundation of modern electric-circuit design and pioneered the application of vectors to physics. He developed and reformulated the electromagnetic theory of Maxwell, discovered the circuit principle that made the long-distance telephone possible, and foresaw television and over-the-horizon

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[571] To dig deeper into the contributions of these "Maxwellians" to Maxwell's heritage, one is advised to read the comprehensive study "The Maxwellians" by B.J. Hunt, Cornell University Press 1994, 266 pp.

[572] For further reading, see:

- Nahin, P.J., *Oliver Heaviside*: *Sage in Solitude*, IEEE Press: New York.

[573] Heaviside was first to apply the *unit-step function* (sometimes named after him). The *delta-function*, $\delta(x)$, was introduced by Dirac in quantum mechanics (1930), but Heaviside had already used it extensively before him (1893). **Cauchy** (1815) knew the unit-step function in the definition $U(t) = \frac{1}{2}(1 + \frac{t}{\sqrt{t^2}})$ which was called by him '*restricteur*'.

radio. He virtually 'invented' the ionosphere to explain Marconi's transmission of radio signals over the Atlantic in 1901.

Stimulated by Maxwell's "*Treatise on Electricity and Magnetism*" (1873) and working independently of Gibbs until 1888, he developed vector analysis from the quaternion system. Consequently, he vastly simplified Maxwell's 20 equations in 20 variables by squeezing their essence into 4 equations in vector form. He eliminated the potentials and emphasized the primacy of the physical field-vectors[574,575] $\boldsymbol{E}$ and $\boldsymbol{B}$ (he also suggested boldface type to distinguish vectors from scalars).

Heaviside was born in a London slum, "among these dark Satanic mills", at the beginning of the mid-Victorian age. His family was at a low social and economic level. The world of his youth was very grim and Heaviside might have been a character straight out of Dickens: the youngest of four sons of a sickly wood engraver who could barely support his family. An early bout with scarlet fever left his hearing permanently impaired, cutting him off from the society of other children. [That handicap molded a confrontational personality and sarcastic style that would sometimes carry him too far in his published attacks on those with whom he disagreed. Years latter he recalled his youth with great bitterness, declaring that it had permanently deformed the course of his life.]

He left school at 16 and thereafter had no formal education, let alone any university training. After teaching himself the Morse Code and the elements of electricity, he went at 18 to Denmark to work for the Northern Telegraph company. He got this job through his uncle, **Charles Wheatstone**, husband of his mother's sister. In Denmark, Heaviside gained practical experience as a telegraph operator and technical trouble-shooter, and was steadily promoted.

He returned to England in 1871 and embarked on an ambitious program of self-education in science and mathematics. A paper he published in 1873 merited mention in the 2^{nd} edition of Maxwell's treatise on '*Electricity and Magnetism*'. His encounter with this book led him to quit his position in 1874 and devote himself entirely to private study. [Heaviside would never again hold any other job in his life; he spent the next 35 years in scholarly research,

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574 To Maxwell, the magnetic vector potential, not the fields, played the central role in electrodynamics [an idea enjoying a comeback in modern Quantum Electrodynamics, and the other gauge theories inspired by it]. Also, Maxwell's original equation display no obvious symmetry in their form.

575 This was done independently by **Hertz** in Germany (1884). However, the two men arrived at the same endpoint by two entirely different paths. Heaviside and Hertz became good friends through an extensive correspondence, but never met.

the publication of technical papers and the carrying on a most interesting correspondence.] This was a momentous decision for a man of 24 without independent means. Since then he was living as a recluse among and off his relatives, devoting himself to the extension of Maxwell's theory.

In 1884, independently of **Poynting**, he described the flow of electromagnetic energy in space [Poynting got into print first, which justifies the modern name of 'Poynting vector']. During 1888–1903, Heaviside pushed Maxwell's theory beyond the limits set by the master himself: he was speculating on 'faster-than-light' charged particles producing a *conical wave* [electromagnetic shock-wave, known today as *Cherenkov radiation* of light in matter].

In 1891 Heaviside was elected a fellow of the Royal Society. Thus, in 17 years he had risen from the obscurity of an unemployed telegraphist to world fame. In 1896 a state pension was awarded him, at the instigation of **FitzGerald** and other distinguished scientists.

Nevertheless, the fact that he was not a university man raised a barrier, a certain antagonism, between him and his contemporaries. The latter reproached him for his notable lack of mathematical rigor. Yet Heaviside did develop an abundance of mathematical and physical methods and results which afterwards, on critical elaboration by various scientists, proved to be substantially true[576]. By 1908 Heaviside moved to Torquay, on the southern

[576] Heaviside used the abbreviations $p = \frac{d}{dt}$, $p^{-1} = \int_0^t \cdot dt$. One of the problems considered by him was a semi-infinite cable in series with impedance r and a voltage source $V_0 H(t)$. Neglecting the self-induction in the cable and denoting the potential and current by $E(x,t)$ and $I(x,t)$, respectively, the circuit equations are

$$-\frac{\partial I}{\partial x} = C\frac{\partial E}{\partial t}; \qquad -\frac{\partial E}{\partial x} = RI,$$

where C, R are the capacitance and resistance per unit length, respectively. Eliminating I and putting $\frac{\partial E}{\partial t} = pE$, Heaviside obtained the ODE:

$$\frac{d^2E}{dx^2} = (RCp)E$$

with the solution

$$E(x,p) = A(p)e^{\lambda x} + B(p)e^{-\lambda x}, \qquad \lambda = \sqrt{RCp}.$$

After the determination of A and B from the boundary conditions at $x = 0$ and $x = \infty$, he obtained the current I_0 and voltage E_0 at the end of the line in the explicit form

$$I_0 = V_0\left[r + \sqrt{R/Cp}\right]^{-1}; \quad E_0 = V_0\left[1 + r\sqrt{\frac{Cp}{R}}\right]^{-1}.$$

coast of England. There, his F.R.S. and other honors meant nothing to his neighbors, who treated him as a joke. He died there and lies buried in his parents' grave, his name visible on the tombstone only when the grass is closely cut.

Homage to Oliver

"Like all creative scientists, he did it because he could not help it. There were ideas pent up in him which demanded expression at any cost. He developed scientific ideas as naturally as a poet writes or a bird sings."

Norbert Wiener, 1936 (1894–1964)

Expanding in ascending powers of p (valid for $t \to \infty$) and proving heuristically that $p^{1/2}H(t) = (\pi t)^{-1/2}$ [a formula known to **Sylvestre Lacroix** (1819)], he obtained

$$E_0 = V_0 - V_0 r\sqrt{\frac{C}{\pi Rt}}\left\{1 - \frac{r^2C}{2Rt} + 3\left(\frac{r^2C}{2Rt}\right)^2 + \cdots\right\}.$$

For small values of t, Heaviside expanded the expression for E_0 in descending powers of p, arriving at

$$E_0 = 2V_0\sqrt{\frac{Rt}{\pi r^2C}}\left\{1 + \frac{2Rt}{3r^2C} + \frac{1}{15}\left(\frac{2Rt}{3r^2C}\right)^2 + \cdots\right\} - V_0\left(e^{\frac{Rt}{r^2C}} - 1\right).$$

Thus, Heaviside solved the problem of the electrical transmission-line *avoiding the use of the Laplace transform* and without using the term 'asymptotic series'. He never employed any of the theories of divergent series which were introduced at the end of the 19^{th} century, but proceeded quite formally. This caused him a great deal of trouble with the "Cambridge mathematicians", who were so indignant at Heaviside's unrigorous use of divergent series, that they stopped the publication of a sequence of his papers. Nevertheless, Heaviside continued to use his '*experimental mathematics*'.

"We should now place the operational calculus with Poincaré discovery of automorphic functions and Ricci's discovery of the tensor calculus as the three most important mathematical advances of the last quarter of the 19^{th} century. Applications, extension, and justifications of it constitute a considerable part of the mathematical activity of today."

Edmund Taylor Whittaker (1928)

"The next time you make a long-distance call and the voice on the other end comes through loud and clear, reflect for a moment on the man who made it possible."

Paul J. Nahin[577] (1988)

1881–1922 CE **Eliezer Ben-Yehuda** (1858–1922, Israel). Philologist and lexicographer. Father of modern Hebrew. Revived the ancient language into a vernacular that served as a basis for current spoken Hebrew. Compiled the great modern Hebrew Thesaurus (17 volumes; 1910–1922[578]), based on biblical, Talmudic and post-Talmudic sources. It was his fanaticism, aided by a combination of fortunate circumstances, which finally made a reality of his dream. If modern Hebrew became a living tongue with time, it was partly because of his innovations and efforts.

Ben-Yehuda was born as Eliezer Isaac Perelman in Lithuania. Studied at the Sorbonne, Paris (1878–1881). Settled in Jerusalem (1881), established the first Hebrew school (1881) and the Hebrew Language Academy (1890).

1881–1925 CE **Francis Ysidro Edgeworth** (1845–1926, Ireland and England). Economist. Made important contributions to mathematical economics and statistics, notably on general equilibrium theory.

Edgeworth was born in Longford, Ireland (now Irish Republic) of mixed Irish, Spanish and Huguenot descent. Became professor of political economy at King's College, London (1888–1891) and Oxford (1891–1922).

His main achievement (not adequately appreciated until the development of *game theory* and related topics after 1944) was to pioneer an approach to general equilibrium based not (like Walras's scheme) on an explicit economy-wide price mechanism, but on direct *cooperation between individual agents* in the absence of prices. This approach has subsequently been shown to yield an optimum effectively identical with the competitive optimum of **Walras** and **Pareto**.

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577 P.J. Nahin: "*Oliver Heaviside: Sage in Soltitude*" IEEE Press, 1988.

578 Completed 1959, by his widow and son.

1882 CE Electric light in New York.

1882 CE The *Albatross expedition*, under direction of U.S. Fish Commission, further extended knowledge of the extent and variety of *marine life*.

1882 CE **Moritz Pasch** (1843–1930, Germany). Mathematician. Pioneer of the pure axiomatic approach to geometry. In his book (1882) *Vorlesungen über neuerer Geometrie* (Lectures on modern geometry), he developed a new method of representation of rigorous deductive structures of projective geometry via axioms. The *Pasch Axiom*[579] is named after him. Hilbert was influenced by these ideas of Pasch.

Pasch was born to a Jewish family. He was a professor at Giessen (1873–1911).

1882–1892 CE **Carl Louis Ferdinand von Lindemann** (1852–1939, Germany). Mathematician. Proved that π is transcendental. This proof ended the long odyssey in quest of the squared circle, started by Anaxagoras[580], ca 434 BCE. Showed (1884, 1892) how to express the roots of an arbitrary polynomial in terms of theta functions.

1882–1897 CE **Friedrich Ratzel** (1844–1904, Germany). Geographer and ethnographer. Had principal influence in the modern development of both disciplines.

Ratzel was born in Karlsruhe, Germany and studied zoology at the University of Heidelberg, graduating in 1868. During 1869–1875 he traveled extensively in the Americas, studying urban and cultural life, which later helped him to lay the foundation of *cultural geography* and *political geography*. His most important works are: *Anthropogeographie* (1882, 1891) and *Politische Geography* (1897).

According to Ratzel, cities are the best place to study people because life is "blended, compressed, and accelerated" in cities, and they bring out the "greatest, best, most typical aspects of people." He believed that once these facts about urban life are examined, they can serve as a great aid in the study of cultural history. His interest in cultural geography would soon inspire him to explore the field of *human geography*.

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579 An axiom of *order*. If a straight line intersects one side of a triangle and does not pass through a vertex, it must intersect another side of the triangle. In the axiomatic system of Pasch, the concepts of point, line and plane are undefined.

580 The side of a square of equal area to a unit circle is $x = \sqrt{\pi}$. Since π is transcendental, the equation $x^2 - \pi = 0$ is not solvable by an algebraic number and hence π has no construction with compass and straightedge.

In his *Anthropogeographie*, he examined the causes of human population distribution, or the dynamic aspects of geography. He also related geography to history. Physical features, such as mountains or bodies of water, are discussed w.r.t. *human migrations*. According to Ratzel, religious, linguistic, and ethnic groupings also determine population distribution.

As an outgrowth of these studies, he began his study of *political geography*. In this book, Ratzel develops the concept that views the state as "a particular spatial grouping on the earth's surface." The state, as defined by Ratzel, consists of "a human group with definite organization and distribution." From these ideas, Ratzel developed the concept of *Lebensraum* or living space, Ratzel hypothesized that the state naturally seeks to increase its size. If the state's neighbors are weak, the state will grow larger and spread into other states. As evidenced, Ratzel believed that space was a great political force.[581]

1882–1911 CE **Adolph Hurwitz** (1859–1919, Germany). Mathematician. Contributed to the theories of special functions, ordinary differential equations, modular functions, number theory, Riemann surfaces, set theory and Fourier series[582]. In 1882 he defined the generalized Zeta function

[581] *Geopolitics* is an approach to understanding international politics that seeks to explain the political behavior of states in terms of geographical variables such as size or location (a kind of 'Social Darwinism').

The ideas of Ratzel in this field were extended by **Rudolf Kjellen** (1864–1922) and **Karl Haushofer** (1869–1946). The latter came to be seen in the 1930s and during WWII, as providing the geopolitical ideas for the *Nazis*. However, Nazi geopoliticians rejected Haushofer's geopolitics because it failed to incorporate the '*race principle*' adequately. Like Ratzel, Haushofer had some of his ideas hijacked by the Nazis. Nevertheless, Haushofer is still accused of providing the academic and scientific support for the expansion of the Third Reich. Incidentally, Haushofer's son, Albrecht, was indicted in the July 20, 1944 attempt to assassinate Hitler and was executed in 1945 by the SS.

Following the war, Haushofer was interrogated by the allies and put to trial before the Nuremberg War Crimes Tribunal, but acquitted. Together with his wife (half-Jewish) Haushofer committed suicide on March 13, 1946, in Pähl, W. Germany.

[582] In 1902, Hurwitz gave an elegant solution to the ancient *isoperimetric problem* of finding a simple closed plane curve of given perimeter with maximum area. Confining himself to continuous piecewise smooth close curves, he put $x = x(s)$, $y = y(s)$, $0 \le s < L$ as the parametric representation of such a curve of perimeter L and area F, s being the arc-length.

$[\zeta(s,a) = \sum_{n=0}^{\infty}(a+n)^{-s}]$ and derived an important formula for it[583].

In 1889 he wrote a fundamental paper on the zeros of Bessel functions, Lommel polynomials and analytic functions in general (*Hurwitz theorem*).

In 1896 he showed that any rotation in 4-dimensional space E_4, could be expressed in the form $q \to \ell q r^{-1}$, where q is a quaternion and ℓ, r are unit quaternions.

Assume the Fourier-series expansions

$$x(t) = \frac{1}{2}a_0 + \sum_{\nu=1}^{\infty}[a_\nu \cos(\nu t) + b_\nu \sin(\nu t)],$$

$$y(t) = \frac{1}{2}c_0 + \sum_{\nu=1}^{\infty}[c_\nu \cos(\nu t) + d_\nu \sin(\nu t)]$$

where $t = 2\pi\frac{s}{L}$. The relations

$$(dx)^2 + (dy)^2 = (ds)^2 = \left(\frac{L}{2\pi}\right)^2 dt^2, \qquad F = \int_0^{2\pi} x\dot{y}dt$$

then lead to

$$L^2 = 2\pi^2 \sum_{\nu=1}^{\infty} \nu^2(a_\nu^2 + b_\nu^2 + c_\nu^2 + d_\nu^2),$$

$$F = \pi \sum_{\nu=1}^{\infty} \nu(a_\nu d_\nu - b_\nu c_\nu).$$

It follows that

$$L^2 - 4\pi F = 2\pi^2 \sum_{\nu=1}^{\infty} \left[(\nu a_\nu - d_\nu)^2 + (\nu b_\nu + c_\nu)^2 + (\nu^2 - 1)(c_\nu^2 + d_\nu^2)\right] \geq 0.$$

For a given L, the *equality* will hold for a maximal F. In that case one must have $a_\nu = b_\nu = c_\nu = d_\nu \equiv 0$ for $\nu = 2, 3, \dots$ and $b_1 + c_1 = 0$, $a_1 - d_1 = 0$. Hence

$$x = \frac{1}{2}a_0 + a_1 \cos t + b_1 \sin t, \qquad y = \frac{1}{2}c_0 - b_1 \cos t + a_1 \sin t,$$

which are the parametric equations of a *circle*. All other curves must satisfy $L^2 > 4\pi F$.

583 $\zeta(s,a) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\left\{\sin\left(\frac{\pi s}{2}\right)\sum_{n=1}^{\infty}\frac{\cos(2\pi a n)}{n^{1-s}} + \cos\left(\frac{\pi s}{2}\right)\sum_{n=1}^{\infty}\frac{\sin(2\pi a n)}{n^{1-s}}\right\}$ for $\Re(s) < 0$.

In 1903 he investigated the properties of Fourier series when the sum does not necessarily converge, and discovered the *Hurwitz-Lyapunov theorem*[584].

He also derived a criterion for the stability of the solutions of ordinary linear differential equations. This criterion (necessary, but not sufficient) required the positivity of certain determinants formed by the coefficients [*Hurwitz criterion*, *Hurwitz polynomials*].

Hurwitz was born at Hildesheim, Germany, into a Jewish family. He studied at Münich, Berlin and Leipzig under **Weierstrass**, **Kronecker** and **Klein**. In 1882 he became privatdocent at Göttingen and in 1892 he was appointed to the vacant chair of **Frobenius** at the polytechnicum of Zürich.

1883–1894 CE **Osborne Reynolds** (1842–1912, Ireland). British engineer, physicist and educator. Known for his work in fluid mechanics. In 1883 he demonstrated that the transition from laminar to turbulent flow depends on a dimensionless characteristic number, known as "Reynolds number"[585]. In 1894 Reynolds introduced *turbulent shearing stresses* or *Reynolds' stresses*, into hydrodynamics in his paper: "*On the Dynamical Theory of Incompressible Viscous Fluids and the Determination of the Criterion*" [*Phil. Trans. A* **186**, 123–164]. These concepts are of great importance in fluid flow modeling experiments and many geophysical phenomena. Reynolds made significant contributions to the theories of heat transfer, turbine pumps, turbulence, tidal motions in rivers and the concept of *group velocity*.

Reynolds was born in Belfast into a family of Anglican clerics. He graduated at Queen's College, Cambridge, in mathematics (1867). In 1868 he became the first professor of engineering at Owen's College, Manchester, a position he held until his retirement in 1905.

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[584] Let $f(x)$ be bounded in the interval $(-\pi,\pi)$ and let $\int_{-\pi}^{\pi} f(x)dx$ exist, so that the Fourier constants a_n and b_n of $f(x)$ exist. Then the series $\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ is convergent and its sum is $\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx$.

[585] In fluid flow through a pipe, for example, Reynolds number R is given by $\{\overline{v}d\rho/\eta\}$, where $\overline{v}$ is the average flow velocity, d the pipe's diameter, ρ the fluid density and η is the fluid viscosity. The transition from laminar to turbulent flow occurs when $1000 < R < 5000$.

Modern Heat Engines; The Turbine

The chief reason for the low efficiency of steam engines is that they burn their fuel outside the cylinder. Much of the heat is absorbed by the bulky equipment that produces the steam. But if the heat is created inside the cylinder, this major source of energy loss is removed and we have an internal-combustion engine (ICE).

Early ICE used gas instead of gasoline as fuel. In a gas engine, the working fluid is a mixture of atmospheric air and an inflammable gas. Early experiments were described already in 1820 *in a paper entitled:*

"On the Application of Hydrogen Gas to produce a Moving Power in Machinery, with a description of an Engine which is Moved by the Pressure of the Atmosphere Upon a Vacuum Caused by the Explosions of Hydrogen Gas and Atmospheric Air".

It was read by ***W. Cecil*** *before the Cambridge Philosophical Society in England. Cecil mentioned earlier experiments at Cambridge by* ***Farish****, who was said to have operated an engine by gun powder.*

Another English inventor, ***William Barnett****, patented (1838) a gas engine which compressed the fuel mixture. Barnett's engine had a single up-and-down cylinder with explosions occurring first at the top, then at the bottom of the piston.*

In France, ***Jean Joseph Étienne Lenoir*** *built the first practical gas engine in* 1860*. It used street-lighting gas for fuel. This single-cylinder engine had a storage battery ignition system. The piston, moving forward for a portion of its stroke by the energy stored in the fly-wheel, drew into the cylinder a charge of gas and air at ordinary atmospheric pressure. At about half-stroke the valves closed, and an explosion, caused by an electric spark, propelled the piston to the end of its stroke. On the return stroke, the burnt gases were discharged, just as a steam engine exhausts. These operations were repeated on both sides of the piston, and the engine was thus double-acting. These engines were quiet and smooth in running. The gas consumption was, however, excessive. By* 1865 *four-hundred of these engines were in use in Paris for such jobs as powering printing presses, lathes, and water pumps.*

To a Frenchman, ***Alphonse Beau de Rochas****, belongs the credit of proposing the idea of a 4-cycle engine. In a pamphlet published in Paris in* 1862 *he contemplated such an engine which was to be built* 14 *years later. Rochas himself did not, however, put his engine into practice, and probably*

had no idea of the practical difficulties to be overcome. ***Siegfried Marcus***[586] *built the first successful petrol-driven 4-stroke cycle engine and carriage in 1875, superseding an earlier model (1864). In the same year,* **Carl Benz** *also developed a gasoline engine. These engines were basically the same as gasoline engines built today.*

In a gasoline engine the explosion of the fuel produces hot, expanding gases which force the piston to move. Four strokes of the piston are required to complete one cycle of operation — intake, compression, power and exhaust. Gasoline engines generally use many small cylinders rather than a large one, because vibration is reduced and the engine can run more slowly without stalling.

A 4-stroke cycle engine in an automobile or airplane can make 30 *revolutions per second. One should imagine the pistons racing up and down, the valves opening and closing, the sparks igniting the gasoline — all at the right time for each cylinder! [It seems a wonder that any mechanism so complex should work at all. Yet such is the skill of modern science and engineering, that we give the matter hardly a thought when we get into a car or airplane: we are confident of arriving at our destination in luxurious comfort and without a hitch.]*

In spite of the advantages of internal combustion, the average automobile engine is only about 15–20 *percent efficient*[587]*. Most of the heat of the explosion is lost in the hot exhaust gases, in the flow of hot air through the radiator, and in the friction of moving parts inside the engine.*

In 1896, ***Rudolf Diesel*** *(1858–1913, Germany) invented and built the diesel engine, which increased the efficiency by a simple method: instead of compressing an explosive mixture of air and gasoline, only the nonexplosive air is compressed. Since compression of air alone cannot result in burning, the air may be compressed to as high as ratio as* $18\colon 1$*. The high compression*

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[586] **Nikolaus August Otto** (1832–1881, Germany) built a small experimental gas engine (1861). He devised the 4-stroke cycle (which bears his name) in 1876 and derived a patent for it in 1877. **Gottlieb Daimler** used petrol (1885) to drive the engine.

[587] The efficiency of a steam engine is $\eta = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}$ where Q_1 is the total heat supplied (for heating the water and vaporizing it) and Q_2 is the heat given up to the condenser. It is less than $\frac{T_1 - T_2}{T_1}$, which is the ideal efficiency of a reversible Carnot cycle which would operate between two reservoirs of temperatures T_1 and T_2.

In an ICE the efficiency is $\eta = 1 - \left(\frac{v_1}{v_2}\right)^{\gamma - 1}$, where $\frac{v_2}{v_1}$ is the *compression-ratio* and $\gamma = \frac{C_p}{C_v} \simeq 1.4$. For $\frac{v_2}{v_1} = 7$ the limiting theoretical efficiency is 55 percent; the actual achievable efficiency is much lower.

heats the air to about $550\,^\circ C$*. Then, at the top of the compression stroke, a powerful pump (fuel injector), squirts fuel into the cylinder and the fuel ignites as it comes into contact with the very hot compressed air. Cheap oil can be used in the diesel engine instead of gasoline and an efficiency of up to* 40 *percent is achieved. However, because of the higher compression and more powerful explosion, the engine requires stronger and heavier cylinders. The extra weight makes the diesel engine less practical for light passenger cars.*

The steam engine, gasoline engine and diesel engine — all require a piston which moves back and forth in a cylinder. These engines are known as reciprocating engines*. A part called a* crankshaft *transforms this reciprocating motion into rotary motion to turn wheels.*

Felix Wankel *(1902–1988,* Germany*) patented in* 1929 *and built during the early* 1950*'s a gasoline engine that uses rotors instead of pistons (known as a* rotary engine *or a* Wankel engine*). The rotors produce rotary motion directly. A Wankel engine operates more quietly and smoothly than a piston engine, needs fewer moving parts, weights less, is smaller than a piston engine of the same power, uses lower octane gasoline but burns more fuel per kilometer. Large-scale production came in* 1968 *with the Japanese 'Mazda* 110 *S'.*

A turbine is a wheel turned by the momentum of a moving fluid such as wind, water, steam or gas. It converts the fluid motion directly into rotary motion [from the Latin turbo*, meaning that which* spins *or* whirls *around]. This rotary motion is used to turn generators that produce electricity and drive huge ocean liners, and serves as an essential component of a jet-propelled aircraft.*

Thus, turbines work on the same principle as the windmill and the water wheel. Unlike the other important types of engines described earlier, they have no pistons or cranks. The motion of a current fluid turns a shaft by means of an arrangement of projecting blades. Steam and internal combustion engines have to change reciprocatory motion into rotatory motion; turbines produce rotatory motion directly.

There are two main kinds of turbines: in some the whole available energy of the fluid is converted into kinetic energy before *the fluid acts on the moving part of the turbine. (In the case of steam, it supplies power* after *it has completely expanded, solely at the expense of its kinetic energy.) Such turbines are termed* Impulse *or* Action Turbines*, and they are distinguished by the fact that the wheel passages are never entirely filled by the fluid.*

In the early de Laval turbines*, the shaft turned at* $30,000$ *revolutions per minute — too rapidly for driving most kind of machinery — so it had to be reduced with gear wheels, thus lowering the efficiency. High speeds, of course,*

limit the size of these turbines. The centrifugal force in a large wheel would become so great that the wheel would break apart.

Turbines in which only part of the available energy is converted into kinetic energy before the fluid enters the wheel are termed pressure *or* Reaction Turbines.

In practice, the fluid (steam, say) passes through a ring of fixed blades *fastened to the turbine casing. These blades are so shaped that the space between one blade and the next* acts like a nozzle; *i.e., the blade spacing at the leading edge is greater than the spacing at the trailing edge. This means that as the steam leaves the blades, the pressure falls and the* steam expands. *The steam now hits a ring of moving blades. These blades are again so designed that the spaces between them function as nozzles.*

The moving blades receive a continuous backward thrust *(reaction) from the steam issuing from them. This makes them rotate in the opposite direction. As it leaves the moving blades, the steam again expands. It goes through another ring of fixed blades, onto another set of moving blades and so on. The function of the fixed blades (fastened to the turbine casing) is to aim the steam so that it strikes the wheels at the correct angle.*

Reaction turbines rotate about 3000 *revolutions per minute. A turbine works more efficiently than a piston engine, because the fluid pushes continuously against the turbine wheel. In a 4-cycle piston engine, the exploding fuel pushes against the piston on only one of the pistons four strokes. The turbine is also more efficient than piston engines because of its faster running speed, which makes it possible to deliver more power for its weight and volume. In addition, they do not have moving valves, spark plugs, carburetors and other parts which, in other engines, frequently require repair.*

The turbine finds its most important use today as a generator of electricity in power stations. These powerhouse turbines use coal[588] *— a cheap source of energy. The burning coal heats water and produces high pressure steam to drive the wheels.*

Water turbines *are used to generate electricity at dams and waterfalls. The power of a water turbine depends on the volume of flowing water and the distance (head) that water falls before it strikes the turbine wheel.*

Reaction turbine wheels are mounted on vertical shafts and are completely under water. *They have either spirally curved vanes or blades with variable slant that adjust the wheel to differing amounts of water flow. The wheels of a reaction turbine work best when a large volume of water falls a short*

[588] Nowadays, *nuclear fuel* is also used.

distance, while impulse wheel work best where a small volume of water falls a great distance.

Steam turbines rank among the most powerful machines in the world: one steam turbine turning a generator can supply all the electricity used by about 3 million people. Steam enters many turbines at temperatures up to 566 °C and may have pressures of up to 140 kg/cm^2. The steam rushes into the turbine at a speed of 1600 km/h. It strikes the first wheel, giving it a push, goes on to the next wheel, and so on. A modern steam engine has as many as 24 wheels mounted on a horizontal shaft. Steam expands to as much as 1000 times its original volume as it passes through the turbine. Therefore, each succeeding pair of nozzles and wheels must be larger than the last one to make use of all the expanding steam. This gives the steam turbine its typical trumpet-like shape. Most modern turbines use both impact-type and reaction-type wheels at different stages along the shaft.

Gas turbines use hot gases (oil, kerosene, natural gas) instead of steam, without first using them to heat water into steam. It has three main parts: first a *compressor*, a special type of fan that sucks in air and compresses it. This compressed air mixes with fuel and burns in a *combustion chamber*. The burning gases expand enormously and rush through a *turbine*, spinning the turbine wheels. Part of the rotary power from the turbine wheels drives the air compressor, that is mounted on the same shaft as the wheels. The rest of the rotary power can turn electric generators, run pumps or drive ships and locomotives. In a jet engine most of the power must rush *out* the turbine's tailpiece to give the plane a forward thrust.

The temperatures generated in gas turbines range from 700 °–800 °C. Thus engineers must make gas turbines from metals or ceramic materials that keep their strength and shape at such temperatures, which would weaken steel. The hotter a gas turbine runs, the more efficiently it operates. This can be a disadvantage when the turbine is used to propel ships or locomotives, which must often move slowly.

Windmills came into use in the Middle East in the 900's and in Europe in the 1100's. In the 1600's people built the first crude gas turbines by mounting fans over a cooking fire to turn roasting meat on a spit. The hot gases from the fire spun the fan, and gears connected the fan to the spit. In 1629 an Italian engineer, ***Giovanni Branca***, built a crude steam turbine which drove a machine. In 1791, the Englishman ***John Barber*** patented a gas turbine that was an ancestor of the turbojet. These first forms of turbines worked inefficiently. In 1832, ***Benoit Fourneyron*** (1802–1867, France) developed the first fully successful enclosed water turbine. It developed 37 KW and drove hammers used to forge metal.

C.G. de Laval *built a successful impulse steam turbine in* 1883. *In 1884* ***Charles Algernon Parsons***[589] *(1854–1931) developed a reaction steam turbine in England. The American inventor* ***Charles G. Curtis*** *(1860–1953) developed (1900) the first steam turbine using many sets of wheels. The first big Curtis turbine was installed in an electric power plant in Chicago in* 1903. *It ran a generator that produced* 5000 *kilowatts of electricity and started a revolution in power production.*

1883 CE, Aug. 26, 1:00 pm and ***Aug. 27***, 10:02 am *The volcanic explosion of Krakatoa* (Sunda Strait, 6.10 °S, 105.42 °E). A volcano island, about halfway between Sumatra and Java, was blown to bits in one of the most stupendous natural explosions ever recorded. About 20 cubic kilometers of material were emitted during the paroxysmal eruption.

An explosion of 150 megaton of TNT is required to produce the equivalent of the ensuing air pressure disturbance, and the total energy released through the Aug. 27 explosion is estimated at 10^{25} ergs. Actual sound from the Aug. 27 explosion was *heard* 5000 km away. Atmospheric ultrasonic *acoustic-gravity waves* circled the earth several times before they attenuated below the recording level. Long-period air-coupled *sea-waves* traveled as far as 18,000 km (through these waves energy was coupled from the atmosphere to the ocean via resonant coupling).

Some 36,000 humans perished in the disaster, mostly by a huge *tidal wave*, 40 meters high, that washed over the shores of nearby islands. A column of stones, dust, and ashes projected from the volcano shot up into the air to a height of 80 km — higher than the ozone layer. The finer particles coming into the higher layers of the atmosphere covered a large part of the surface of the earth, and gave rise to beautifully brilliant sunset glows and multicolored twilight effects, that were observed for 3–4 years after the eruption.

1883–1889 CE **Carl Gustaf Patrik de Laval** (1845–1913, Sweden). Engineer and inventor. Built the first practical *impulse steam turbine* (to power a cream separator of his invention).

Attempts to design a steam turbine had been made by numerous inventors, but all fell short of practical success — mainly because of the difficulty

[589] Son of the Irish astronomer **William Parsons** (1800–1867), 3^d Earl of Rosse, the first to observe a spiral nebulae (1845).

of arranging for a sufficiently high velocity in the working parts to utilize a reasonably large fraction of the steam kinetic energy [the principle involved requires that, for optimal efficiency, the velocity of the blades should approximate to half the velocity of the jets which strike them]. There was a further difficulty of getting the energy of the steam collimated in a single direction without undue dispersion, when it is allowed to expand through an orifice from a chamber at high pressure.

Laval overcame these difficulties, partly by the special shape of the nozzle used to produce the steam jet speed and partly by features of design which allowed an exceptionally high speed to be reached in the wheel carrying the vanes against which the steam impinged. To increase the velocity of gas in the nozzle beyond the speed of sound, he designed it such that after converging to a minimum cross-sectional area, the nozzle was expanded to a larger area. Laval's principle of nozzle design is widely employed in contemporary turbines and jet engines.

1883–1892 CE **George Francis FitzGerald** (1851–1901, Ireland). Physicist. Concluded, on the basis of Maxwell's equations that an oscillating electric current would produce electromagnetic waves (1883). This finding was later verified experimentally by **Heinrich R. Hertz** (1886) and used in the development of wireless telegraphy.

Independently of **H.A. Lorentz**, FitzGerald studied the results of the Michelson-Morley experiment[590](1887) and suggested that when in motion, a body is shorter (along its line of motion) than when at rest and that such a shortening, or contraction, affects also the instruments used in the experiment. Lorentz arrived at this idea independently (1895) and developed it considerably. The theory is known as the *Lorentz-FitzGerald contraction*, which **Albert Einstein** used in his special Theory of Relativity (1905).

FitzGerald was born and died in Dublin.

1883–1906 CE **Francis Edgar Stanley** (1849–1918, USA) and his identical twin brother **Freelan O. Stanley** (1849–1940, USA). Inventors and manufacturers. Invented (1883) a photographic *dry plate process* and operated a firm to manufacture the plates. Built (1896) the first *steam-engine powered automobile*; founded and directed (1902–1917) Stanley Motor Co. to produce *Stanley Streamers*; broke world's record for fastest mile (28.2 s) in a steam car (1906). Francis was killed in an automobile accident.

[590] The experiment was an attempt to measure the earth's motion relative to the pervasive luminiferous ether postulated as the medium in which light waves were propagated. The attempt failed to detect any such motion.

1883–1892 CE **Ludwig Gumplowicz** (1838–1909, Poland and Austria). Influential economist and sociologist. Maintained that human history is a result of a continued conflict, first among different ethnic groups, then between states (that were formed as a result of the conquest of habitable lands by the strong groups who subdued the weaker groups), and finally between classes inside the states.

Gumplowicz was born in Krakow, Poland, to a family of Jewish Rabbis, but later converted to Christianity. Professor (at Graz) from 1883.

1883–1908 CE **Svante August Arrhenius** (1859–1927, Sweden). Physicist and chemist. One of the founders of physical chemistry. Won the 1903 Nobel prize for his pioneering contributions to the *electrolytic theory of dissociation*[591] (1883–1887) and *chemical kinetics* (1889–1899).

In 1886 he established the importance of carbon dioxide to the earth's heat balance (maintaining that the doubling of the concentration of CO_2 in the atmosphere would result in an average global temperature increase of about 6 °C). He went on to describe the "*greenhouse effect*" brought about by CO_2 in the atmosphere.

According to the *kinetic theory of gases*, the pressure exerted by the gas on the walls of the container is determined by the bulk of molecules, whose energy is of order the *mean value*. This means that molecules with very large energies have practically no perceptible effect on the pressure and also on the total reserve of energy of the gas. In the case of chemical reactions the converse is often true. It turns out that precisely the rare molecules with high energy often determine the course of chemical reactions.

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591 His theory states that salts, upon dissolution in water, separate into mobile oppositely charged ions, even in the absence of an applied electric field. He used this to rationalize many seemingly puzzling properties pertaining to the behavior of solutions, e.g. the hydrolysis of salts, acids and bases; the solubility of salts and its variation with temperature; the constancy of the observed heat of neutralization of strong acids and strong bases. His theory's greatest triumph was that it could render a physical meaning to the parameter n in the van't Hoff equation for *osmotic pressure* $PV = nkT$, n being the *number of ions in the solution*.

Arrhenius did not know about electrons at the time (1883). Disbelieved by most of the senior chemists in Sweden, his thesis received the lowest grade that the University of Uppsala could bestow. But Arrhenius seems to have been a stubborn fellow, not easily put down by rebuffs. He circulated copies of his work to leading scientists through Europe, and as the years passed the theory gradually became more and more accepted. Eventually it was judged respectable enough for Arrhenius to be elected a member of the Swedish Academy of Sciences.

The mystery of chemical reaction timescales stems from the fact that molecules entering into a reaction collide every 10^{-10} sec whereas a reaction frequently requires several minutes (sometimes hours). It means that at any given time only an extremely small portion of all collisions result in a chemical reaction.

Arrhenius theorized that reactions are initiated only by collisions of molecules whose energy exceeds a definite critical value, the so-called *activation energy* E_A. Taking into account a thermodynamic equation of van't Hoff, he arrived at an equation which expresses the *reaction time*, t is proportional to $\tau e^{\frac{E_A}{kT}}$, where τ is the time between collisions, k is the Boltzmann constant and T is the absolute temperature.

For instance, when molecules of hydrogen and iodine collide, they form two molecules of hydrogen iodide HI. Taking $\tau = 10^{-10}$ sec, $T = 0\,^\circ\mathrm{C} = 273$ K, $E_A = 3 \times 10^{-12}$ erg, $kT = 3.8 \times 10^{-14}$ erg, we get $t = 3 \times 10^{17}$ years! This result accords with the fact that at $0\,^\circ$C, the reaction $H_2 + I_2 \to 2HI$ is practically unobservable. A characteristic feature of Arrhenius' formula is the extremely sharp decrease in reaction-time and increase in reaction rate for slight lowering of the temperature.

The arrhenius formula can also be recast in the form:

$$\textit{probability of reaction} \approx \textit{const.} \times \exp\left[-\frac{E_A}{kT}\right].$$

For example, the reaction $2H_2 + O_2 \to 2H_2O + 117$ kcal/mole although exotermic does not take place even at several hundreds degrees. A spark, or other such disturbance is required to ignite them: As a result of ignition, a violent explosion takes place, the hydrogen and oxygen being practically instantaneously converted into water vapor (the explosion is an adiabatic process: the heat cannot escape, but will go into raising the temperature, and consequently the pressure of the resulting water vapor will rise enormously).

The spark increases T in a localized region, and by Arrhenius' law, this increases the probability of the reaction. Once started, the heat liberated by the reaction near the spark raises the gas in the neighborhood to such a high temperature, that it in turn can react. This liberates more heat, and allows gas still further away to react, and so on.

The magnitude of E_A for each reaction can be determined *theoretically*, by solving the associated *Schrödinger equation*.

It frequently happens that chemical reactions are much more involved because they may proceed via diverse intermediate stages (reaction pathways). By way of illustration, the reaction $H_2 + Cl_2 \to 2HCl$ does *not* proceed via

collisions of a molecule of hydrogen and a molecule of chlorine, but by the scheme

$$H_2 + Cl_2 \rightleftharpoons H_2 + Cl + Cl; \quad Cl + H_2 \rightleftharpoons HCl + H; \quad H + Cl_2 \rightleftharpoons HCl + Cl.$$

As a result, the actually observed reaction rate involves complicated relationships. However, for each separate reaction, say, for $Cl+H_2 \rightleftharpoons HCl+H$, the Arrhenius law holds true, and the reaction rate is proportional to $e^{-\frac{E_A}{kT}}$, the activation energy E_A having different values for each reaction[592].

The energy of activation E_A, characterizes the *energy barrier* that must be overcome by the system for the reaction to occur. The dependence of the reaction rate on $\frac{E_A}{kT}$ follows from the Boltzmann distribution among energy levels: the exponential indicates the fraction of molecules that possess sufficient energy E_A for the reaction to take place. A chemical reaction proceeds spontaneously (i.e. is exergenic) only if it is accompanied by a *decrease of the free energy* $\Delta G < 0$ (a necessary but not sufficient kinetic condition).

Ahrenius' theory explains how a persence of a *catalyst* can accelerate a chemical reaction by many orders of magnitude – by lowering the activation energy.

Arrhenius was born at Vik, near Uppsala, the son of a surveyor and land agent. He studied at Uppsala (1876–1881), Stockholm (1881–1884), and the Universities of Graz, Amsterdam and Leipzig (1886–1888).

In 1895 he became professor of physics at Stockholm. In 1896 Arrhenius discovered that the amount of CO_2 in the atmosphere determines the global temperature and theorized that the ice ages occurred because some process had reduced the level of CO_2. [In 1938, **G.S. Callendar** determined that human activities were causing *increase* in the amount of CO_2 in the earth's atmosphere.] Similar views of the role of CO_2 in the earth's atmosphere were expounded in 1899 by **Thomas Chamberlin** (1843–1928, U.S.A.).

In 1907 he put forward the so-called '*panspermia theory*' in his book *Worlds in the Making.* In it he lifted the rather undeveloped ideas of **R.E. Richter** (Germany) and **Lord Kelvin** (1881), to a level of a serious scientific theory. Accordingly Arrhenius suggested that life did not originate on earth. He imagined that simple living forms may have drifted from world to world, propelled by *radiation pressure* through interstellar space.

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592 In chemical and biochemical reactions, the fraction $\frac{E_A}{kT}$ in the exponent is usually written as $\frac{E}{RT}$, where the universal gas constant $R = kN_A$ and $E = E_A N_A$ relate to a *mole* of substance rather than to a single molecule $[N_4 = 6.02205 \times 10^{23}/\text{mole}]$.

Panspermia and the Quest for Life's Origin[593]

In the physical sciences, the question of the origin of life is the study of the nature in which life evolved from non-life sometime between 3.9 and 3.5 billion years ago. This topic also includes theories and ideas regarding possible extra-planetary or extra-terrestrial origin of life hypotheses, thought to have possibly occurred over the last 13.7 billion years in the evolution of the known universe since the big bang.

Origin of life studies has a profound impact on biology and human understanding of the natural world. Progress in this field is generally slow and sporadic, though it still draws the attention of many due to the eminence of the question being investigated. A few facts give insight into the conditions in which life may have emerged, but the mechanisms by which non-life became life are still elusive.

Astronomers now believe that the universe began at least 15 billion years ago, when the first clouds of the elements hydrogen and helium were formed. Gravitational forces collapsed these clouds to form stars. These stars converted hydrogen and helium into heavier elements, including those such as carbon, nitrogen, and oxygen, which are necessary for life. These elements were returned to interstellar space by explosions of some of these stars to form clouds in which simple molecules such as water, carbon monoxide, and hydrocarbons were formed. These clouds then collapsed to form a new generation of stars and solar systems. In at least one solar system, our own, a variety of objects were formed, including comets (believed to be the most primitive objects in our solar system), meteorites, asteroids, and planets. One of the planets, the earth, formed at a distance from the sun where conditions were

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[593] For further reading one is referred to the following books:

- Hoyle, Fred and C. Wickramasinghe, *Evolution from Space*, Simon and Schuster, 1981.
- Dyson, Freeman, *Origins of Life*, Cambridge University Press, 1989.
- Goldsmith, D. and Tobias Owen, *The Search for Life in the Universe*, Addison-Wesley, 1992.
- De Duve, C., *Vital Dust: The Origin and Evolution of Life on Earth*, Basic Books, 1996.
- Hazen, R.M., *Genesis: The Scientific Quest for Life's Origins*, Joseph Henry Press, 2005.

favorable and the necessary chemical ingredients were available for the origin of life.

The final, most important events leading to the origin of life are perhaps the least understood of the story. Life began during the first billion years of an earth history which is 4.5 billion years old. In the early earth, volcanoes, a gray, lifeless ocean, and a turbulent atmosphere dominated the landscape. Vigorous chemical activity generated heavy clouds, which were fed by volcanoes and penetrated both by lightning discharges and solar radiation. The ocean received organic matter from the land and the atmosphere, as well as from infalling meteorites and comets. Here, substances such as water, carbon dioxide, methane, and hydrogen cyanide formed key molecules such as sugars, amino acids, and nucleotides. Such molecules are the building blocks of proteins and nucleic acids, compounds ubiquitous to all living organisms.

A critical early triumph was the development of RNA and DNA molecules, which directed biological processes and preserved life's "operation instructions" for generations. DNA and RNA first appeared as fragments, then a fully assembled helices. These helices formed some of the living threads. However, other threads derived from planetary processes such as ocean chemistry and volcanic activity. This evolving bundle of threads thus arose from a variety of sources, illustrating that the origin of life was triggered not only by special molecules such as RNA or DNA, but also by the chemical and physical properties of the earth's primitive environments.

The evolution of the plants and animals most familiar to us occurred only in the last 550 million years. The illustration depicts the appearance of marine invertebrates (such as shell-making ammonites), then fish, amphibians, reptiles, mammals, and humanity. The life thread which continues on in the oceans reminds us that the evolution of aquatic life continues even today. The development of land plant communities was manifested in the relatively ancient clubmosses, horsetails, and ferns, and the more recent gymnosperms (for example, conifers) and angiosperms (flowering plants).

Perhaps the most recent significant evolutionary innovation has been humanity's ability to record and build upon its experience, thus triggering the rise of civilization and technology. These developments bring us to the present, and, as the thread of life reaches the summit of a tree-covered hill, we ponder our future.

Most of life's history involved the biochemical evolution of single-celled microorganisms. We find individual fossilized microbes in rocks 3.5 billion years old, yet we can conclusively identify multicelled fossils only in rocks younger than 1 billion years. The oldest microbial communities often constructed layered mound-shaped deposits called stromatolites, whose structures suggest that those organisms sought light and were therefore photosynthetic. These

early stromatolites grew along ancient seacoasts and endured harsh sunlight as well as episodic wetting and drying by tides. Thus it appears that, even as early as 3.5 billion years ago, microorganisms had become remarkably durable and sophisticated!

Many important events mark the interval between 1 and 3 billion years ago. Smaller volcanic terrains were joined by larger, more stable granitic continents. Life learned how to release oxygen from water, and it populated the newly expanded continental shelf regions. Finally, between 1 and 2 billion years ago, the eukaryotic cells with their complex system of organells and membranes developed and began to experiment with multicelled body structures.

Given the huge number of stars known to exist in the universe, life has very likely also developed elsewhere. If this "other" life can control and transmit energy such as light and radio waves, we just might be able to detect it.

As NASA develops its mission to build a space station and to visit other solar system bodies such as comets, planets, and moons, it responds to humanity's need to return to the cosmos, both to understand life's origins as well as to expand its horizons.

Panspermia

Since the dawn of history, man has speculated about the possibility that intelligent life may exist on other worlds beyond the earth. Perhaps the earliest record of this tradition is found in Genesis **6**, *1–4.*

As astronomy developed, concepts akin to panspermia *(ubiquitous life) were propounded by various philosophers and scientists:* ***Anaximander*** *(ca 560 BCE) asserted that worlds are created and destroyed.* ***Anaxagoras*** *(ca 460 BCE), one of the first proponents of the heliocentric theory, believed that invisible seeds of life were dispersed throughout the universe.* ***Epicuros*** *(ca 300 BCE) and his school taught that many habitable worlds, similar to our world, existed in the vast reaches of space. The Roman philosopher* ***Lucretius*** *(ca 65 BCE) was a firm believer in the existence of 'other earths' inhabited by 'other people'.*

The concept of life on other worlds was incompatible with the Ptolemaic-Christian cosmology, and until the 15^{th} century CE the geocentric doctrine excluded all 'heretical' notions of panspermia. The first telescopic observations by ***Galileo*** *(1609) opened a new era in astronomy and dealt a mighty blow to the ideas of many of his contemporaries. It became evident that the planets were similar to the earth in many respects, and this similarity evoked the questions of the existence of cities inhabited by intelligent beings there.*

These bold ideas were advanced by ***Giordano Bruno*** *(1584) who wrote: "Innumerable suns exist; innumerable earths revolve around these suns. Living begins inhabit these worlds".*

During 1650–1800, *such writers, philosophers and scientists including* ***Cyrano de Bergerac*** *(1619–1655,* France*),* ***Christiaan Huygens*** *(1629–1695,* The Netherlands*),* ***Bernard de Fontenelle*** *(1657–1757,* France*), and* ***Voltaire*** *(1694–1778,* France*) published works dealing with life on other planets. Scientists and philosophers such as* ***Immanuel Kant*** *(1724–1804,* Germany*),* ***Pierre Simon de Laplace*** *(1749–1827,* France*), and* ***William Herschel*** *(1738–1822,* England*) advocated the hypothesis of the plurality of habitable worlds. By the end of the* 18^{th} *century, this hypothesis had gained almost universal acceptance by scientists and intellectuals.*

In 1854, ***William Whewell*** *(1794–1866,* England*) , in his book* Of the Plurality of Worlds*, argued against the probability of planetary life. During the late* 19^{th} *and early* 20^{th} *centuries various modifications of the panspermia hypothesis received wide circulation. According to this hypothesis, life in the universe exists eternally; living organisms never arise from nonliving matter, but are transmitted from one planet to another.*

At the turn of the 20th century, the Swedish chemist, ***Svante Arrhenius*** *(1907), conjectured that microorganisms — spores or bacteria, probably adhering to small specks of dust — are propelled by the* starlight radiation pressure *from one planet to another. If, by chance, they should land on some planet where conditions for life are favorable, these spores were thought to germinate and initiate the local evolution of life.*

Although such transmission of life from one planet to another within a single planetary system cannot be completely discounted, the feasibility of propagation of panspermia from one planetary system to another has divided the scientific community: most biologists and astronomers consider the hypothesis to be highly unlikely because ultraviolet light and cosmic X-rays would prove lethal to Arrhenius' spores. A minority of diehards, however, still believes that the complexity of terrestrial life cannot have been caused by a sequence of random events, but must have come from elsewhere.

Chief among these was the astronomer ***Fred Hoyle*** *(1915–2001), who resumed the debate in* 1981 *with a new theory of cosmic creationism, overriding most of the objections of the negativists. Others have pointed out that radiation pressure is not the only conceivable mechanism for interstellar transport of living things. One may assume that the Galaxy is populated here and there by advanced technical civilizations that have discovered and exploited space travel. A survey party from such a civilization, landing on a clement planet, may 'contaminate' it with diverse microorganisms — intentionally or otherwise. Alternatively, some have speculated that colonizing starships might*

themselves be nanotechnologically designed vehicles, carrying equipment for automated re-creation of favorable planets. Indeed, it has been suggested that humankind itself *engage in such nano-robotic assisted colonization projects!*

Our assessment of the probabilities for the existence of extraterrestrial life *are based on knowledge gained from terrestrial biology and the laws of physics and chemistry, which make some scenarios more probable than others. These laws (the summary of our experience in studying the universe around us) appear to be valid as far as we can test them, including the analysis of light from stars and from the most distant galaxies.*

Can we make any definite statement about the possible chemistry of alien life — the molecules that form living organisms in faraway places? Life that is based on chemical reactions (i.e. on the interaction of atoms to form complex molecules) appears to require carbon *as its key structural element. Only carbon can form chemical bonds with hydrogen, oxygen, and nitrogen (as well as other less abundant elements) in a way that readily promotes the development of a wide variety of information-bearing, structure forming, energy converting polymers.*[594]*If carbon is the crucial element in all chemical life, we are still not much restricted, since carbon has a high abundance everywhere in the universe — and indeed, some types of stars would not shine without the catalytic effect of carbon nucleus.*

Life also seems to require a solvent*, a fluid medium in which atoms, ions and molecules can encounter one another and undergo chemical reactions. The great ability of* water *to dissolve other polar substances makes it one of the most favored solvents. In addition, water's heat capacity, its heat of vaporization, its ability to remain liquid in a temperature range appropriate for many chemical reactions, its cosmic abundance, and its chemical stability — all single it out as exceptionally well suited for use by living organisms*[595].

For all types of life, it seems important to have a variety of individuals, and thus some sort of chemical mutation, starting from the simplest progenitor molecules. Otherwise, the processes of natural selection will not have enough material on which to operate, as it discriminates among various living creatures on the basis of reproductive success.

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[594] *Silicon* can also form polymers, but these are too stable under ordinary conditions to serve as a basis for life. The chemical affinity of silicon for oxygen implies that at temperatures low enough for complex molecular structures to exist, silicon will bound up as silicates.

[595] *Ammonia* or *methyl alcohol* might serve as a solvent instead of water under certain highly specialized conditions, but that would restrict the temperature range that life can tolerate.

The Greenhouse Effect

The atmosphere is largely transparent to visible light (3000–7000 Å) which occupies the peak of the solar spectrum (blackbody radiation at $5800°$ *K). The earth, warmed by solar light, emits a 'blackbody' spectrum (at* $300°$ *K) peaking in the infrared, at a wavelength of about* $10\,\mu m$*. This radiation cannot escape immediately because it is absorbed by the atmosphere, particularly by water vapor*[596] *and to a lesser extent by* CO_2*. The atmosphere therefore acts in the same way as the glass in a greenhouse*[597]*, letting through light but not allowing infrared radiation to escape. It is this infrared radiation that is mostly responsible for heating the atmosphere whereas the visible radiation is an inefficient heater.*

When gas molecules absorbs light waves, this energy is transformed into internal molecular motion, which is detectable as a rise in temperature. The warmed gas eventually radiates this energy away. Some of this reradiated energy travels upwards, where it may be reabsorbed by other gas molecules, a possibility less likely with increasing height, because the concentration of water vapor decreases with altitude. The remainder travels downward, and is again absorbed by the earth. Thus, the earth's surface is supplied continually with heat from the atmosphere, as well as from the sun. Without these absorptive gases in our atmosphere, the earth would not be a suitable habitat for humans and numerous other life-forms.

Of all the gases which comprise our atmosphere, N_2 *is a poor absorber of all types of incoming radiation.* O_2 *is an efficient absorber of shorter ultraviolet*

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596 The first scientist to describe the *greenhouse effect* (for *water vapor*), was the physicist **John Tyndall** (1820–1893, Ireland) in a paper "*On Radiation Through the Earth's Atmosphere*" (1863).

597 *Greenhouse effect*: short-wave solar radiation passes through the glass and is absorbed by the objects in the greenhouse. The long wavelengths radiated by these objects (e.g. infrared) cannot penetrate the glass and are trapped, thus warming the greenhouse. An important factor in keeping the greenhouse warm is the fact that it prevents mixing of the air inside with cooler air outside. In other words, greenhouses are warm mainly because air is not allowed to escape and *convect* heat away; a crop of standing corn, a walled garden, a tree-bordered enclosure, act in the same way and become warmer than their surrounding by reducing free circulation of air.

waves in the high atmosphere. Ozone (O_3) absorbs longer ultraviolet radiation in the stratosphere (10–50 km), which accounts for the high temperature there. Altogether H_2O, O_2 and O_3 absorb 20% of the total solar radiation.

Any reduction in the water vapor or CO_2 content of the atmosphere would weaken the greenhouse effect and allow the earth to cool. However, the concentration of CO_2 in the atmosphere is now increasing, slowly but steadily.

When humans started to burn fossil fuel on a large scale with the onset of the Industrial Revolution, they caused a great deal of "locked up" carbon to be released, and this trend is now being aggravated by burning tropical forests.

Whereas the level of CO_2 in 1850 *amounted to* 265 *ppm, it has now grown to* 340 *ppm, and unchecked it could well reach* 600 *ppm by* 2050. *The result is a steady warming of our planet, projected to rise by $3\,^\circ C$ above normal within* 50 *years. While there will be little change at the equator, the poles may well become $7\,^\circ C$ warmer.*

Thus, carbon dioxide, that gas that puts fizz into soft drinks, is one of the most important components of the atmosphere, and plays a key role in determining the earth's climate, even though it amounts to a mere 0.03 *percent*[598].

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598 Models and analyses of global warming generally agree that human economic activity makes the earth warmer than it would otherwise be. Yet discrepancies between theory and observation persist (1994); the predicted warming based on recent increases in concentrations of greenhouse gases is slightly more than the observed warming of the atmosphere. In addition, the warming trend in North America does not appear to follow the global pattern.

The answer is ironic. In all probability, aerosols primarily composed of sulfates, themselves the result of commercial activity, enhance the ability of the atmosphere to reflect sunlight back into space before it can reach the planet's surface and participate in the warming process. The sulfate particles, about 0.1–1.0 micron in diameter, are particularly concentrated over the industrial areas of the Northern Hemisphere. Their capacity to *cool by scattering sunlight* has become a recognized force in climatic change only recently. Clearly, both the cooling effects of aerosols and the warming caused by greenhouse gases must be taken into account if we are to attain accurate climate models.

In contrast to industrial effects, *agriculture* is directly or, at least in some cases, indirectly responsible for releasing a substantial proportion of greenhouse gases (CO_2, methane, nitrogen oxide and chlorofluorocarbons).

Now, global warming could either *enhance or impede* agriculture: warmer air holds more water vapor, and so global warming will bring about more evaporation and precipitation. Areas where crop production is limited by acid condi-

In addition to absorption, the overall radiation balance depends also on the part of the solar energy that is reflected back into cosmic space. This reflection takes place from the outer layers of the atmosphere, clouds, dust, and the earth's surface. The coefficient of reflectivity (i.e. the percentage of solar energy flux reflected back into space) is called the albedo, and for the earth it averages approximately 0.310. *The most important component affecting the earth's albedo are the clouds*[599].

When both reflection and absorption are taken into account an overview of the radiation balance in the earth-atmosphere system can be set up. Certain complication arise due to the following factors:

- *There are subsystems which can store thermal energy in the hydrosphere and atmosphere, and these can mutually exchange some of the absorbed solar energy.*
- *A significant part of the solar energy is converted to other forms (e.g. mechanical energy) in the moving atmosphere (winds), energy in the movement of the hydrosphere (waves), the chemical energy of photosynthesis, and heat which evaporates water from the biosphere.*
- *There are some small (but non-negligible) sources of energy which are completely independent of solar energy (geothermal, gravitational, radioactive).*

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tions would benefit from a wetter climate. Moreover, given sufficient water and light, increased ambient CO_2 concentrations absorbed during photosynthesis could act as a fertilizer and facilitate growth in certain plants.

If, however, increased evaporation from soil and plants does not coincide with more rainfall in a region, more frequent dry spells and droughts would occur, and a further rise in temperature will reduce crop yields in tropical and subtropical areas.

Finally, global warming will precipitate a thermal swelling of the oceans and melt polar ice. Higher sea levels may claim low-lying farmland and cause higher salt concentrations in the coastal groundwater.

The most recent analysis of the impact of climatic change on the world food supply (1992), concluded that average global food production will decline 5 percent by 2060.

599 Land has an albedo of about 20%, calm sea 8%, stormy sea 40%, fresh snow 80%, clouds 70%, ice 70%, and forest 12%. *Man* has changed 17% of the continent's surface, that is some 5% of the earth's surface. This probably changed the global albedo from 0.305 to the present value of 0.310, corresponding to a decrease in solar flux of approximately 8.6×10^{11} KW with a resulting *global cooling* of $\sim 1\,^{\circ}$K.

All these factors result in a rather complex (and still imperfectly understood) system of energy flows which has existed in stable form for million of years. However, since the Industrial Revolution, these have become somewhat influenced by man's own technology.

A blackbody's radiation is governed by the Stefan-Boltzmann radiation law.

The average total flux (insolation) from the sun, known as the solar constant, is about 1.36 kW/m^2 [Watt = Joule/sec = 0.239 cal/sec, of which 40% is in the visible light range (0.4–0.7 micron)]. On a sunny day 75% of insolation may reach the earth's surface; on an overcast day only 15%. On average 51% of insolation is absorbed by the surface as thermal energy – 29% as direct radiation, and 22% as diffused radiation. The latter comprises light scattered by atmospheric dust, water vapour, and air molecules. About 4% of the radiation reaching the surface is directly reflected at the same wavelengths,from the surface back into space. Surface reflectance values (albedo) depend on materials (e.g. 5–10% soils, 15–25% grass, 40–90% snow, etc.) Of the mean $(100 - 51 - 4 = 45\%)$ of insolation not reaching the surface at all, the breakdown is as follows: 6% (in insolation units) scatters from the atmosphere back into space; 20% reflected from cloud tops; 3% absorbed by clouds, and 16% absorbed by the atmosphere. Eventually, all of the visible (and other) optical radiation absorbed by the atmosphere and earth's surface is re-radiated back into space as infrared rays (3–30 microns) peaking at 10 microns. The average result of radiation absorption, scattering, reflection and re-radiation is that the mean atmospheric surface temperature is maintained at about 15°C.

Ignoring the atmosphere, and assuming a steady state blackbody radiation, where the influx of energy from the sun equals the outflux of energy radiated by the earth, the terrestrial temperature T_e assumes a value given by the equation

$$e_a \sigma T_e^4 4\pi R_e^2 = f_e e_s \sigma T_s^4 \left(\frac{R_s^2}{d^2}\right) \pi R_e^2,$$

e_a = *emissivity of earth,* e_s = *emissivity of sun,* σ = *Stefan's constant,* $T_s = 5800\,^\circ K$ = *sun's absolute temperature,* $R_s = 6.96 \times 10^8$ *m* = *radius of sun,* d = *sun-earth distance* $= 1.5 \times 10^{11}$ *m,* f_e = *fraction absorbed by earth. Therefore,*

$$T_e = T_s \left[f_e \frac{e_s}{e_a} \frac{R_s^2}{4d^2}\right]^{1/4}.$$

To qualify as a perfect black body, the surface of the planet must be nonreflecting; that is, all the sunlight reaching it must be absorbed and later

reradiated as planet light $(f_e = 1)$. *Also, there can be no gases in its atmosphere that absorbed outgoing planet light. Leaving aside such subtleties the above equation renders a mean surface temperature of about* $278\,^\circ K = 5\,^\circ C$.

If we take into account the sunlight reflected back by clouds, ice caps, and deserts, the proper value for f_e *is* 0.65. *If this be the entire story, the earth's surface temperature would instead be* $-20\,^\circ C$ *(all water is frozen).*

However, the greenhouse blanketing of the absorbing atmospheric gases keeps the earth's surface significantly warmer than $250\,^\circ K$. *To estimate how much warmer, we note that the application of the Stefan-Boltzmann law to the earth-atmosphere system yields* $T^4 = 2(250)^4$, *where the factor* 2 *arises from the fact that the atmosphere radiates outward into space* and *inwards to the earth's surface, whereas the earth's surface radiates only upwards. Therefore* $T = 297\,^\circ K = 24\,^\circ C$, *which is about right. The difference between* $-20\,^\circ C$ *and* $24\,^\circ C$ *is a measure of the greenhouse effect; the empirical result, as noted above, is actually* $15\,^\circ C$.

The energy exchange between the atmosphere and the surface is not entirely radiative: air which is warmed by contact with the surface rises, and transports heat upward by convection. *Also, evaporation of water from the ocean cools the surface, and when water droplets condense, heat (latent heat of condensation) is passed into the atmosphere. Thermal* conduction *also transports heat from the surface into the air, though to a smaller extent than convection and evaporation.*

For every 100 *units of solar radiation incident upon the upper atmosphere, only* 22 *reach the surface directly and are absorbed. Of the rest,* 35 *are reflected back into space (mostly within the troposphere),* 21 *are absorbed by the atmosphere (mostly above the troposphere), and the remainder of* 22 *reaches the surface after scattering and diffusing through clouds.*

The earth radiates some 118 *units: 11 escape directly, and* 107 *are absorbed by the atmosphere.*

The atmosphere radiates 158 *units: 54 travel into space and* 104 *return to earth.*

The outgoing thermal radiation $(11 + 54 = 65)$ *therefore just balance the* $100 - 35 = 65$ *units of solar energy that entered altogether into the system.*

Radiative processes alone, however, leave the atmosphere with a debit *of* 30 *units* $(21 + 107 - 158 = -30)$ *and the surface with a* credit *of the same amount* $(44 + 104 - 118 = 30)$. *A net balance is achieved via the processes of convection, evaporation and conduction.*

1883–1909 CE **Wilhelm Maybach** (1846–1929, Germany). Automobile builder. Produced with **Daimler** (1883) one of the first gasoline motors. Constructed the first Mercedes automobile (1900–1901); credited with the invention of spray-nozzle *carburetor*, *honeycomb radiator*, and change-speed gear. With his son Carl established at Friedrichshafen a company to build aircraft engines (1909). Maybach automobiles were produced from 1922 to 1939.

1883–1923 CE **Alfred Pringsheim** (1850–1941, Germany). Mathematician. made important contributions to the theory of convergence and divergence of infinite series and infinite products. Studied the position of singularities of power series, derived a new test of convergence of complex series and established theorems on multiplication of infinite series and the convergence and summability of Fourier series.

Pringsheim was born of Jewish parents and baptized in order to obtain a university position. All his property was confiscated by the Nazis (1939) and he was forced to flee to Switzerland.

1883–1932 CE **Konstantin Eduardovitch Tsiolkovsky**[600] (1857–1935, Russia). A pioneer of astronautical and space travel theory. In an article written in 1898 [appeared in 1903 in *Nautschnoje obozrenije* (Science Survey)] he described a streamlined, rocket-driven vehicle for space travel which used liquid oxygen and hydrogen as propellants. He was perhaps the first man to base this project on sound principles[601]. His proposal included such practical innovations as gyroscopic control, a jet deflector for navigation in space, proper rocket shapes, nozzles to ensure supersonic exhaust velocities, a multistage operation to escape the earth's gravitational field, the protection of the passenger chamber from atmospheric frictional heating and a rotating

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[600] For further reading, see:

- Kosmodemianskii, A.A., *Konstantin Tsiolkovsky: His Life and Work*, Translated from the Russian by X. Danko, Foreign Language Pub. House: Moscow, 1956, 101 pp.
- Von Braun, W. and F.I. Ordway, *History of Rocketry and Space Travel*, 1967.
- Celnikier, L.M., *Basics of Space Flight*, Editions Frontieres: Gif-sur-Yvette Cedex: France, 1993, 356 pp.

[601] Before him (1881), the German engineer **Hermann Ganswindt** had recognized the fundamental importance of the *escape velocity* $\left(11.3\ \frac{\text{km}}{\text{sec}}\right)$ and had conceived (with the physicist **R.B. Gostkowsky**) vehicles that would propel themselves with a series of chemical explosions and thus escape terrestrial attraction. The name *astronautical* was coined (1912) by the Frenchman **Robert Esnault-Pelterie**.

space station. Most of his suggestions are realized today in the design of space vehicles.

In 1895, Tsiolkovsky introduced the new concept of *space elevator*, He imagined placing a "celestial castle" at the end of a spindle-shaped cable, with the "castle" orbiting the earth in a *geosynchronous orbit*. The tower would be built from the ground up to an altitude of 35,790 kilometers above mean sea level (geostationary orbit).

Tsiolkovsky was born in Izhevsk, southern Siberia, the son of a forester. At the age of nine he became almost completely deaf following a serious illness. He was self-educated and made his living as a science teacher during 1880–1918, first in a sequestered country school in the Borovsk district and then at Kaluga. Throughout this period he devoted most of his free time to scientific investigations.

His advanced ideas were slow to gain acceptance; he was met with indifference and disbelief. However, in 1918 he became a member of the Academy, and in 1921 he was allotted a personal pension. In the mid-1920's, Tsiolkovsky's works on rocket engineering and space flight began to win international recognition. **Hermann Oberth** wrote to him in 1929: "*You have ignited the flame, and we shall not permit it to be extinguished*".

One of the largest craters, on the dark side of the moon, discovered in 1959 by the Soviet spaceship *Lunik 3*, was named after Tsiolkovsky.

Space Elevator — or, Climbing to the Stars[602] *(1895–2005)*

The Biblical Jacob [Genesis **28***, 12] saw in his dream "a ladder set up on the earth, and the top of it reached to heaven, and . . . the angels of God ascending and descending on it." Since his time, ladders have been replaced*

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602 For further reading, see:

- Celnikier, L.M., *Basics of Space Flight*, Editions Frontieres: Gif-sur-Yvette Cedex: France, 1993, 356 pp.

- Pearson, J., *The Orbital Tower: a spacecraft launcher using the earth's rotational energy*, Acta Astronautica **2**, 1975, pp. 785–799.

by elevators, and so a modern Jacob might well dream of building an elevator along which astronauts could ride to space.

Indeed, we know that a satellite at altitude of 35,790 km, whose orbit is in the plane of the equator, will appear stationary from the earth's surface: an electromagnetic signal can be sent from one to the other without adjusting the direction of the antenna. This is called a geostationary orbit[603]*, much used by the telecommunication industry.*

In principle, one may draw a material cable from a geostationary satellite to the point immediately below it on the terrestrial equator, effectively creating a physical track along which vehicles could move, driven by an earth-bound electrical generator, thereby lowering rocket-launch costs and risks to a minimum.

Next, we expound the basic physics of such an endeavor: Let m *be a mass point in a circular orbit at distance* r *from the earth's center, and orbital angular velocity* $\omega = \frac{v}{r}$. *No net force acts on the satellite, being in dynamic equilibrium under the opposing centrifugal force* $m\omega^2 r$ *and the gravitational force* $G\frac{M_E m}{r^2}$.

Consider now a second mass $\Delta m \ll m$ *attached to* m *by a weightless wire or cable (tether) of length* l*, the link being aligned along the radius vector to the earth's center, with* Δm *closer to earth. Since* $\omega = \frac{v}{r}$ *is common to both* m *and* Δm *(a rigid link constraint), the forces acting on* Δm *are:*

- *a centrifugal force* $(\Delta m)\omega^2(r-l)$ *away from the earth's center*
- *a gravitational force* $GM_E(\Delta m)/(r-l)^2$ *towards the earth's center*

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[603] *A Geosynchronous Satellite*: consider a satellite of mass m in a circular orbit around the earth at a constant speed v at an altitude h above the earth's surface. Since its centripetal acceleration is furnished by the gravitational force, Newton's second law yields $G\frac{M_E m}{r^2} = m\frac{v^2}{r}$, where G is the gravitational constant, M_E is the earth's mass and $r = R_E + h$ is the distance of the satellite from the earth's center.

In order to appear to remain over a fixed position on the earth, the period of the satellite must be $T = 24$ hours and the satellite must be in orbit directly over the equator. Combining the above equation with $v = \frac{2\pi r}{T}$ we obtain $r = \left(\frac{GM_E T^2}{4\pi^2}\right)^{1/3}$. Substituting numerical values $G = 6.67\times10^{-11}\,\frac{\mathrm{Nm}^2}{\mathrm{kg}^2}$, $T = 86,400\,\mathrm{sec}$, $M_E = 5.98\times10^{24}\,\mathrm{kg}$, we obtain $r = 42,250\,\mathrm{km}$, or $h = 35,790\,\mathrm{km}$.

Thus, there is a net force tending to pull Δm towards the earth:

$$\begin{aligned} \Delta F &= \frac{GM_E(\Delta m)}{r^2}\left[\left(1-\frac{l}{r}\right)^{-2}-\left(1-\frac{l}{r}\right)\right] \\ &\approx \frac{GM_E(\Delta m)}{r^2}\left[3\frac{l}{r}\right] \qquad \text{for } l \ll r, \end{aligned} \tag{1}$$

where the relation $m\omega^2 r = \frac{GM_E(\Delta m)}{r^2}$ was used. The mass Δm is prevented from falling towards the earth by a tension ΔF in the connecting link.

Instead of a small mass joined to a larger one by a mass-less wire, consider next a long, massy cylindrical tether sticking out of m towards the earth, the entire system moving as a rigid body at an angular velocity ω around the earth. Across an elementary segment of tether, mass dm and distance r from the earth, there will be a difference in tension dF given by

$$dF = \frac{GM_E dm}{r^2} - \omega^2 r dm = \frac{GM_E \rho A dr}{r^2} - \omega^2 r \rho A dr, \tag{2}$$

where $dm = \rho A dr$, ρ being the density, A the cross-section, and dr the length of the segment. Suppose that the cable has a uniform cross-section and density, and stretches from a geosynchronous satellite to a point whose distance is a from the earth's center. The tension F_r at a point on the cable at distance r from the terrestrial center is obtained by integrating the last equation:

$$F_r = GM_E A\rho\left(\frac{1}{a}-\frac{1}{r}\right) - \frac{1}{2}\omega^2 A\rho(r^2-a^2). \tag{3}$$

Since $GM_E \approx 10^{14}$, $\omega^2 \approx 5\times 10^{-9}$, the second term in the above equation is negligible compared to the first for cables whose length is a finite fraction of the geostationary distance. Thus

$$F_r \approx GM_E A\rho\left(\frac{1}{a}-\frac{1}{r}\right) \approx \frac{GM_E A\rho}{a} \qquad \text{for } r \gg a, \tag{4}$$

directed towards the earth's center. If the material of which the cylinder is made must not rupture under its own weight, we have

$$\sigma A > GM_E A\rho\left(\frac{1}{a}-\frac{1}{r}\right), \tag{5}$$

where σ is the strength. This imposes a limit to the length of the cable which can be suspended in this way. It can be shown that for a uniform tether to hang all the way from a geostationary satellite to the surface of the earth, the

strength of the material from which the cable is made must be in the region of the theoretical upper limit, namely $10^{11}\,N/m^2$, *or* $100\,GPa$[604].

Since the tension in the tether is greatest at the height of the geosynchronous satellite and tapers down towards earth, it is advantageous to taper the structure from the point of greatest force to that of least force, in such a way that the stress per unit cross-sectional area is constant. Thus, replacing dF *by* σdA *we recast Eq.* (2) *in the form*

$$dF = \sigma dA = \frac{GM_E}{r^2}\rho A dr - \omega^2 r \rho A dr. \tag{6}$$

Dividing by A *and integrating from* ξ *to* r *we obtain*

$$A(r) = A(\xi)\, e^{\frac{\rho}{\sigma}\left[\frac{1}{2}\omega^2(\xi^2 - r^2) + \frac{GM_E}{\xi}(1 - \frac{\xi}{r})\right]} \tag{7}$$

But $\frac{GM_E}{\xi^2} = g(\xi) =$ *the acceleration of gravity at level* ξ. *Also the first term in the exponent (centrifugal acceleration) is much smaller than the gravitational term. Choosing* $\xi = r_0 =$ *earth's equatorial radius* $= 6378\,km$, $g_0 = g(r_0) =$ *acceleration due to gravity at the cable's base (earth's surface)* $= 9.780\,m{\cdot}s^{-2}$; $A(r_0) =$ *cross-sectional area of the cable on the earth's surface,* (7) *yields*

$$A(r) = A(r_0)\, e^{\frac{\rho}{\sigma} g_0 r_0 (1 - \frac{r_0}{r})}. \tag{8}$$

This equation gives a shape where the cable thickness initially increases rapidly in an exponential fashion, but slows at an altitude a few times the earth's radius, and then gradually plateaus when it finally reaches maximum thickness at geostationary orbit. The cable thickness then decreases again out from geosynchronous orbit.

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[604] By comparison, most steel has a tensile strength of under 2 GPa, and the strongest steels no more than 5.5 GPa, but steel is dense, The much lighter material *Kevlar* has a tensile strength of 2.6-4.1 GPa, while *quartz* fiber can reach upwards of 20 GPa; the tensile strength of *diamond* filaments would theoretically be minimally higher.

Carbon nanotubes appear to have a theoretical tensile strength and density that are well above the desired minimum for space elevator structures. The technology to manufacture bulk quantities of this material and fabricate them into a cable is in early stages of development. While theoretically carbon nanotubes can have tensile strengths beyond 120 GPa, in practice the highest tensile strength ever observed in a single-walled tube is 52 GPa, and such tubes tensile strength ranges between 30 and 50 GPa. Even the strongest fiber made of nanotubes is likely to have notably less strength that its components. Improving tensile strength depends on further research on purity and different types of nanotubes.

Thus the taper of the cable from base to the satellite at $\xi = r_G = 42,250\,km$ is:

$$\frac{A(r_G)}{A_0} = \exp\left[\frac{\rho}{s} \times 4.832 \times 10^6\ m^2 \cdot s^{-2}\right].$$

Using the density and tensile strength of steel, and assuming a diameter of 1 cm at ground level, yields a diameter of several hundred kilometers (!) at geostationary orbit height, showing that steel, and indeed most materials used in present day engineering, are unsuitable for building a space elevator.

The equation shows us that there are four ways of achieving a more reasonable thickness at geostationary orbit:

- *Using a lower density material. Not much scope for improvement as the range of densities of most solids that come into question is rather narrow, somewhere between 1000 kg m^{-3} and 5000 kg m^{-3}.*
- *Using a higher strength material. This is the area where most of the research is focused. Carbon nanotubes are tens of times stronger than the strongest types of steel, hugely reducing the cable's cross-sectional area at geostationary orbit.*
- *Increasing the height of a tip of the base station, where the base of cable is attached. The exponential relationship means a small increase in base height results in a large decrease in thickness at geostationary level. Towers of up to 100 km high have been proposed. Not only would a tower of such height reduce the cable mass, it would also avoid exposure of the cable to atmospheric processes.*
- *Making the cable as thin as possible at its base. It still has to be thick enough to carry a payload however, so the minimum thickness at base level also depends on tensile strength. A cable made of carbon nanotubes (a type of fullerene), would typically be just a millimeter wide at the base.*

A space elevator cannot be an elevator in the typical sense (with moving cables) due to the need for the cable to be significantly wider at the center than the tips. While designs employing smaller, segmented moving cables along the length of the main cable have been proposed, most cable designs call for the "elevator" to climb up a stationary cable.

Climbers cover a wide range of designs. On elevator designs whose cables are planar ribbons, some have proposed to use pairs of rollers to hold the cable with friction. Other climber designs involve moving arms containing pads of

hooks, rollers with retracting hooks, magnetic levitation (unlikely due to the bulky track required on the cable), and numerous other possibilities.

Power is a significant obstacle for climbers. Available energy storage densities, barring significant technological advances, are unlikely to be able to store the energy for an entire climb in a single climber without making it weigh too much. Some potential solutions have involved laser *or* microwave power beaming*, and solar power. Other possible designs involve the use of:*

- *energy from regenerative braking of down-climbers passing energy to up-climbers as they pass,*
- *magnetospheric braking of the cable to dampen oscillations,*
- tropospheric *heat differentials in the cable,*
- ionospheric *discharge through the cable.*

The primary power methods (laser and microwave power beaming) have significant problems with both efficiency and heat dissipation on both sides, although with optimistic numbers for future technologies, they are feasible. Electrical power transmitted from earth or from the geostationary station through the tether cable might require the use of yet to be developed superconducting materials which could complicate the cable design and add potential corrosion and microscopic cracking issues. Carbon nanotubes, while not superconducting, can be extremely conductive and may represent a solution to this problem.

Planetary engineering on a scale needed to realize a working space elevator may seem today hardly more than a science fiction writers' domain. Yet the point is that it is not excluded by any physical law!

Space Elevator Timeline

1895 ***Konstantin Tsiolkovsky*** *imagined the idea of a* space elevator*. Comments from* ***Nikola Tesla*** *at about the same time suggests that he may have also conceived this idea.*

1957 ***Yuri N. Artsutanov*** *suggested to extend a* counterweight *from the geosynchronous satellite in a direction away from earth, keeping the center of gravity of the cable motionless relative to earth. He also proposed* tapering the cable thickness *so that the tension in the cable be*

kept constant. This renders a thin cable at ground level, thickening up towards the geosynchronous satellite.

There have been two dominant methods proposed for dealing with the counterweight need: a heavy object, such as a captured asteroid or a space station, positioned past geosynchronous orbit, or extending the cable itself well past geosynchronous orbit. The latter idea has gained more support in recent years due to relative simplicity of the task and the fact that a payload that went to the end of the counterweight-cable would acquire considerable velocity relative to the earth, allowing it to be launched into interplanetary space.

1966 *American engineers found that the strength required for a cylindrical tether would be twice that of any existing material including graphite, quartz, and diamond.*

1975 *An American scientist,* ***Jerome Pearson****, designed a tapered cross section that would be better suited to building the elevator. The completed cable would be thickest at the geosynchronous orbit, where the tension was greatest, and would be narrowest at the tips to reduce the amount of weight per unit area of cress section that any point on the cable would have to bear. He suggested using a counterweight that would be slowly extended out to 114,000 kilometers (almost half the distance to the moon) as the lower section of the elevator was built. Without a large counterweight, the upper portion of the cable would have to be longer than the lower due to the way gravitational and centrifugal forces change with distance from earth. His analysis included disturbances such as the gravitation of the moon, wind and moving payloads up and down the cable. The weight of the material needed to build the elevator would have required thousands of Space Shuttle trips, although part of the material could be transported up the elevator when a minimum strength strand reached the ground or be manufactured in space from asteroidal or lunar ore.*

1977 ***Hans Moravec*** *published an article called "A Non-Synchronous Orbital Skyhook," in which he proposed a modification of the space elevator idea into a more feasible tether propulsion system (Journal of the Astronautical Sciences, Vol. 25, Oct.–Dec. 1977).*

1978 ***Arthur C. Clarke*** *introduced the concept of a space elevator to a broader audience in his novel, The Fountains of Paradise, in which engineers construct a space elevator on top of a mountain peak in the fictional island country of Taprobane (which is actually an early name for Sri Lanka).*

1982 *In* ***Robert A. Heinlein****'s novel Friday the principal character makes use of the "Nairobi Beanstalk" in the course of her travels.*

1999 ***Larry Niven*** *authored the book Rainbow Mars which contained a "Hanging Tree" — an organic 'Skyhook' which was capable of interstellar travel. The book skillfully discussed several merits/demerits of such an approach to the Beanstalk — the primary demerit being that the water necessary to sustain such an enormous 'tree' would require the drying up of all of its host planet's water bodies — which is used as a plot device to explain the drying up of Mars.*

2000 ***Min-Feng Yu*** et al., *publish: "Tensile Loading of Ropes of Single Wall Carbon Nanotubes and their Mechanical Properties," Phys. Rev. Lett. 84, pp. 5552–5555.*

T. Yildirim et al., *publish: "Pressure-induced interlinking of carbon nanotubes," Phys. Rev. B 62, pp. 12648–12651.*

David Smitherman *of NASA / Marshall's Advanced Projects Office has compiled plans for such an elevator that could turn science fiction into reality. His publication, "Space Elevators: An Advanced Earth-Space Infrastructure for the New Millennium" is based on findings from a space infrastructure conference held at the Marshall Space Flight Center in 1999.*

2002 *American scientist,* ***Bradley C. Edwards****, suggests creating a 100,000 km long paper-thin ribbon, which would stand a greater chance of surviving impacts by meteors. The work of Edwards has expanded to cover: the deployment scenario, climber design, power delivery system,* orbital debris *avoidance, anchor system, surviving atomic oxygen, avoiding lightning and hurricanes by locating the anchor in the western equatorial pacific, construction costs, construction schedule, and environmental hazards. Plans are currently being made to complete engineering developments, material development and begin construction of the first elevator. Funding to date has been through a grant from* NASA Institute for Advanced Concepts*. Future funding is sought through NASA, the* United States Department of Defense*, private, and public sources. The largest holdup to Edwards' proposed design is the technological limits of the tether material. His calculations call for a fiber composed of epoxy-bonded* carbon nanotubes *with a minimal tensile strength of 130 GPa (including a safety factor of 2); however, tests in 2000 of individual single-walled carbon nanotubes (SWCNTs), which should be notably stronger than an epoxy-bonded rope, indicated the strongest measured as 52 GPa. Multi-walled carbon nanotubes have been measured with tensile strengths up to 63 GPa.*

1884–1892 CE **Heinrich Rudolf Hertz** (1857–1894, Germany). The last of the great physicists of the 19^{th} century. Discovered electromagnetic waves and opened the way for the development of radio, radar and television. **James Clerk Maxwell** had predicted such waves in 1864.

Hertz used a rapidly oscillating electric spark to produce waves of ultra-high frequency, and showed that these waves induced similar oscillations in a distance wire loop. He measured the velocity of electromagnetic waves and demonstrated that their speed, the transverse nature of their vibrations and their susceptibility to reflection, refraction and polarization are all in complete correspondence with the properties of light waves and infrared radiation. He thus established beyond doubt the electromagnetic nature of light.

Hertz discovered the effect of ultraviolet radiation upon electric discharge and thus laid the foundation to the discovery of the photoelectric effect. In 1892 he experimented with the passage of cathode-rays (electrons) through thin layers of metals. These experiments were crucial for the eventual identification of these rays.

Hertz is remembered today for his experimental discovery of electromagnetic radiation predicted by Maxwell, but he was a gifted theoretician too. Independent of **Heaviside**, he reformulated the 20 original Maxwell equation (in 20 variable), in a compact set, removed the potentials and emphasized the fields $\boldsymbol{E}$ and $\boldsymbol{B}$.

Hertz was born at Hamburg to a Jewish family that had converted to Christianity[605] in 1838. He began to study engineering at Münich in 1877, but soon abandoned it in favor of physical science at Berlin's University. During 1877–1878 Hertz prepared himself for his future work by reading the original works of Laplace and Lagrange and attending the lectures of **G.R. Kirchhoff** and **H. von Helmholtz**. In 1880 he submitted his dissertation and became assistant to Helmholtz in the physical laboratory of the Berlin Institute. In 1883 he became acquainted with Maxwell's theory. He made his discoveries in the Karlsruhe Polytechnic, where he was professor of physics.

In 1889 Hertz was appointed to succeed **R.J. Clausius** as professor of physics at the University of Bonn. He died prematurely, a few weeks short of

[605] Nevertheless, **Gustav Hertz** (1887–1975), a nephew of Heinrich and a physics Nobel Laurete in 1925, was still considered a Jew by the Nazis in 1933. Consequently he was expelled from the Berlin Polytechnical University.

his 37^{th} birthday, from jaw cancer — complicated by blood poisoning caused by surgery.

Standard Time

1884 CE *Worldwide Time Zones were established. The meridian of longitude passing through the Greenwich Observatory in England was chosen as the fiducial point for the world's time zones. [The mean solar time at Greenwich is called the Greenwich Mean Time (GMT) or Greenwich Civil Time (GCT). Astronomers call it Universal Time (UT).]*

The international conference in Washington D.C. (1884) set up 12 *time zones west of Greenwich and* 12 *to its east. These zones divide the world into* 23 *full zones and two half-zones. The* 12^{th} *zone east and the* 12^{th} *zone west are separated by an imaginary line called the International Date Line (IDL). This IDL is halfway around the world from Greenwich. A traveler crossing this line while headed west, toward China, loses a day. If he crosses it traveling eastward, he gains a day. [In his book 'Around the World in Eighty Days",* **Jules Verne** *(1828–1905, France) made use of this fact to dramatize the conclusion of his story.] A few places, such as the polar regions, use GMT but not standard time zones.*

Before the adoption of standard time, each city in the U.S.A. kept the local time of its own meridian. With the growth of railroads, these differences caused difficulties: Railroads that met in the same city *sometimes ran on different times. In* 1883 *the railroads of the United States and Canada adopted a system for standard time.*

In 1918, *the U.S. Congress authorized the establishments of time zones in the United States. Today, nearly all nations keep standard time.*

1884 CE **Paul Gottlieb Nipkow** (1860–1940, Germany). Inventor. Discovered the basic scanning principle of television, in which the light intensities of small portion of an image are successively analyzed and transmitted. This he accomplished through the invention of a rotating disc with one or more spirals of apertures that passed successively across the picture.

Nipkow was born in Lauenburg, Pomerania, and died in Berlin during the second World War.

1884–1885 CE **William Le Baron Jenney** (1832–1907, USA). Architect. Designed Home Insurance Co. Building, Chicago, with type of steel skeleton construction, making it the forerunner of modern *skyscraper*.

1884–1885 CE **John Henry Poynting** (1852–1914, England). Physicist. Defined a vector that quantified the direction and magnitude of energy flow of electromagnetic waves (*Poynting vector*).

Poynting also discovered a theorem that states the conservation of energy for the electromagnetic field[606]. Poynting was a professor of physics at Mason Science College, Birmingham, from 1880 until his death.

1884–1886 CE **Ottmar Mergenthaler** (1854–1899, Germany and USA). Clockmaker and inventor. Invented the Linotype typesetting machine, regarded as the greatest advance in printing since the development of movable type 400 years earlier.

It went through many stages of experimental development and was first successfully used commercially in New York City by *The Tribune* (1886). It gave a great impact to the development of printing.

Mergenthaler was born in Hachtel, Württemberg, Germany and was trained as a watch and clockmaker. He arrived in Baltimore USA (1872) and took a job in a machine shop, eventually working his way up into a partnership.

His device consisted of a keyboard that composed *matrices* (molds) for letters, and then cast an entire line of type at once. He demonstrated the device and patented the Linotype in 1884. Many improvements have been

[606] The time rate of change of electromagnetic energy within a certain volume, plus the energy flowing out through the boundary surfaces of the volume per unit time, is equal to the negative of the work done by the field on the sources within the volume (provided there are no dissipative effects).

made in the design of the machine since then,[607]and more than 1500 separate patents have been taken out in connection with it. The present Blue Streak Linotypes can set type in all sizes from the very smallest to the larger display sizes and in thousands of designs. More than 850 languages and dialects are set on Linotype machines in all parts of the world.

1884–1888 CE **Max Eastman** (1854–1932, U.S.A.). Inventor. Made it possible for millions of people to become amateur photographers. Perfected flexible roll-films and roll holders for winding them. Produced the first lightweight camera.

1884–1903 CE **Gottlob** (**Friedrich Ludwig**) **Frege** (1848–1925, Germany). Mathematician, logician and philosopher.

Played a crucial role in the emergence of modern logic and analytical philosophy. His writings on the philosophy of logic, mathematics and language are of major importance. His logical works mark a break between contemporary approaches and the older Aristotelian tradition. Created the first fully axiomatic system of propositional and first-order logic and also represented the first treatment of higher-order logic. His theory of *meaning*, especially his distinction between the *Sense* (*Sinn*) and *Reference* (*Bedeutung*) of linguistic expressions, was important in semantics and the philosophy of language.

His major works are: *Begriffsschrift* (Concept-Script, 1879); *Funktion und Begriff* ('Function and Concept', 1891); *Über Sinn und Bedeutung* ('on Sense and Reference, 1892); *Grundgesetze der Arithmetic* ('Basic Laws of Arithmetics, 1893–1902).

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[607] The operator sits before the keyboard which resembles that of a typewriter but has 90 keys. He touches a letter key that releases a *matrix* (brass mold) from the magazine (metal case) at the top of the machine. A moving belt carries the matrix to its proper place in the line. Spaces between words are formed by wedge-shaped spacebands, which are automatically inserted when the operator presses a key. When the operator has composed the line of matrices they are transferred to the casting mechanism. Here they are automatically *Justified* (spaced) and molten metal forced into the faces of the matrices. The metal hardens into a *slug* with raised letters into a shallow, sideless tray that is called *galley*. The slugs are used to print books, newspapers, magazines and other kinds of printed materials. The machine immediately and automatically return the matrices to their places in the magazine, where it will be used over and over again.

Frege's lifelong project, of showing that mathematics was reducible to logic, was not successful[608].

Frege's ideas influenced **Dedekind**, **Zermelo**, **Husserl**, **Russell**, **Carnap** and **Wittgenstein**. He in turn was influenced by **Leibniz**, **Boole** (1847), **de Morgan** (1847), **Cantor** (1872), and **C.S. Peirce** (1878). It seems that Frege and **Peano**, working in parallel along the same time-window 1884–1904, had influenced each other in the field of axiomatic arithmetic and symbolic logic. Frege rederived the *Peano Axioms* (governing the natural numbers) from *Hume's Principle.*

Frege was born in the coastal city of Wismar in Northern Germany, and lived there until 1869. He studied at the University of Jena (1869), receiving his Ph.D. there in 1873. He then spent all his working life at that university, rising to a final level of associate professor in 1894. He married (1880) Margaret Liesenburg (1856–1905). They had two children who died young. His work was unfavorably reviewed by his contemporaries and then completely ignored for 20 years. In his own country he long remained an obscure professor of mathematics[609]. He was known to be rather anti-semitic and wanted to see all Jews expelled from Germany. Had he lasted for another decade he would surely become a Nazi sympathizer. This feature in his personality has gravely disappointed some of Frege's intellectual progeny.

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608 In 1903, while the second volume of his *Grundgesetze* was still in Press, Frege received a letter from **Bertrand Russell** which left him 'thunderstruck': in it Russell informed him of a contradiction in his logical system (the '*Russell Paradox*'), Frege never did manage to amend his axioms to his satisfaction. After Frege's death, **Kurt Gödel** showed (1930) in his *incompleteness theorems* that Frege's logistic program was impossible.

609 In her book '*Frege*', Joan Weiner (Oxford University Press 1999, p.3) quotes the diary of Frege, written in 1924:
"One can acknowledge that the Jews are of the highest respectability, and yet regard it as a misfortune that there are so many Jews in Germany and that they have a complete equality of political rights with citizens of Arian descent; but how little is achieved by the wish that the Jews in Germany should lose their political rights, or better yet, vanish from Germany.
If one wanted laws past to remedy this evil, the first question to be answered would be: how can one distinguish Jew from non-Jew for certain? That may have been relatively easy 60 years ago. Now it appears to me to be quite difficult. Perhaps one must be satisfied with fighting the ways of thinking which show up in the activities of the Jews and are so harmful, and to punish exactly their activities with the loss of civil rights, and to make the achievement of civil rights more difficult."

1884–1918 CE **Alexandr Mikhailovich Lyapunov** (1857–1918, Russia). Mathematician and mechanical engineer. Initiated the modern theory of stability of autonomous systems of nonlinear differential equations (both ordinary and partial). By his method, one gains information about the location of the solution in phase-space even without solving the equation. The procedure consists of finding a nonnegative functional of the configuration ("*Lyapunov function*") which has a non-positive time derivative[610]. Then the solution of the differential equation will remain in a region described by the functional and the initial conditions. To date, certain constructive methods are known for obtaining analytic expressions for Lyapunov functionals.

In today's terminology, *Lyapunov's theorem* (1892) asserts that the equilibrium state will be an *attractor* if the Lyapunov function is zero, iff the configurations is at equilibrium and if the time-derivative of this function has a fixed sign opposite to that of the function itself. This function plays an important role in thermodynamic stability theory, since it is identified with the *entropy production function* and is a useful concept even for a system away from thermodynamic equilibrium. Lyapunov's theory is also important in nonlinear dynamics and control theory.

Lyapunov was born in Yaroslavl, a son of the astronomer Mikhail Vasilievich Lyapunov, who worked at Kazan University. Lyapunov's brother

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[610] The concept of the stability of an equilibrium is familiar from elementary mechanics. It is known, for example, that in a system whose mechanical energy is conserved ('conservative system') — an equilibrium position corresponding to a local minimum of the potential energy is a stable equilibrium position (Lagrange, 1788).
This idea was generalized by Lyapunov into a simple but powerful method for studying stability problems in a broader context.
A simple example is provided by the autonomous system

$$\frac{dx}{dt} = f(x,y), \qquad \frac{dy}{dt} = g(x,y)$$

which is assumed to have an isolated critical point at (x^*, y^*), i.e. $f(x^*, y^*) = 0$; $g(x^*, y^*) = 0$.
Let $u = x - x^*$, $v = y - y^*$, (u, v) small. Expanding in Taylor series about (x^*, y^*) we have

$$\dot{u} = \dot{x} = f(x^* + u, y^* + v) = u\frac{\partial f}{\partial x}\Big|_{x^*} + v\frac{\partial f}{\partial y}\Big|_{y^*} + O(u^2, v^2, uv)$$

$$\dot{v} = \dot{y} = g(x^* + u, y^* + v) = u\frac{\partial g}{\partial x}\Big|_{x^*} + v\frac{\partial g}{\partial y}\Big|_{y^*} + O(u^2, v^2, uv),$$

where $\frac{\partial f}{\partial x}\big|_{x^*}$ etc are numbers, *not* functions. Hence the *linearized* system can

be written as

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

or simply $\dot{\boldsymbol{u}} = A\boldsymbol{u}$ where $\boldsymbol{u} = (u, v)$ and

$$A = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}_{(x^*, y^*)} = \textit{Jacobian matrix}$$

at the fixed point (also known as the *stability matrix*).

The general solution of this linearized ODE system is written in terms of the *eigenvalues* (λ_1, λ_2) of A and the corresponding eigenvectors $(\boldsymbol{v}_1, \boldsymbol{v}_2)$ which are solutions of $A\boldsymbol{x} = \lambda\boldsymbol{x}$, namely:

$$\boldsymbol{u}(t) = c_1 e^{\lambda_1 t}\boldsymbol{v}_1 + c_2 e^{\lambda_2 t}\boldsymbol{v}_2$$

under the initial condition $\boldsymbol{x}_0 = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2$. [Note that not every matrix A *has* two independent eigenvectors – e.g. $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ has $\lambda_1 = \lambda_2 = 1$ but only a *single* eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, up to a multiplicative constant. In such a case, the analysis presented here must be slightly revised.]
If $\tau =$ trace of $A = \lambda_1 + \lambda_2$ and $\Delta = \det A = \lambda_1\lambda_2$, one finds

$$\lambda_{1,2} = \frac{1}{2}[\tau \pm \sqrt{\tau^2 - 4\Delta}],$$

where λ_1, λ_2 may or may not be real. Note that the transformation $u = x - x^*$, $v = y - y^*$ has virtually moved the isolated critical point to the origin.
If $\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{x}^*$, $\boldsymbol{x}^*$ is called an *attractor*; this is guaranteed within some neighborhood of $\boldsymbol{x}^*$ (for $\boldsymbol{x}_0$) provided $\operatorname{Re}\lambda_j < 0$, $j = 1, 2$. Assume that we have shifted $\boldsymbol{x}^*$ to $(0, 0)$.
Let $x = x(t)$, $y = y(t)$ be a general solution of the above DE system and let $V[x(t), y(t)] \equiv V(t)$ be a function with continuous first partial derivatives in the neighborhood of the origin, such that $V(0, 0) = 0$. Then V is said to be *positive definite* if $V(x, y) > 0$ for $(x, y) \neq (0, 0)$ in the neighborhood and *negative definite* if $V(x, y) < 0$ for $(x, y) \neq (0, 0)$.
Similarly V is called *positive semidefinite* if $V(x, y) \geq 0$ for $(x, y) \neq (0, 0)$ and *negative semidefinite* if $V(x, y) \leq 0$ for $(x, y) \neq (0, 0)$. Clearly, along trajectory $(x(t), y(t))$ that solves the DE system,

$$\frac{dV}{dt} = \frac{\partial V}{\partial x} f + \frac{\partial V}{\partial y} g.$$

A positive definite function $V(x, y)$ with the property that $\frac{dV}{dt}$ is negative semidefinite is called a *Lyapunov function* of the above DE system.

The *Lyapunov stability theorem* states: If there exist a Lyapunov function $V(x, y)$ for the system $\{\dot{x} = f, \dot{y} = g\}$ in some neighborhood of $(0, 0)$, then the critical point $(0, 0)$ is stable. If V has the additional property that

$$\frac{dV}{dt} = \frac{\partial V}{\partial x} f + \frac{\partial V}{\partial y} g$$

is negative, then the critical point $(0,0)$ is *asymptotically stable.*
Loosely speaking, a critical point is *stable* if all paths that get sufficiently close to the point stay close to the point at all times. Our critical point is said to be asymptotically stable if it is stable *and* there exist a circle $x^2+y^2=r_0^2$ such that every path inside it for some $t=t_0$, approaches the origin as $t\to\infty$.
In the case of a system of n first order ODE

$$\dot{x}_i = f_i(x_1\cdots x_n;t), \qquad i=1,2,\dots,n,$$

we write it in a vector form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}[\boldsymbol{x}(t);t].$$

The system is *autonomous* if explicit time dependence of $\boldsymbol{f}$ is absent. *Any* solution of this system is denoted by $\boldsymbol{x}(t)=\boldsymbol{x}(\boldsymbol{x}_0,t_0;t)$ with $\boldsymbol{x}_0=\boldsymbol{x}(\boldsymbol{x}_0,t_0;t_0)$. A *specific* solution $\boldsymbol{x}^*(t)=\boldsymbol{x}(\boldsymbol{a},t_0;t)$ is said to be *Lyapunov-stable* for $t\geq t_0$ if, for any $\epsilon>0$ there exists $\delta(\epsilon,t_0)>0$ such that for any general solution, $|\boldsymbol{x}(t_0)-\boldsymbol{x}^*(t_0)|=|\boldsymbol{x}_0-\boldsymbol{a}|<\delta$ implies

$$|\boldsymbol{x}(t)-\boldsymbol{x}^*(t)|<\epsilon \qquad \text{for all times } t\geq t_0.$$

In the case of instability, there always exists *some* $\epsilon>0$ and some $\boldsymbol{x}_0$ in an arbitrary small neighborhood of $\boldsymbol{a}$ such that $\boldsymbol{x}(\boldsymbol{x}_0,t_0;t)$ will leave the 'ϵ-tube' for some $t>t_0$ (thus, stability is nothing more than a uniformly continuous dependence on the initial conditions). One and the same DE may have both stable and unstable solutions (linear DE are an exception).
If a solution is stable for $t\geq t_0$ and δ is independent of t_0, the solution is *uniformly stable* for $t\geq t_0$. A solution $\boldsymbol{x}^*(t)$ is *attractive* if there exists $\eta>0$ such that $|\boldsymbol{x}(t_0)-\boldsymbol{x}^*(t_0)|<\eta$ implies $\lim_{t\to\infty}|\boldsymbol{x}(t)-\boldsymbol{x}^*(t)|=0$. If $\boldsymbol{x}^*(t)$ is also stable it is said to be *asymptotically stable.*
The transformation $\boldsymbol{y}(t)=\boldsymbol{x}(t)-\boldsymbol{x}^*(t)$ yields

$$\dot{\boldsymbol{y}} = f[\boldsymbol{y}+\boldsymbol{x}^*(t)]-f[\boldsymbol{x}^*(t)]=g(\boldsymbol{y},t).$$

The solution $\boldsymbol{x}^*(t)$ now corresponds to the trivial solution $\boldsymbol{y}=0$ and the stability of this solution corresponds exactly to that of $\boldsymbol{x}^*(t)$. Thus, it is possible to reduce the concept of stability of a motion to that of a treatment of the special case of the stability of an *equilibrium position.* One may therefore restrict oneself to the treatment of the trivial solutions of $\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x};t)$ i.e. assume that $\boldsymbol{f}[0;t]=0$.
The *Lyapunov stability theorem* for this system then states the following: If there is a positive definite function $v[\boldsymbol{x}(t),t]$ for the system $\dot{\boldsymbol{x}}=f[\boldsymbol{x}(t),t]$ such that for $\boldsymbol{x}(t)=(x_1,\dots,x_n)$

$$\frac{dv}{dt}\equiv\dot{v}[\boldsymbol{x}(t),t]=\frac{\partial v}{\partial t}+\nabla v\cdot\boldsymbol{f}[x(t),t]\leq 0,$$

then the trivial solution is stable.

Sergei was a composer; another brother, Boris, was a specialist in Slavic philology and a member of the Soviet Academy of Sciences.

After graduating from the Gymnasium in Nizhny Novgorod in 1876, Lyapunov enrolled in the Physics and Mathematics Faculty of St. Petersburg University, where **P.L. Chebyshev** greatly influenced him. His master's thesis, suggested to him by Chebyshev, led him to become interested in the stability of ellipsoidal forms of equilibrium of rotating fluids (1884–1885). This, in turn, led to his classical paper on the general problem of the stability of systems having a finite number of degrees of freedom (doctoral thesis, 1892)[611].

In 1885 Lyapunov moved to the University of Kharkov, where he became a professor in 1893. In 1901 he occupied the vacant chair of Chebyshev at the St. Petersburg Academy of Sciences. In a series of papers written between 1903 and 1918, Lyapunov returned to the problem of the figure of equilibrium of rotating nonhomogeneous fluids. He came to the conclusion that 'pear-shaped' figures, that branch off from the *Jacobi ellipsoids*, are unstable. (This instability was confirmed in 1917 by **J. Jeans**, who used Lyapunov's results in his astrophysical models.)

In the summer of 1917 Lyapunov went to Odessa with his wife, who suffered from a serious form of tuberculosis. On the day of her death on 31 October 1918 he shot himself, and died three days later.

Lyapunov and **A.A. Markov**, who had been schoolmates at St. Petersburg University and, later, colleagues at the Academy of Sciences, were Chebyshev's most prominent students, and representatives of the St. Petersburg mathematics school. Both were outstanding mathematicians and both exerted a powerful influence on the subsequent development of science.

1885 CE, Aug. 20 **Ernst Hartwig** (Germany). An astronomer. Observed a new 'star-light' from a supernova explosion near the center of the Andromeda Galaxy that happened ca 2 million years ago.

1885 CE **Gottlieb Wilhelm Daimler** (1834–1900, Germany) and **Carl Benz** (1844–1929, Germany). Engineers. Experimenting separately, they developed successful gasoline engines.

Daimler powered a two-wheeled motorcycle with his engine. Benz installed his engine in a three-wheeled carriage. His vehicle had electric ignition, a water-cooled engine and a differential gear, all of which are still common in

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[611] **Poincaré** tackled similar problems by making wide use of geometrical and topological concepts, while Lyapunov used purely analytical methods. Both works are fundamental to the qualitative theory of ODE.

automobiles today. He also designed a float-type carburetor and a transmission system.

1885 CE **Charles Sanders Peirce** (1839–1914, U.S.A.). Mathematical logician. Introduced the concept of *truth values* of a proposition, the forerunner of later truth tables. Son of Benjamin Pierce. Studied at Harvard. Worked as a physicist and mathematician in the United States Coast and Geodetic Survey (1861–1891). He retired from the USCGS without a pension, to devote himself to writing, and consequently suffered financial hardships during his retirement. After his death, several hundreds of unpublished manuscripts were found.

Pierce worked on the *4-color problem* and problems of knots and linkages. He then extended his father's work on associative algebras and set theory. Invented a map projection using elliptic functions.

1885–1893 CE **William Seward Burroughs** (1855–1898, USA). Inventor. Developed mechanical calculating machine (1885). It could do addition and listing.

He was born in Rochester, N.Y. and began his career as a bank clerk. This made him aware of the need for labor-saving device in accounting. His poor health necessitated a move to a warmer climate and he relocated to St. Louis (1882) where he devoted the next few years to devising an efficient calculating machine. He improved the machine in 1893, including an oil-filled hydraulic governor.

By 1898, the year Burroughs died, more than 1,000 machines had been sold, and by 1926 his company had produced a million machines.

1885–1896 CE **Karl Martin Leonhard Albrecht Kossel** (1853–1927, Germany). Biochemist. Discovered the nucleic acids *adenine* (1885), *thymine* (1894) and the amino acid *hystidine* (1896). Investigated the chemistry of proteins, the cell, and the cell nucleus. One of the first to apply methods of analytical chemistry to examine chemical processes in living tissues. Awarded the Nobel Prize for physiology or medicine (1910). He was a professor at Heidelberg (1901–1924).

His son **Walther Kossel** (1888–1956, Germany) is known for his theory of the physical nature of chemical valence (1916).

1885–1904 CE **Carl Freiherr Auer von Welsbach** (1858–1929, Austria). Chemist and inventor. Discovered (1885) the metallic *rare earth*

elements[612] *neodymium* (Greek: "new twin") and *praseodymium* (Greek: "green twin"). Invented (1898) first metallic filament for incandescent gas lamps, which, for a while, competed successfully with Edison's electric light (1879).

Auer separated the so-called element didymium into neodymium (Nd; $A = 60$) and praseodymium (Pr; $A = 59$). The *ceramic industry* uses salts of neodymium to color glass and in glazes. The metal is present in *mich metal* (1904), an alloy with many uses.

1885–1909 CE **Edward Herbert Thompson**(1860–1935, USA). Explorer and archaeologist. Discovered Yucatan Maya remains at *Chichen-Itza*, including Sacred Well, Great pyramid and astronomical observatory. Unearthed many objects of archaeological significance.

Thompson was born in Worcester, Mass. Was US Consul in Merida, Mexico (1885–1909). On March 04, 1904, Thompson began dredging the Cenote of Sacrifice at the ancient Maya city Chichen-Itza and eventually substantiated legends[613] describing this natural, water-filled, limestone well as a repository for the precious objects and human victims offered to the gods by the ancient Maya (ca. 600 AD).

1886 CE The Woods Hole *Biological Station* was established (on Cape Cod, Mass., U.S.A.). Out of it emerged (1930) the oceanographic institution for research and study of marine science.

1886 CE **Albert Ladenburg** (1842–1911, Germany). Chemist. Made the first laboratory synthesis of a natural alkaloid, *coniine*[614]. Coniine is the toxic component of *hemlock*, the poison that ended the life of **Socrates** in 399 BCE.

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612 A group of 14 elements with atomic numbers $A = 58–71$. The name *rare earth* is a misnomer, since they are neither rare nor earths. Rare earths have 3 electrons in the outer shells of their atoms that take part in valence bonding. Because of this property, all rare earths have similar properties in water solutions, and all can exist in the 3-valent state. In nature they are always found in the form of phosphates, carbonates, fluorides and silicates.
The rare earths have many scientific and industrial uses. Tiny amounts of separated rare earths are used in *lasers*.

613 According to the book *Relacion de les Cosas de Yucatan* (1566) by Fray **Diego de Landa** (the bishop of Yucatan) and another book by Don **Diego Sarmiento de Figueroa** (1579): in times of drought and disaster, beautiful maidens were cast into the deep and muddy well as offerings to appease the god of rain. The well was some 6 m across and 30 m deep.

614 $C_8H_{17}N$: one of the simplest alkaloids; found in the plant *Conium maculatum* which was known to the ancient Hebrews as *Rosh* [*Deut* **29**, 17; **32**, 33; *Jer* **8**,

1886 CE An economical way was discovered to make *aluminum* from abundant *alumina* and electric power.

14; **9**, 14; **23**, 15; *Psalms* **69**, 22; *Lament* **3**, 19; **3**, 4; *Hoshea* **10**, 4; *Amos* **6**, 12]. Described also by **Theophrastos** (320 BCE), **Dioscorides** (70 CE) and **Pliny** (75 CE).

Aluminum

Aluminum is the most abundant metal in the earth's crust (about 7.3 percent), and is the third most abundant element. It occurs as silicates in almost all crystalline rocks[615]*, clay and slate*[616]*.*

Alum,[617]*from which the element takes its name, was known to the Greeks and Romans.*

The German chemist **Andreas Marggraf** *(1709–1782) was first able (1754) to isolate alumina (Al_2O_3) from clay.* **Humphry Davy** *(1809), isolated the impure metal, which he called* aluminum. *A purer metal was obtained by* **H.C. Oersted** *(1824) by heating Aluminum chloride with potassium.*

Although aluminum currently costs less than $2.00 per kg, it was considered the most valuable metal in 1827 (16 million dollars for one kilogram!). Indeed, it was so cherished by royalty in the early to mid 1800s that they alone ate with aluminum spoons and forks while their lower class guests dined with cheaper gold and silver service. Why was it originally so expensive?

Aluminum was first prepared by **Friedrich Wöhler** *(1827), using the following reaction*

$$AlCl_3 + 3K \rightarrow Al + 3KCl,$$

where potassium was obtained by passing an electric current (from a voltaic cell) through molten KCl. But copper and zinc used in voltaic cells were expensive in the early 1800's and in addition the great cost of energy required

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615 *Feldspar* [$KAlSi_3O_8$, or K_2O, Al_2O_3, $6SiO_2$] is a constituent of primary rocks such as granite, and by the disintegration of these rocks, either by simple hydrolysis or by combined action of moisture and atmospheric CO_2, soluble alkali salts and insoluble aluminum silicate (clay) pass into the soil; *Cryolite* (Na_3AlF_6); *Hornblende*; *Tourmaline*; *Augite* and Micas.

616 Common clay is a mixture of *kaolin* [Al_2O_3, $2SiO_2$, $2H_2O$] with limestone, quartz, and oxide of iron. The oxide *alumina*, Al_2O_3 is found either anhydrous as *corundum*, or hydrated as *diaspore* [Al_2O_3, H_2O], *gibbsite* [Al_2O_3, $3H_2O$] and *bauxite* [an ore which is a mixture of the minerals $AlHO_2$, $Al(OH)_3$. It contains also some iron oxide, titanium oxide and other impurities]. *Slate* is clay hardened and laminated by pressure.

617 $KAl(SO_4)_2{\cdot}12H_2O$; a mixture of clay and limestone constitutes *marl*, whilst a mixture of clay and sand is called *loam*. *Bauxite* was discovered (1821) by the mineralogist **P. Berthier** near Les Baux in Provence, France.

to melt large quantities of KCl was prohibitive. Thus, it was impractical to produce aluminum by passing an electric current through molten Al_2O_3 because it has a high melting point of ca 2000°C. This high temperature is difficult to achieve and maintain, and even so, the components of most voltaic cells melt below this temperature (zinc at 420°C and copper at 1083°C).

The cost of aluminum began to drop in the late 1800s as a result of two major advances:

- *The invention of the commercial direct current electric generator (1869) by the Belgian inventor* ***Zénube Théophile Gramme****, which could produce electricity by steam or water. Although this mode of production of electricity was much less costly than electricity generated by voltaic cells, aluminum still cost more than $200,000 a kg.*
- *Chemists discovered[618] (1886) the electrolytic method of producing aluminum: they could lower the melting point of aluminum oxides by mixing it with salts such as cryolyte.*

Since 1886, the price of aluminum has decreased markedly because of lower electrical costs, improved production techniques, and recycling of discarded aluminum products.

Aluminum is not only the most abundant metal but also one of the most useful because of its unique combination of properties: low density (2.699$\frac{\mathrm{g}}{\mathrm{cm}^3}$ at 20°C), high resistance to corrosion (by forming a very thin protective layer of Al_2O_3), good thermal and electrical conductivity, attractive luster, and lack of toxicity. Its uses range from electric transmission lines to kitchen foil and cooking utensils. Alloys with magnesium, manganese, and copper have high mechanical strength and are easily machined, so that they have come into widespread use in the construction of buildings, automobiles, airplanes, and ships.

Nowadays, aluminum is of inestimable value in energy conservation: Around homes one finds storm doors and windows, insulation backed with aluminum foil, and aluminum siding, Because vehicle weight significantly affects gas mileage, substituting aluminum for heavier metals in cars, trucks, trains, and aircraft helps preserve petroleum supplies. Aluminum thus helps

[618] **Charles Martin Hall** (1863–1914, USA) and independently, in the same year, by **Paul Louis Toussaint Heroult** (1863–1914, France): A carbon-lined iron box, which serves as a cathode, contains the electrolyte, which is the molten mineral *cryolite* (Na_3AlF_6) in which aluminum oxide Al_2O_3 is dissolved. The aluminum oxide is obtained from the ore *bauxite*. To make a ton of aluminum in this way requires about 20,000 KW-hours of electricity.

conserve energy and improve standard of living at the same time. The largest producers are currently the United States (ca 2 million metric tons in 1990), Russia and Canada.

Alkaloids — Elixirs of Life and Death

Primitive man had no knowledge of the cause of his physical ailments, nor did he have effective means of alleviating his suffering. His life span was relatively short. The slightest injury or the smallest infection frequently brought him great suffering and death. Little did he know that healing and pain-killing medicinal substances lay within arm's length.

Primitive man attributed his ills so "evil spirits" that had invaded and taken control of his body. He sought the services of the tribal medicine man, who he believed was endowed with supernatural powers. The medicine man had many magical ways of "healing" the members of his tribe. Often he danced and chanted around his patient's prostrate body, shaking rattles filled with animal teeth and small bones. Sometimes the patient was instructed to kill a small animal and wear a string of its teeth around his neck as a "charm". Or the "witch doctor" might prepare a foul-tasting concoction of toad's eyes, the dried blood of a bird, and a pulverized bone of a deceased enemy soaked in the urine of a newborn baby. Then he would administer the concoction in various ways; he might have the patient drink it, or he might rub it on the patient's body.

However, as man roamed the land in search of anything edible, he tasted roots, leaves, stalks, and tree barks — almost anything he thought might provide nourishment. Through trial and error, he undoubtedly encountered many plants with both beneficial and malevolent properties. Many of these plants contained alkaloids *that are still valuable in modern medicine. Other plants had a negative effect on him. For example, the heady fragrance and delicacy of the lily of the valley with its white bell-shaped flowers belies its*

toxic properties. The plant can cause severe skin lesions in a susceptible individual. However, modern physicians have found beneficial uses for this plant as well. It contains a substance that sometimes is incorporated into diuretics *(fluid reducers) and* cardiac *(heart) tonics.*

Many decorative garden plants that we take for granted, such as the rhododendron, also have toxic properties. Children who have sucked the nectar from the colorful flowers have suffered severe shock-like symptoms. At one time rhododendrons were used as an insecticide because of their toxicity. Daffodils, eucalyptus, oleander, azaleas, hyacinth, poinsettia, and bleeding heart are other common plants that can be deadly if eaten. Undoubtedly early man tasted them all in his quest for food.

Thus, the curative, narcotic, hallucinogenic and poisonous effects of certain plant extracts were known already in prehistorical times: The use of psychoactive drugs is very ancient. The peoples of India (ca 1000 BCE) were using a potent psychoactive drug called soma *(possibly derived from mushrooms), and Herodotos records the Scythians inhaling the smoke from burning hemp seeds.*

The linkage between chemistry and the art of healing also goes back to ancient times. Recipe books or antidotaries *of various mixtures believed to have curative powers were known in medieval Europe. The Arabs preserved an essentially unbroken contact with ancient medical science; portions of the works of* ***Hippocrates*** *and other early medical practitioners such as* ***Dioscorides*** *and* ***Galen*** *had been translated into Arabic by the* 9^{th} *century. The first Arabic pharmacopoeia was brought to Europe in the* 11^{th} *century.*

Paracelsus *(1493–1541), who was part charlatan and part scientist, played an important role in furthering the medical applications of chemistry and in urging a search for new drugs.*

Poisons play an important part in history. Every historical figure in power lived with the constant fear that his life could be abruptly ended with his next cup of wine or morsel of food. The Duchess of Ferrara, the infamous ***Lucretia Borgia*** *of Italy (1480–1519), was notorious for her use of herb poisons, which she carried concealed in hinged finger rings. The Duchess disposed of an untold number of persons who threatened the power of her family, or persons whom she considered menacing.*

Alchemists were often members of the royal courts. One of their functions was to prepare special poisonous potions, as well as to prepare antidotes for them.

In addition to making protein, carbohydrates, fats, and other compounds familiar to most people, plants synthesize a huge array of substances that

are usually found in relatively low quantities. The general name of these substances is secondary plant products*, which gives no idea of the chemical range of the materials or any idea of their importance. The medicine products, the natural dyestuffs, products for the chemical industry (gums, resins, etc.), and a wide variety of common, everyday substances used as flavorings and essential oils (perfumes, peppermint) are secondary plant products.*

Among the secondary plant products that have long played important roles in human life are the alkaloids*. They can be extracted from the roots, leaves and seeds of certain plants (Table 4.8) and possess marked physiological activity and are also valuable curative agents. The word alkaloid is a purely chemical word defined as an organic chemical molecule — one containing carbon atoms — which also contains at least one atom of nitrogen. All can be crystallized and, when dissolved in water or in alcohol, they give an alkaline reaction to the solution. The nitrogen in an alkaloid is usually found in combination with a ring of carbon atoms, the so-called* heterocyclic ring*.*

Other common replacement for carbon in these structures are oxygen and sulfur. These compounds are contrasted to carbocyclic *compounds, which contain only carbon in the ring. When an alkaloid is mixed with an acid, such as hydrochloric, the very water-soluble hydrochloride is formed; this is the usual way of getting alkaloids into simple solutions. In solutions or as the crystal, they are colorless, are most soluble in alkaline solutions such as weak sodium hydroxide, and almost invariably have a bitter taste. The flavor of tonic water is due to quinine, one of the alkaloids.*

There are today about 300 *known natural alkaloids and their number increases with the growth of biochemical research of plants. Among the types in common use are the sedatives, and analgesics, the narcotics, the stimulants, the antidepressants, the tranquilizers, and the hallucinogens. Some of these alkaloids are listed in Table 4.12.*

The exploration of the chemical nature of alkaloids has been one of the boldest and most difficult challenges of analytic organic chemistry, occupying the minds of the greatest chemists over the past two centuries. The 19^{th} *century in particular, saw an accelerated growth in the discovery and understanding of drugs.*

A good deal of study of the specific effects of various drugs and of the relationship between chemistry and physiology took place. An interest in the synthesis *of drugs arose, as opposed to simply isolating them from naturally occurring materials.*

Table 4.12: Common Alkaloids and their properties

Alkaloid	Source	Activity or Application
Aconitine	Aconite root (Wolfsbane) (*Aconitum napellus*)	Poison, sedative, useful in all febrile and inflammatory diseases
Atropine	*Atropa belladonna*	Mydriatic, antispasmodic
Caffeine	Coffee (*Caffea arabica*)	Stimulant, diuretic
Cinchonine	Bark of the Quina tree (*Cinchona officinalis*)	Antipyretic
Cocaine	Coca leaf (*Erythroxylom coca*)	Local anesthetic
Codeine	Opium poppy (*Papaver somniferum*)	Cough control, analgesic
Colchicine	(*Colchicum autumnale*)	Anti-cancer teraphy; treatment of gouty-arthritis
Coniine	Hemlock (plant) (*Conium maculatum*)	Poison
Digoxigenin	Purple Foxglove (plant) (*Digitalis purpurea*)	*Digitalis* for treatment of cardiac insufficiency
Ephedrine	Ma huang (*Ephedra vulgaris*)	Decongestant (respiratory ailments); Mydriatic
Epinephrine	Body adrenal medulla	Hormone controls metabolism, and adjustment to stress
Ergonovine	Sclerotia of rye grain fungus (*Claviceps purpurea*)	Ergotism
Ergotamine	Sclerotia of rye grain fungus (*Claviceps purpurea*)	Ergotism
Etoposide	Mandrake (plant) (*Mandragora officinarum*)	Analgesic, soporific, chemotherapy
Himbacine	(*Galbulimima belgraviana*)	Hallucinogenic
Hyoscyamine	Henbane (*Hyoscyamus niger*)	Hallucinogenic, narcotic, mydriatic

Table 4.12: (Cont.)

Alkaloid	Source	Activity or Application
Lobeline	Indian tobacco (*Lobelia inflate*)	
LSD	Sclerotia of rye grain fungus (*Claviceps purpurea*)	Hallucinogenic
Mescaline	Peyote cactus (*Lophophora williamsii*)	Hallucinogenic
Morphine	Opium poppy (*Papaver somniferum*)	Cough control, analgesic
Nicotine	Tobacco (*Nicotiana tabacum*)	Insecticide
Papaverine	Opium poppy (*Papaver somniferum*)	Cough control, analgesic
Piperine	Pepper (*Piper nigrum*)	Spice, seasoning
Psilocybin	Mushroom fungus (Soma) (*Stropharia cubensis* and *Psilocybe mexicana*, *Amanita muscaria*)	Hallucinogenic
Quinine	Bark of the Quinea tree	Antimalarial
Reserpine	Indian snake root (*Rauwolfia serpentina*)	Tranquilizer, sedative
Scopolamine	Jimson weed (*Datura stramonium*)	Sedative, hypnotic, mydriatic soporific, depressant
Serotonin	Body blood platelets, mid brain and enterochromaffine cells	Hormone; prevent bleeding
Strychnine	*Nux Vomica* (plant)	Poison, tonic
Taxol	Bark of yew tree	Ovary and lung cancer
THC	Hemp (Hashish, Marijuana) (*Cannabis Sativa*)	Hallucinogenic, fiber
Thebaine	Opium poppy (*Papaver somniferum*)	Cough control, analgesic

Table 4.12: (Cont.)

Alkaloid	Source	Activity or Application
Tryptamine	Yakee, Yato (*Virala calophylla*)	Hallucinogenic
Tubocurarine	Curare (*Chondodendron*)	Poison, treatment of tetanus and hydrophobia
Valium (diazepam)	Synthetic	Minor tranquilizer
Vinblastine	Periwinkle plant	Chemotherapy
Vincristine	Periwinkle plant	Chemotherapy
Vindesine	Periwinkle plant	Chemotherapy

Medical vocabulary:

Analgesic: pain-reducing without impairing consciousness.

Anesthetic: capable of producing loss of bodily sensations with or without loss of consciousness; used in surgery. Whereas *general* anesthetics produced a state of coma, *local* anesthetics work by depressing sensory endings or blocking the conduction of impulses through the nerves.

Antipyretic: reducing fever.

Antispasmodic: preventing or curing spasms.

Antitoxin: a serum serving to neutralize a toxin.

Aphrodisiac: exciting the sexual organs.

Barbiturate: derivative of barbituric acid, used especially as sedative or hypnotic. Affects all levels of the central nervous system. Can be addictive.

Carminative: easing gripping pains and expelling flatulence.

Decongestant: a drug that reduces excessive circulation in an organ by constricting blood vessels; usually taken to drain nasal passages and alleviate cold symptoms.

Depressant: an agent that reduces activity of bodily function.

Diuretic: fluid-reducing.

Emetic: cause vomiting.

Emollient: having softening and soothing effect.

Hallucinogenic: having the capability of the perception of objects or the experiencing of feelings that have no cause outside one's mind; caused especially as a result of mental disease or effects of a drug. It has been suggested that hallucinogens permit people to enter the "real" world, closed off from childhood by the many layers of culture that surround us from birth.

Medical vocabulary (cont.)

Haemostatic: drugs used to control bleeding.

Hypnotic: (= *Soporific*) sleep-inducing agent.

Mydriatic: a drug that produces dilation of the pupils.

Narcotic: an addictive drug that in moderate doses blunts the senses, relieves pain and induces sleep. In excessive doses causes stupor, coma or convulsions.

Sedative: tending to calm or relieve tension and irritability. Both sedative and hypnotic drugs depress the higher brain centers, decreasing excitement and activity.

Stimulant: an agent that temporarily increases the functional activity or efficiency of a tissue or an organ. Energy producing.

Sudorific: producing copious perspiration.

Therapy: mode of medical healing. Perhaps from the Hebrew *trufa* = medicine; may also be linked to *teraph* = ancient Hebrew household god.

Tonic: substance producing a feeling of well-being.

Tranquilizer: drug used to reduce mental disturbance such as anxiety or tension.

The highly competitive nature of our culture often necessitates the quick lunch, the fast freeway, the "urgent" telephone call — in general, a "burning the candle at both ends" way of life. This fast pace tends to play havoc on many people's "nerves". Consequently some people use tranquilizers to relieve anxiety and tension.

Physicians' offices are lined daily with victims of modern life who find that relaxation is difficult to obtain. Patients complain of many symptoms, from queasy stomach to "sledgehammer" headache. Some fear that they are developing major diseases because their tensions "translate" into symptoms of actual diseases.

Vulnerary: used in healing wounds.

Table 4.13: MILESTONES OF PROGRESS OF ALKALOID RESEARCH

1776	***William Withering*** *recognized the importance of digitalis in the treatment of heart and kidney diseases.*
1805	***Friedrich Sertürner*** *extracted morphine from opium and used it to relieve pain.* ***Carl Gauss*** *used morphine to relieve pain of mothers in difficult child-birth.*
1816	***Pierre Pelletier*** *and* ***Joseph Caventou*** *isolated strychnine and quinine.*
1817	*The name alkaloid coined by the pharmacist* ***W. Meissner****.*
1818–1840	*Discovery of caffeine, atropine, codeine, curarine and other important alkaloids.* ***Gerhardt****,* ***Regnault****,* ***Laurent****,* ***Andrews*** *and* ***Berzelius*** *developed new methods for the investigation of alkaloid structure.* ***Liebig****,* ***Würtz*** *and* ***Hoffmann*** *(1848) considered alkaloids as acids of amonia in which atoms of hydrogen were replaced by organic radicals.*
1886	***Albert Ladenburg*** *synthesized coniine — the first alkaloid to be synthesized in the laboratory.*
1905	***Robert Willstätter*** *discovered the chemical structure of many alkaloids and synthesized some of them (atrophine, cocaine).*
1925	***Robert Robinson*** *discovered the chemical structure of morphine and other alkaloids. Explained the formation of alkaloids from condensation of amonia, formaldehyd and amino-acids. Suggested that plant-alkaloids are end waste-products of their metabolic chain.*
1938	***Hoffmann*** *and* ***Stoll*** *discovered LSD.* ***Dustin*** *discovered that colchicine was cytotoxic (blocking cell division).*

Table 4.13: (Cont.)

1943	**Hoffmann** *discovered that LSD is hallucinogenic.*
1944–1956	**Robert Woodward** *synthesized quinine (1944), strychnine (1947), lysergic acid (1954)* and *reserpine (1956).*
1965	*First synthesis of the active hallucinogen THC.*

Additional historical, folkloristic and scientific data concerning alkaloids is given below.

I. THE FOUR GENERA: *Atropa, Datura, Hyoscyamus* AND *Mandragora* BELONG TO THE *tomato* FAMILY (SOLANACEAE) AND EACH CONTAINS ONE OR MORE OF THE TROPANE ALKALOIDS: ATROPINE, SCOPOLAMINE AND HYOSCYAMINE.

Atropine is a stimulant of the central nervous system and depressant of the parasympathetic nervous system.

In minute quantities, atropine is used as an antidote to other poisons; in moderate doses it causes loss of motor coordination; in higher concentrations it leads to hallucinations, delirium, stupor; in large quantities it is a deadly poison.

Greece and Rome knew it as a sedative and an hallucinogen. Bacchanalian orgies utilized new wine spiked with small amounts of the sap. With the spread of Christianity, bacchanalian orgies became a lamented aspect of the golden age of Rome, but belladonna's star rose again as witchcraft and demonology captured people's imagination.

The name belladona *dates from the late Middle Ages: Italian ladies put drops of diluted nightshade sap in their eyes to induce* mydriasis *— the deep, dark mysterious look caused by dilated pupils. Indeed, since atropine blocks the normal transmission of signals across synaptic junctions between nerves, opthalmologists used it to prevent the autonomous closing of the pupil in bright light.*

At one time the jimsonweed, which contains the alkaloid scopolamine, was used in childbirth to alleviate pain. However, it is extremely toxic, and it

often resulted in coma and death for those women who ingested it. Today, scopolamine is incorporated into some motion-sickness remedies. It is also used in the treatment of some symptoms of Parkinson's disease.

White henbane, another herb with poisonous properties, contains the alkaloids hyoscyamine, hyoscine, and atropine. Ancient men found this herb to be useful in warfare. For example, two enemy camps might hold a "truce" celebration. The conquered army would serve wine laced with this poison to the invading army as a "goodwill" gesture, and the two armies would exchange toasts. The conquered became the conquerors after the first cup of wine. In modern times this drug has been used for treating mercury poisoning and morphine addiciton. In small amounts it produces sleep; in larger amounts it produces death.

Scopolamine as a truth serum *may or may not still be used, depending on whose national intelligence agency is being asked.*

The third tropane alkaloid, hyoscyamine*, is similar in action to atropine, but the clinical responses are sufficiently different to suggest that the receptor sites are not the same. The amounts used medically are very small; practitioners know that even slightly higher dosages, particularly when administered internally, have grave consequences and may lead to death.*

Mandrake, known in biblical times [Gen **30**, 14–17; Cant **7**, 14] also contains large quantities of the alkaloids hyoscyamine and scopolamine, which are capable of causing prolonged stupor and alleviating severe pain. Relatives of victims being crucified brought sponges soaked with a solution of mandrake and other herbs to help numb the victim's pain. This was a merciful release from the tortures of the slow death of crucifixion. It is believed that Christ was administered this drug by his disciples to help relieve his agony in his last hours.*

Long before the Hebrews, the mandrake had been associated with sexuality and sins of the flesh. The Ebers medical papyrus of 1500 BCE *listed it as* dudajm*, the fruit that excites love; Pharaoh Tutankhamen was buried with* 11 *mandrake roots in the sixth row of his floral collarette to ensure his potency in the next world. The Greeks named it* circeium *after Circe who, Homer reported, lured men to her and changed them into swine, that is, into sexual pigs. They referred to Aphrodite as* Dios Mandragoritis*. Mandrake roots, carved into big-hipped and -breasted fertility figurines, have been unearthed at Antioch and Damascus and from tombs in Constantinople and Mersina.*

Further evidence for the medicinal use of mandrake is found in the writings of ***Hippocrates***, ***Plato***, ***Pliny***, ***Theophrastos***, ***Galen*** *and Dioscorides. Even as late as the* 13^{th} *century, a mixture of opium, and mandrake juice compounded in vinegar was taken up in sponges and inserted into the nostrils of patients undergoing surgery, at the Bologna medical school.*

Datura (Jimsonweeds) contains the three tropane alkaloids mentioned above. It has been used ritually in India as far back as records have been kept. It is also known to be used in puberty rites of passage in South and Central America and from there it diffused into the British colonies in North America.

Datura was the original knockout drops, and thieves in India and in Europe used it for centuries. Up until at least the beginning of this century, juice of datura leaves was added to milk given young Indian girls who were to be initiated into prostitution. The drink was narcotic and, it was asserted, aphrodisiac, so that the victim was believed to have actively contributed to her own downfall. The Chinese, believing that datura was a sexual stimulant, administered it to brides on their wedding night to calm their nerves and to make them more sexually receptive.

The European attitude towards datura parallels that in the Far East. Apollo's priests drank datura to achieve sedated, prophetic, and oracular states. The sacerdotal plant of Delphi was undoubtedly datura; the mumbling speech, trance states, and known fears of over-dosage are consistent with datura intoxication. Greek physicians knew it as nuxmetal *or* neura*, a reference to its sedative action, and when extended unconsciousness was desirable, as during surgery, datura was mixed with opium.*

Rome followed Athens' lead in ritual and in medicine and added the drug-induced orgy in which datura mixed with wine was used to induce hallucinogenic states and to heighten sexual activity. Avicenna, a tenth-century Arabian physician, recommended datura *not only for surgery, but as an excellent treatment of anxiety.*

From Arabia, the medical and aphrodisiacal use of datura spread to Spain and to Western Europe; Northern Europe was too cold to support the growth of datura with high tropane content.

Datura was an important ingredient in the poisonings that pervaded Southern Europe from 1400 to 1700. Nobles and merchants sent members of their family to schools teaching the art of poisoning for much the same reasons as we send our children to graduate schools of business administration. No love potion worth paying good money for was without datura, usually supplemented with extracts of other solanaceous plants and, for good measure, attar of roses, marjoram, other herbs, and a newt's tongue. Witches' Sabbats, the infamous black masses that so intrigued prurient Victorians, utilized ointments and unguents containing datura. Whole leaves were inserted into the rectum or vagina where tropanes are quickly absorbed; the broomstick, developed as a symbol of this part of the ritual probably because of the "flying" hallucination experienced by the users of the drug.

Cocaine, like atropine, belongs to the pyrrolidine *family of alkaloids. It is found in the leaves of the South American coca bush, known to the natives before the discovery of America.*

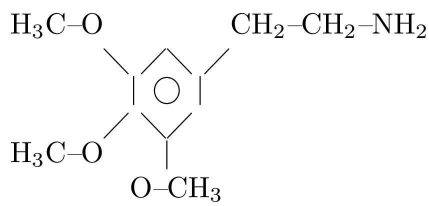
Cocaine acts as a stimulant to the central nervous system. It produces euphoria and insensitivity to pain and is a potent antifatigue agent. Although cocaine has been used as a local anesthetic, its toxic properties have caused a decline in its medicinal use. Many of its derivatives such as Novocaine and Lidocaine are widely used in dentistry and plastic surgery and in nerve blocks to reduced severe pain.

II. The alkaloid family of *isoquinoline* includes *quinine, mescaline, chinchonine* and the opiates — *morphine, codein, thebaine* and others.

Quinine is obtained from the bark of the chinchona tree, found primarily in the rain-forests on the eastern slopes of the Andes of Peru, Bolivia and Columbia, at heights of 1000–8000 *m. The name of the genera was coined by Linnaeus (1628) in honor of Count Chinchona, viceroy of Peru. Natives of Peru used it to cure fevers. After Lima was founded (1520), this became known to the Spaniards who carried the curative bark to Europe, whereby* 1640*, it was widely used as an antimalarial.*

During WWII, when Japanese overrun the plantations of south-east Asia, Quinine shortage urged American and British chemists to develop synthetic antimalarial substitutes.

Mescaline is a hallucinogenic alkaloid. It causes hallucinations, sense distortions, elevated blood pressure, and profuse sweating, although it is about 7500 *times less potent than LSD. Mescaline was obtained from the small cactus plant, the peyote, by the Indians in Mexico as early as the* 16^{th} *century for use in their religious ceremonies.*



Mescaline

Peyote grows in the Rio Grande River area. It has buttons or tufts, which are from a small cactus plant. They are dried and sometimes crushed and brewed into a beverage. The Aztec Indians used the brew during their religious festivals, and the early American Indians used it as a hallucinogen to "communicate" with their "divine spirits". The drug reached the North-American Apache Indians (1870), who adopted it as a cult object.

The botanical family Papaveraceae contains 25 genera and 120 species of flowering plants. Most are herbaceous annuals, although a few die back to the ground each year and form new shoots from a perennial rootstock. The family originated in Asia Minor. The genus *Papaver* contains ten species, several of which, the Iceland poppy (*P. nudicaule*) and the oriental poppy (*P. orientalis*), are common garden plants. The California poppy (*Eschscholzia*), well known to most gardeners, is a member of another genus in the family. None of these plants produce alkaloids of medical interest. The one that does is the opium poppy (*P. somniferum*) whose specific name was chosen by Linneaus because of the sleep-inducing properties of the gum produced in the young seed capsule of the plant.

The opium poppy can be grown in many parts of the world where the growing season is sufficiently long with warm, sunny weather. Because of the need for cheap labor, it is presently grown in relatively few countries of the world. India and Pakistan each produce about 100 metric tons of opium each year, most of it consumed locally, Afghanistan produces the same amount, but the crop is smuggled into other countries of the Middle East. Turkey has traditionally been the major supplier of legal (medicinal) opium for the West, with between 60 and 100 metric tons produced each year. Mexico has only recently become an opium-producing country, although its production is still under ten metric tons. Less is known about production in the area of southeast Asia called the Golden Triangle, an area embracing parts of Burma, Laos, and Cambodia, but it may well be the greatest producer. Report on production vary from 250 to over 500 metric tons per year. The world's production of raw opium is close to 1000 metric tons per year, of which less than 250 tons enters legal medical channels; the United States processes about 150 metric tons to isolate morphine and codeine.

Of the alkaloids found in raw opium, three are of medical importance. Close to 11 percent of opium is the single alkaloid morphine, while codeine constitutes about 2 percent and thebaine a bit less than 1 percent of the weight of the opium gum.

Morphine ($C_{17}H_{19}O_3N$) is the principal component of *opium* [from the Greek word *opion*, for "poppy juice"; mentioned in the Jerusalem Talmud: *Avoda Zara*, page 40, side 4], which is obtained as the milky juice that exudes from unripe poppy seed capsules.

Morphine acts on the central nervous system and can induce addiction. The specific action of morphine appears to be related to the ability of the molecule to fit into and block specific receptor site on a nerve cell; the benzene group of the morphine molecule fits snugly against the flat part of the protein that acts as a receptor site, and the neighboring group of carbon atoms is at the correct distance and orientation to fit into the groove. Beyond the groove is a group with a negative charge, which can attract the positive charge of the nitrogen atom. By fitting the shape of the receptor and binding to it, the incoming morphine molecule eliminates its action. In this respect the molecule mimics the body's natural pain killers, the enkephalins.

Raw opium has used medically for centuries. Summerian tablets of 2500 *BCE noted that when small balls of opium were eaten or taken after mixing with wine, the drug induced sleep and relieved pain. Homer spoke of nepenthe, a substance which will "lull pain and bring forgetfulness of sorrow". Hippocrates, Theophrastos, Pliny, Dioscorides, and other ancient medical writers recommended opium, and it was the most used analgesic up to the* 20^{th} *century.*

In the later half of the 19^{th} *century, pharmaceutical chemists started altering the morphine molecule in order to make a compound which would be more effective than the natural alkaloid and less addictive. They came up with heroin (more "heroic" than morphine), in which morphine's hydrogen atoms of two –OH groups have been replaced by acetyl groups* ($–CO–CH_3$)*. This replacement made heroin more soluble among the hydrocarbon chains of fats and less soluble in water. When injected directly into the blood, it passes more rapidly through the blood-brain barrier, the barrier that prevents water-soluble and large molecules from passing between the two. As a result, it is more potent than morphine, but its effect does not last as long. Once heroin is absorbed into the body, the acetyl groups are removed, forming morphine, which provides its analgesic and euphoric action.*

III. The *piperidine* alkaloid group; (includes *coniine* and *nicotine*).

Nicotine, along with about ten other alkaloids, is found in tobacco. When it is taken into the body through smoking, nicotine increases the blood pressure and pulse rate and constricts the blood vessels. Nicotine contains both a piperidine ring and a pyrrolidine ring. In concentrated form, it is extremely toxic and is therefore often used as an insecticide.

IV. THE *ergot alkaloid* GROUP; includes *lysergic acid* (*LSD*), *strychnine* AND *THC* (DELTA-TETRA-HYDRO-CANNABINOL) WHICH IS THE ACTIVE ELEMENT IN THE RESINOUS EXUDATE OF THE HEMP PLANT (CANNABIS).

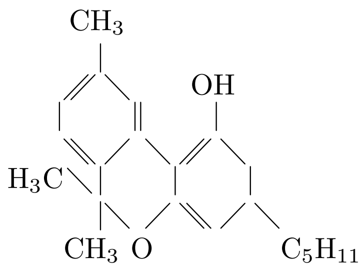
Nux Vomica, a strychnine-type drug, was once widely used to kill rats whose parasitic fleas were responsible for spreading plague diseases, such as the bubonic plague that killed millions of people in Europe for three centuries. These fleas also spread typhus, another deadly disease. Other strychnine-type drugs were used to eradicate unwanted spouses, competing heirs, meddling relatives, and so on. These unfortunate people were often served their last meal with a "seasoning" of this drug.

Although the use of cannabis *as a fiber and food crop dominated its production in the Orient for a long time, its medical uses were known and exploited. In a medical book ascribed to the legendary Emperor Shen-Nung in 2737 BCE (but probably written in the Han dynasty about 100 BCE) , ground, dried leaves of cannabis, known as* ma-yo*, were recommended for malaria, beriberi, constipation, as an anesthetic in surgery, and for that disease of old age — absent mindedness. Shen-Nung noted that its primary medical value was in calming hysterical women. In India and throughout southeast Asia, cannabis was also used medically, and its capacity to ease anxiety was noted in both Hindu and, later, in Buddhist writings, where it was referred to as the "soother of grief".*

It seems that cannabis entered India about 1500 *BCE, where it was first formally exploited as an hallucinogen. Knowledge of its hallucinogenic properties spread from India throughout Asia and Asia Minor about* 500–1000 *BCE. Herodotos described its uses by the Scythians in* 500 *BCE. Thebans, Greeks, Arabs and natives of Africa succumbed in turn to the drug.*

Marijuana entered the New World via Mexico when her French rulers introduced its cultivation for hallucinogenic purposes in the nearly ideal hot, dry lands around Mexico City. Its use spread very slowly to the native peoples because the peon preferred native hallucinogenic plants that were sanctified by usage dating back to Aztec times — marijuana was good for a mild smoke after a long hard day in the fields or mines. The product entered the United States through the Southwest and by 1860 *was introduced to the Eastern Seaboard via the immigrants from the Caribbean Islands. In New York, it was used by the black poor and by the socially chic.*

Marijuana is composed of the dried leaves and flowering tops, stems, and leaves of the female Indian hemp plant. The active component of marijuana is tetra-hydro-cannabinol (THC). Some slang names for this drug are grass, pot, and weed.



Marijuana

Hashish has the same active principle (THC) as marijuana, but it is considered to be about ten times more potent than the marijuana grown in America. Hashish comes from pure resinous exudate of the female hemp plant. The Tunisian cannabis is considered three times more potent than the American varieties. Like marijuana, it is not considered addictive, but can produce psychic dependence, hallucination, and distortion of time and space.

The word "hashish" is derived from the name of a Persian prince, Hasan-ibn-Sabbah, whose pirate army, the Hashashins, were paid off partly with resin [our word "assassins" is derived from these mercenaries].

The introduction of cannabis to "polite" society came when Napoleon's army brought hashish back from conquered Egypt. In 1844 a private club, the Club de Hachichins opened in Paris, dispensing hashish in candy or mixed with wine. Its founders included ***P. Gautier***, ***C. Baudelaire***, ***Dumas****, and others of the socially elite literary set. Other clubs sprang up and introduced smoking. At the same time, French medical authorities began recommending it as a calming agent for the hysterical, justifying the recommendation of Emperor Shen-Nung.*

Ergot alkaloids

For uncounted centuries, ergot has been one of the cursed scourges of mankind. It has plagued the body and mind ever since we began to use grasses for their edible seeds. In Europe it was called the holy fire (ignis sacer), St. Anthony's Fire, the ignius beatae, Virginis invisibilis, or the infernalis.

Three major cereal grains have been used to make bread. The common bread was an unleavened product made from barley, but only wheat and rye flour make a raised or leavened bread. It is likely that Thrace and Macedonia, but not Greece, grew some rye, but the plant was not introduced into much of Europe until the Christian era. France began to grow rye about 300 CE,

and Britain obtained her starting seed when the Teutons invaded the island. Rye was not grown as a major cereal grain in Europe and European Russia until the fifth century, and attempts to pinpoint just when historical records of ergotism began is difficult.

Thus, an epidemic suspiciously like ergotism broke out among the Spartans in 430 BCE*, and a plague of* 857 *CE in the Rhineland also matches the clinical symptoms. A disease "like fire" was reported in Paris in* 943*, from Aquitaine-Limousa in* 994 *with* 4000 *deaths, and from Rheims in* 1041 *with* 2000 *deaths. From that time on, instances of ergotism have been recorded in sufficient detail so that we can be sure of its cause.*

Ergotism results from the ingestion of sclerotia of ergot ground up in rye flour. Two major types of ergotism are known, gangrenous and convulsive. In the former, severe constriction of the blood vessels results in swelling as blood accumulates in the hands or feet, with burning sensations alternating with intense cold. Numbness follows within a few days and this, in turn, is followed by blackening of the limb, horrible odors, and eventually merciful, but unbearably painful death.

Convulsive ergotism accurately describes the symptoms. Twitching of head, arms, and hands is followed by contractions of muscles throughout the whole body. The afflicted typically roll themselves into a ball and then stretch themselves out at full length, the actions accompanied by terrible pains. Vomiting, deafness, blindness, and hallucinations usually follow. Feats of superhuman strength, and the conviction that flying is possible have been noted. If the victim recovers, and about 30–40 *percent do, hallucinations can continue aperiodically for up to a year. Domestic animals who eat ergot-contaminated grain or table scraps exhibit identical responses. It is said that dogs will tear bark from trees until their teeth fall out and that ducks will strut like roosters, attacking people and other animals.*

Depending upon the weather, the genetic constitution of the host and the fungus, the care taken to eliminate sclerotia before milling grain into flour, and the amount of bread eaten, devastating outbreaks of ergotism could occur. And they did occur in Europe on an average of once every five to ten years.

Innumerable people had ergotism before its etiology was recognized in 1673 *by a Parisian lawyer-physician,* **Denis Dodart***. Up to the middle of the* 11^{th} *century, over* 20 *massive epidemics were reported in France alone, and by the middle of the* 14^{th} *century, over* 50 *epidemics had been reported from central Europe. Lacking any knowledge of its cause, it was reasonable to call upon the saints to intercede with heaven for succor. But which one? St. Anthony the Great was born in Egypt in the first century CE and established the idea of monastic life. Long the patron saint for erysipelas, a bacterial disease of the*

skin which causes swelling and burning, it seemed logical for him to become the intercessor for this disease as well.

In 1039, a French nobleman, ***Gaston de la Vollaire****, built a hospital in the Rhone Valley, obtained relics of St. Anthony, and asked monks to serve in the hospital. These men formed the Order of St. Anthony and dedicated themselves to nursing the survivors of ergotism. In the* Book of Hours *by the master of Mary of Burgundy*[619]*(1480), St. Anthony is asked for protection against the disease. The Holy Fire disease was soon called St. Anthony's fire. The fantastic paintings of the Dutch painter* ***Hieronymus Bosch*** *(1450–1516) depict victims of ergotism: crawling cripples and "flyers" out of high windows.*

Although Dodart's identification of the cause of ergotism was known among the few educated physicians of the 17^{th} *century, the direct connection between* ergot *and St. Anthony's fire did not become general knowledge until the* 18^{th} *century. The dark, heavy, sour, but very nourishing bread of central and eastern Europe contained so much ground-up weed seed that the dark sclerotia went unnoticed. Since bread was truly the staff of life, the persistence of the peasants in eating ergot-contaminated bread is not surprising. When a high probability of starvation had to be weighed against possible ergotism, people made the only logical choice.*

Between 1580 *and* 1900 *there were* 65 *major ergot epidemics in Europe and the United States. In* 1722*, Peter the Great mounted an invasion of Turkey to obtain for Russia the still-coveted ice-free port to the seas. His cavalry ate ergotized black bread and* 20,000 *men and horses were stricken; the invasion was called off. Between* 1770 *and* 1780*, epidemics raged through Germany and France with over* 8000 *documented deaths. In the winter of* 1812–1813*, Napoleon's troops and horses ate bread baked from rye commandeered from the Ukraine, and the resulting epidemic of ergotism contributed to his Russian defeat and turned the retreat from Moscow into a horror. In* 1812*, Austria passed a law stating that inadequately cleaned rye would be confiscated, and other European countries quickly passed similar legislation.*

There was a severe outbreak on the Soviet Union in 1926 *and a smaller one in England in* 1928–1929 *when the Jewish community imported rye from central Europe. During the well-studied Soviet Union epidemic in* 1926*, flour*

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619 **Mary of Burgundy** (1457–1482) was the daughter of **Charles the Bold** (1433–1477) by **Isabella of Bourbon**. The marriage of Mary to the irascible Maximilian of Austria was a major event in European history. Her accidental death at the age of 25 took place while out hunting with a falcon. The *Book of Hours* was made for her by an anonymous painter.

containing 2 percent ergot was found to be enough to cause convulsive ergotism and some samples of rye flour contained up to 7 percent sclerotia. It has been claimed that convulsive and hallucinogenic ergotism struck a town in Provence in 1951, but the French government denied this, stating that there was an inadvertent contamination of the flour by an insecticide; this bureaucratic explanation does not conform to the symptoms noted.

Lysergic acid diethylamide (LSD) is a product of ergot, a parasitic black fungus that grows on rye. Because it acts on the central nervous system and has unpredictable effects, LSD is sometimes referred to as a "mind-bending" drug. Many users have experienced visual and perceptual distortions, strange sensations, and difficulty in distinguishing between illusion and reality. Occasionally some users have experienced acute terror and other unpleasant psychological effects.

LSD is one of the most potent drugs known to man. Twenty micrograms (1 microgram is one millionth of a gram) will cause physiological changes. Some alleged LSD tablets have been known to contain up to one thousand micrograms of the drug. In addition other substances have been mixed with LSD, such as arsenic, strychnine, and atropine. These substances often add to the many bad LSD "trips".

LSD is believed to be structurally similar to serotonin, which is a compound found in the brain tissue that may play an important role in thinking processes. There is some evidence that LSD either replaces or blocks serotonin activity in the brain, which may help explain the variety of eccentric and bizarre symptoms some users experience. A compound called DMPEA, which is similar to mescaline, has been found in the urine of 65 percent of schizophrenic mental patients and LSD users, which suggests an explanation for the bizarre behavior of some LSD users.

Researchers are now experimenting with one possible medical use for LSD. They are trying to determine whether it can be used to make the last weeks, or even months, more tolerable for terminal cancer patients. They believe that under controlled conditions, LSD could lessen the harsh reality of impending death and make possible reduced dosages of analgesic drugs.

V. *Reserpine* has been used in India for a long time under the name CHANDRA *(moon)* TO TREAT "LUNATIC" PEOPLE.

It was also known as an effective reducer of fever, sedative and a curer of dysentery. Since 1940 it is used in Western medicine to reduce blood pressure. Today it is used successfully to treat and control nervous disorders such as

schizophrenia and for calming psychotic patients so that they can undertake psychotherapy.

VI. *Psilocybin (sometimes called the magic mushroom)* IS A WILD FUNGUS DERIVATIVE, WHICH HAS ALSO BEEN USED IN RELIGIOUS RITES.

Aztec, Inca and Mayan priests, since 1000 *BCE, used amanita under the name teonanacatl ("flesh of the gods"). The same drug, under the name soma was introduced by the Aryan people that entered India from the north in ca* 1500 *BCE. The Spaniards brought the drug to North America and it then became known to the American pioneers.*

Early Europeans recognized that fly agaric, a fungus parasite of the amanita mushroom, could also act as a potent insecticide. They boiled these mushrooms in milk and placed saucers of the mixture on their windowsills and in the doorways of their homes and marketplaces. The flies and other insects that spread disease from home to home and from village to village ingested this lethal mixture and died. Unfortunately children, dogs, and cats were also attracted to it and poisoned.

Those addicted to the mushroom cult experience visions, muscular relaxation, hilarity, alteration in perceptions of time, feeling of total isolation from one's environment. Under its influence, priests would chant the "truth" about health, disease, success or failure, and how to remedy the affairs of everyday life. For the Indians of Central America, Psilocybe experience awakened the forces of creation.

Psychoactive components of hallucinogenic mushrooms are related to those found at the junction of nerve cells in the body and the brain. If too much is taken, death by respiratory failure may occur.

VII. *Digitalis is a mixture of several naturally occurring cardiac glycosides synthesized by Digitalis purpurea and related species in the figwort (Scrophulariaceae) family. Native to Europe, Western Asia, and Central Asia, it is grown all over the world.*

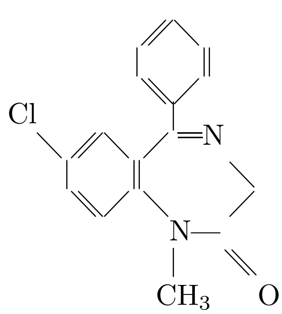
The Latin generic name, Digitalis, means "little finger" from the Latin digitis and is directly derived from the German name for the plant, fingerhut. The Latin term was applied by ***Hieronymus Tragus*** *in* 1539 *and was repeated in* ***Leonhard Fuchs****'s De Historia Stirpium in* 1542. *Several species*

are native to Europe, and all bear the common name foxglove or variants on the same theme.

Digitalis was a medicinal herb for centuries; ***Dioscorides*** *praised it as a plant whose leaves, applied to the skin, could cure many diseases. Juice pressed from the leaves became an ingredient of salves applied to cuts, bruises, and the leg ulcers common in an era of inadequate diets and lack of soap. Rural people made hot water infusions of leaves and drank foxglove tea to experience an inexpensive but dangerous intoxication.*

Among the ills to which flesh is heir is cardiac insufficiency in which a weakened heart fails to pump enough blood through the body. Heartbeat is irregular and fluids collect in the arms, legs, and abdomen because the kidneys cannot perform their normal function. The swelling is known as dropsy or, more formally, as edema. This disease syndrome is not new. Ancient physicians knew of it, but lacking knowledge of the circulation of the blood discovered by ***William Harvey*** *in* 1628 *and information on the function of the kidneys, treatment was limited to usually unsuccessful attempts to reduce edema with medicines which increased urine production (diuretic agents). Today, millions of people pop a small pill which regulates and strengthens the heartbeat and allows the kidneys to expel excess fluid quickly; cardiac insufficiency kills few people since the discovery of digitalis.*

VIII. MINOR TRANQUILIZERS SUCH AS DIAZEPAM *(Valium) are used to alleviate anxiety tensions; they work as skeletal-muscle relaxants and control muscle spasms. Other tranquilizers, such as meprobamate (Miltown), have an antiemetic action that is useful in the treatment of nausea and vomiting, or "morning sickness", of early pregnancy.*



Diazepam (Valium)

IX. *Curare*, A SUBSTANCE OBTAINED FROM A NATIVE SHRUB IN THE SOUTH AMERICAN AMAZON REGION, *was used for centuries by Indians as a weapon in their hunt for small game, such as monkeys. Blowgun darts,*

containing curare-tipped arrows, fatally paralyzed small animals and caused their respiratory muscles to stop functioning. In modern surgery, curare is often used when the complete relaxation of the abdominal muscles is required.

X. *Colchicin* and *acunitin* are AMONG THE MOST POTENT AND POISONOUS ALKALOIDS*: few milligrams of pure substance can cause a human's death.*

Alkaloids are at the junction of four major sciences — chemistry, botany, physiology and medicine.

In spite of a great deal intensive interdisciplinary research, the physiological mode of action of alkaloids in the animal body is poorly understood. Some, like the caffeine alkaloids in tea and coffee, are stimulants. Others, like the alkaloids in ergot, cause constriction of smooth muscle, and still others, like those in the opium poppy, are powerful pair-killers.

It is likely that all operate on or in some part of the central nervous system and that the responses reflect alterations in control over cellular function by the brain and peripheral nerve network.

The physiology, pharmacology, and psychology of addiction to alkaloids like morphine and heroin is even less well understood. Certainly, not all alkaloids are addictive or even habit-forming. One can *get along without a morning cup of coffee without experiencing withdrawal symptoms. Even for those which are addictive, the nature of the addiction and its consequences are poorly understood, and the same can be said for the response to withdrawal from the substance. There are certainly psychological factors as well as biochemical and physiological factors which must be evaluated.*

1886–1897 CE **Gustave Victor Robin** (1855–1897, France). Mathematician. Made significant contributions to potential theory (1886) and thermodynamics. Named after him are:

- A third boundary condition of partial differential equations. (*Robin's boundary condition*; the corresponding Green's function is known as *Robin's kernel*.

- The logarithmic *capacity* of a compact set E (*Robin's Constant*). This is related to his original solution to a problem in electrostatics (*Robin's Problem*), where he established a remarkable connection between potential theory and the capacity concept in point-set topology. From it developed later (**N. Wiener**, 1924) the concept of *capacity* in point-set topology.

- A method for evaluating a single-layered charge distribution over a closed bounded surface of a conductor. This leads to *Robin's integral equation* for the charge density which is solved by successive approximation (*Robin's potentials*; *Robin's Principle*, *Robin's function*).

Robin was a professor of mathematical physics at the Sorbonne in Paris. His idiosyncrasies and early death left him almost unremembered and he died in obscurity. His collected works were published posthumously (1899–1903) by his friend and colleague **Louis Raffy**.

1886–1904 CE **Giuseppe Peano** (1858–1932, Italy). Mathematician, linguist and logician. One of the founders of symbolic logic. Endeavored to develop a formalized language which could be used in mathematical logic and mathematics in its entirety. His major work in this field is '*Formulaire de mathémathiques*' (1894–1908) which he wrote with his students and colleagues at the University of Turin. This work was intended to flow from its fundamental postulates using his logic notation. Parts of his method and notation were accepted in the mathematical world and profoundly changed the outlook of mathematicians.

Following the work of **Dedekind**, he established in 1899 a system of axioms for natural numbers that bears his name. In topology, he discovered a continuous function whose points completely fill the unit square (*Peano's curve*).

He made important contributions to the theory of ordinary differential equations. Peano presented an abstract form of the theory of vectors based on Grassmann's calculus of extensions, and introduced the concept of 'Riemann content'.

He created an artificial international language, later called '*Interlingua*'. It is based upon a synthesis of vocabulary from Latin, French, German and English, with a greatly simplified grammar.

Peano was born in Sardinia. He became a professor at the University of Turin in 1890 and was also a professor at the Military Academy in Turin.

1886–1906 CE **Ferdinand-Frederic-Henri Moissan** (1852–1907, France). Inorganic chemist. First to isolate the element *fluorine* (F) in 1886. Introduced an improved arc furnace for metallurgy (1892). Discovered silicon carbide (1893). Was awarded the Nobel Prize for chemistry (1906).

Fluorine, the lightest of the halogens, is the most reactive of all the elements, and it forms compounds with all the elements except the lighter inert gases. Because its *electronegativity*[620](4) is greater than any other element, it cannot be prepared by reaction of any other element with a fluoride. It was by the *electrolysis* of a solution of KF in liquid HF that fluorine was first obtained by Moissan.[621]

Moissan was born in Paris to Jewish parents and studied at the laboratory of the Natural History Museum. Professor, Ecole de Pharmacie, Paris (1886–1900), Sorbonne (1900). Fluorine's poisonous nature is believed to contribute to his early death at the age of 54.

1886–1908 CE **Elie Metchnikov** (1845–1916, Russia). Immunologist. Pioneer 'microbe hunter'. Hypothesized the role of *phagocytes* in vertebrate blood to fight invasion of bacteria. Was awarded the Nobel Prize for physiology or Medicine (1908) jointly with **Ehrlich**.

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[620] A measure of the relative tendency of an atom to attract electrons to form *anions*. Elements with low electronegativities (metals) often loose electrons to form cations. Oxygen is the second most electronegative element (3.5). Then come chlorine (3.0), Nitrogen (3.0), Bromine (2.8), Iodine (2.5), Sulfur (2.5), Carbon (2.5).

Fluorine occurs in large quantities in the minerals *fluorspar* (CaF_2); *cryolite* (Na_2AlF_6); and *fluoroapatite* [$Ca_5(PO_4)_3$]. It also occurs in small amounts in sea water, teeth, bones, and blood. Fluorinated organic compounds, called *fluorocarbons* are stable and nonflammable. They are used as refrigerants, lubricants, plastics (such as Teflon), insecticides, and aerosol propellants. Stannous fluoride, SnF_2, is used as toothpaste.

[621] Only in 1986 was Moissan's method of fluorine production superseded by the discovery of **Carl O. Christe** that F_2 can be obtained in better than 40% yield by the reaction

$$\begin{aligned}&2K_2MnF_6 \text{ (solid) } + 4SbF_5 \text{ (liquid)}\\ &\qquad \to 4KSbF_6 \text{ (solid) } + 2MnF_3 \text{ (solid) } + F_2 \text{ (gas)}.\end{aligned}$$

Born in Ivanovka, the Ukraine to Jewish parents. Graduated from the University of Kharkov (1864) and received a doctorate from the university of St. Petersburg (1867). Left Russia (1887) to work with **Pasteur**, who offered him the directorship of a laboratory at the Pasteur Institute in Paris.

1887 CE, September to October Yellow River (Huang-ho) in Honan province, China, overflowed, submerging 130,000 km^2 of land and killing about a million people. Flooding was caused by rain.

1887–1890 CE **Augustin** (**Louis Aimé August**) **Le Prince** (1841–1890, France, England and U.S.A.). Engineer, artist and inventor. Constructed the first *moving-picture machine* (camera and projector), predating Edison's claim, and made short moving pictures in Leeds in 1888. He was not the first to have the idea but the first to succeed.

Le Prince was born in Metz, France. His father was a major in the service of Louis-Philippe. He studied chemistry and optics at Leipzig and was trained as a painter. In 1866 he came to Leeds, England to work for a firm that manufactured components for the local locomotive industry.

He first became interested in moving pictures in 1869 under the inspiration of **Muybridge** photography and **Houdin**'s 'magic lanterns' in Paris, but started to realize his ideas upon his immigration to New York in 1882. His serious experiments began in 1885. In 1887 he returned to Leeds to avoid New York's industrial spies and to take advantage of his father in law's offer of support. At first he developed a camera with 16 lenses. The 16 shutters were operated by electromagnets and armatures controlled by a circuit closer. The lenses were made to converge on a single point.

He later developed a single-lens camera but did not have the advantage of celluloid; the paper film from the camera had to be developed into a negative. The negative frames were then stripped from their paper backings and positive transparencies were made of them. These were very flimsy and needed a stronger transparent backing to survive the heat and jerking of the projector, through which they were transported at a rate of 16 frames per second. He used gelatin or glass, but non of these could roll without cracking and only glass was transparent enough. Le Prince had to mount each individual frame onto a specially designed picture-belt which made it all very heavy.

In 1889 he finally got hold of synthetic celluloid which came in coated sheets at a foot square but not in long rolls. He had to make his own. When operating his single-lens camera, the sensitive paper film was intermittently activated at the rear of the lens by providing it with a properly timed intermittently operated shutter.

Le Prince died under mysterious circumstances: he disappeared on Sept. 16, 1890 during a train trip from Dijon to Paris. He never arrived to Paris and was never seen again. No trace of him was ever found. Documents, discovered recently in the city archives of Leeds, point to the possibility that he engineered his own disappearance. Financial and technical difficulties were probably the cause. His failure to patent his single-lens camera in sufficient detail was a fatal oversight and caused his family to loose their legal claim against Edison in 1901.

In 1930 a plaque commemorating Le Prince's pioneering invention was unveiled in the city of Leeds. A second plaque in that city was unveiled in 1988, commemorating the centennial of his great achievement — the first moving picture ever, taken on Leeds bridge on Oct. 14, 1888.

1887–1893 CE **Paul Tannery** (1843–1904, France). The first modern historian of science. He wrote: *Pour l'histoire de la science hellene* (1887), *La géometrie grecque* (1887) and *Recherches sur l'histoire de l'astronomie ancienne* (1893).

Tannery was born at Mantes-la-Jolié and died at Pantin (both localities near Paris). He entered the École Polytechnique (1860), and graduated among the ranking members of his class. For the next 40 years he was in the service of the state monopoly of tobacco, but his evenings and holidays were devoted to the study of the history of science.

It was only in relatively recent times that the importance and centrality of the history of science was realized. There were a few pioneers beginning with the end of the 17th century.

Such men were: **Albrecht von Haller** (1708–1777); **Joseph Priestley** (1733–1804); **Adam Smith** (1723–1790); **Jean Etienne Montucla** (1725–1799) and **Jean Sylvain Bailly** (1735–1793).

But the first man to introduce this theme in a broader context and to increase its circulation was the French philosopher, **Auguste Comte**, who developed it in his *Cours de philosophie positive* (1830–1842). His views were discussed by another French philosopher, **Antoine Augustin Cournot**, in 1861, but the real inheritor of Comte's thought and the first great teacher of the history of science was **Paul Tannery**.

Tannery's philosophy is very different from Comte's, but the greatest difference between them is that Comte's knowledge of the history of science was very superficial, whereas Paul Tannery, being extremely learned and having at his disposal a mass of historical research work which did not exist in the thirties, knew more of the history of science than anybody else in the world. Certainly no man ever was better prepared to write a complete history of science, at least of European science, than Paul Tannery. It was his dream

to carry out this great work, but unfortunately he died, before realizing his ambition. During the 20th century his example has been followed by many scholars, notably **George Sarton**.

1887–1898 CE **Woldemar Voigt** (1850–1919, Germany). Mathematician. First to write down, in 1887, a mathematical transformation which leaves the scalar wave-equation, and consequently Maxwell's equations, invariant (later known as the *Lorentz transformation*). The next pre-relativistic mention of this transformations was given by **Hendrik Antoon Lorentz** (1853–1928, Holland) in 1895, and then in 1898 by **Joseph Larmor** (1857–1942, England).

Voigt established the stress-strain relation in a *viscoelastic solid* in which the stress is related to a linear combination of the strain and the rate of strain, known as a Kelvin-Voigt substance[622] (1892). In 1898 he reinstated Hamilton's term '*tensor*' as the entity representing the local state of stress in an elastic continuum.

Voigt was a tall, thin man with a red beard. He was a truly quiet scholar. His lectures were like his book on crystals — very hard to understand, but deep and knowledgeable. He drew beautiful sketches on the blackboard, polishing and correcting them for five or six minutes. He talked in short, concise sentences, never looking at his audience. He had reputation for calculating incredibly and magnificently.

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[622] It is a generalized *Hooke's law* is 3 dimensions:

$$\mathfrak{T} = \overset{4}{C} : \mathfrak{E} + \overset{4}{D} : \frac{\partial \mathfrak{E}}{\partial t},$$

where $\mathfrak{E}$ is the strain tensor and $\{\overset{4}{C}, \overset{4}{D}\}$ are two fourth order tensors. In isotropic homogeneous materials, the above stress-tensor takes the simplified form:

$$\mathfrak{T}(\boldsymbol{r}, t) = \left(\lambda + \lambda' \frac{\partial}{\partial t}\right) \mathfrak{I} \operatorname{div} \boldsymbol{u} + 2\left(\mu + \mu' \frac{\partial}{\partial t}\right) \mathfrak{E}.$$

For $\lambda' = \mu' = 0$ we fall back on the *elastic solid*, while in the limit

$$\lambda \operatorname{div} \boldsymbol{u} = -p, \qquad \mu = 0, \qquad \mu' = \eta, \qquad \lambda' = \overline{\lambda} - \frac{2}{3}\eta$$

($\overline{\lambda}$, η — viscosity coefficients), we fall back on the *Newtonian fluid*.

Likewise, the limiting case $\lambda' = \mu' = 0, \quad \lambda = \mu \to \infty$ leads us back to a *rigid body*, and the limit $\lambda \operatorname{div} \boldsymbol{u} = -p, \quad \mu = \mu' = \lambda' = 0$ renders the *ideal fluid*.

1887–1889 CE **Lois-Gustave Binger** (1856–1936, France). West-African explorer. The first European to cross the watershed of the *Volta River*. Traveled widely in the Ivory Coast, Senegal, and (todays) Upper Volta, Ghana, Guinea and Gambia.

Binger was born to a Jewish family in Strasbourg and became governor of Ivory Coast (1893).

1887–1901 CE **Ernesto Cesàro** (1859–1906, Italy). Mathematician. Contributed mainly to differential geometry, summability of divergent series (1890) and the theory of numbers. Formulated '*Intrinsic geometry*' [*Lezioni di geometria intrinseca*, 1896], in which he derived coordinate-free ("natural") equation of curves by means of the arc length and curvature variables. This goes back to **Euler** (1736) who used it for special curves.

Cesàro was born in Naples and continued his studies at Liege (Belgium) and Paris under **Hermite** and **Darboux**. He received his doctorate from the University of Rome (1887). He held the chair of mathematics at Palermo until 1891, moving then to Rome, where he held the chair until his death.

1887–1907 CE **Vito Volterra** (1860–1940, Italy). One of the top mathematical physicists of his time. Made major contributions in the general theory of functionals[623] (1887–1889), partial differential equations, integral equations, integro-differential equations, theory of dislocations (1907), mathematical biology (1920) and other topics in mathematical physics. His work had strong influence on the general development of modern calculus.

Volterra contributed to the solution of linear equations in multi- or infinite-dimensional linear spaces by means of his multiplicative integral. Developing the general theory of the functional calculus, Volterra invented a way to reduce calculations with functionals to calculations with usual functions with many variables.[624]

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623 For further reading, see:

- Volterra, V., *The Theory of Functionals*, Blackie and Sons, 1930, 225 pp.

624 In this procedure one has to divide the interval from the initial time t_{in} to the final time t_{fin} into a large, but finite number N of time instants t_i, and then to approximate the functional $F[x(t)]$ with the functions $F(\dots, x_{i+1}, x_i, \dots)$, where x_i gives the value of $x(t_i)$. Then, one has to work with this function, instead of the functional $F[x(t)]$.

This process is called *finite-dimensional approximation* or discretization, of the functional $F[x(t)]$, which itself may be considered as a function of infinitely many variables $x(t)$, with a continuous label t. After performing operations on the function $F(\dots, x_{i+1}, x_i, \dots)$ in the final result one has to take the limit

Volterra was born in Ancona, Italy, to a poor Jewish family. His interest in mathematics started at the age of eleven. At the age of 13 he began to study

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$N \to \infty$, keeping t_{in} and t_{fin} fixed.

R.P. Feynman (1942) utilized Volterra's theory of functionals in his new calculations of averages of *quantum mechanical quantities*. His formulas give us the expectations of certain functionals on the paths $x(t)$ in the configuration space of the classical mechanical system, where the time t runs from some initial time t_{in} to some final time t_{fin}. As a weight in the averaging procedure, one uses the complex phase with argument equalling classical action divided by $\hbar$.

Volterra introduced, for the first time, the beautiful mathematical idea of a *functional derivative* (1887), which was developed further by **Gateaux** (1919) and **Fréchet** (1925) in the framework of functional analysis.

The functional $F[x(t)]$ gives a *number* for each *function* $x(t)$ that we may choose.Volterra asked: How much does this number change if we make a very small change in the argument function $x(t)$? Thus, for a small $\eta(t)$, how much is $\delta F \equiv F[x(t)+\eta(t)] - F[x(t)]$?

To evaluate this, suppose time is divided into very many steps of small intervals ϵ, the values of the time being t_i where $t_{i+1} = t_i + \epsilon$. The function $x(t)$ can then be approximately specified by giving the values x_i that it takes on at each of the times t_i, namely $x_i = x(t_i)$. The functional $F[x(t)]$ is now a number depending on *all* the x_i, that is, it becomes an ordinary function of the variables x_i,

$$F[x(t)] \qquad \to \qquad F(\dots, x_i, x_{i+1}, \dots).$$

If we alter the path from $x(t)$ to $x(t)+\eta(t)$, we change each x_i to $x_i + \eta_i$, where $\eta_i = \eta(t_i)$. Then, the first-order change in our *multivariable function* is

$$dF \equiv F(\dots, x_i+\eta_i, x_{i+1}+\eta_{i+1}, \dots) - F(\dots, x_i, x_{i+1}, \dots) = \sum \frac{\partial F}{\partial x_i}\eta_i,$$

according to the ordinary rules of partial differentiation.

In the limit $\epsilon \to 0$ (assuming it exists, etc.)

$$dF \to \delta F; \qquad \sum \frac{\partial F}{\partial x_i}\eta_i \to \int \frac{\delta F}{\delta x(s)}\eta(s)\, ds,$$

where $\delta x(s)$ is the differential change in path at $x(s)$, and the functional derivative is taken at the point $t_i = s$. Thus, one can show, for example, that if $S = \int_{t_1}^{t_2} L(\dot{x}, x, s)\, ds$, then for any s inside the range t_1 to t_2

$$\frac{\delta S}{\delta x(s)} = -\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial x},$$

where the partial derivatives are evaluated at $t = s$.

the Three Body Problem and made some progress. He attended the University of Pisa (1878–1882) and was appointed professor of rational mechanics there (1883). During that time he began to develop the theory of functionals, which he applied to the solutions of integral and integro-differential equations. The important idea of *harmonic integrals* derives essentially from his functional analysis.

In 1892 Volterra became professor of mechanics at the University of Turin, and from 1900 onward he occupied the chair of mathematical physics at the University of Rome.

In 1905 he became a senator of the Kingdom of Italy. In WWI he joined the Italian Air Force and was first to propose the use of Helium in airships. In 1922 Fascism seized Italy and Volterra fought against it in the Italian Parliament. However by 1930 the Parliament was abolished and Volterra refused to take an oath of allegiance to the Fascist Government. As a Jew in Fascist Italy, he was forced to leave the University of Rome (1931) and resign from all Italian scientific academies. He died in Rome during WWII.

History of Integral Equations[625]

An integral equation is an equation in which an unknown function appears under an integral sign and the problem of solving the equation is to

[625] For further reading, see:

- Polyanin, A. D. and A. V. Manzhirov, *Handbook of Integral Equations*, CRC Press: New York, 1998, 787 pp.
- Hamel, G., *Integralgleichungen*, Springer-Verlag: Berlin, 1949, 166 pp.
- Kanwal, R. P., *Linear Integral Equations*, Academic Press, 1971, 296 pp.
- Moiseiwitsch, B. L., *Integral Equations*, Longman: London, 1977, 161 pp.
- Chambers, Ll. G., *Integral Equations*, 1976, 198 pp.
- Tricomi , F. G., *Integral Equations*, Dover: New York, 1985, 238 pp.
- Kondo, J., *Integral Equations*, Oxford University Press, 1991, 440 pp.

determine that function. The term 'integral equation' was coined by ***Du Bois-Reymond*** *(1888).*

At first, solving integral equations was described as inverting integrals. Long before the subject acquired a distinct status and methodology, ***Laplace*** *(1782) considered the integral equation for $g(t)$ given by $f(x) = \int_{-\infty}^{\infty} e^{-xt}g(t)\,dt$, now called the Laplace transform of $g(t)$.*

Poisson *(1811) discovered its solution, namely,*

$$g(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{xt}f(x)\,dx$$

for large enough a.

Another result stems from ***Fourier's*** *(1811) paper on the theory of heat conduction*

$$f(x) = \int_0^\infty u(t)\cos(xt)\,dt$$

and the inversion formula

$$u(t) = \frac{2}{\pi}\int_0^\infty f(x)\cos(xt)\,dx$$

The first conscious direct use and solution of an integral equation go back to ***Abel*** *(1823). He considered a mechanical problem which led him to the equation*

$$f(x) = \int_a^x \frac{u(\xi)d\xi}{(x-\xi)^\lambda} \qquad 0 < \lambda < 1$$

He then found the solution

$$u(z) = \frac{\sin(\lambda\pi)}{\pi}\frac{d}{dz}\int_a^z \frac{f(x)\,dx}{(z-x)^{1-\lambda}}$$

Liouville *(1832) showed that the solution of the differential equation*

$$y'' + [p^2 - \sigma(x)]y = 0 \quad \text{or} \quad y'' + p^2y = \sigma(x)y$$

$$a \le x \le b, \qquad y(a) = 1, \qquad y'(a) = 0 \qquad p = \text{ parameter}$$

is also the solution of the integral equation

$$y(x) = \cos p(x-a) + \frac{1}{p}\int_a^x \sigma(\xi)y(\xi)\sin p(x-\xi)\,d\xi.$$

The conversion of differential equations to integral equations became a major technique for solving initial and boundary-value problems of ODE and PDE, and this was the strongest impetus for the study of integral equations.

Volterra *(1896) is the first founder of a general theory of integral equations. He set out to solve*

$$f(s) = \Phi(s) + \int_a^b K(s,t)\Phi(t)\,dt$$

for $\Phi(s)$*, where* $f(s)$ *is known and* $K(s,t) = 0 \quad$ *for* $t > s$*. His solution can be written in the form*

$$\Phi(s) = f(s) + \int_a^b \overline{K}(s,t) f(t)\,dt$$

where

$$\overline{K}(s,t) = -K(s,t) + \int_a^b K(s,\tau)K(\tau,t)\,d\tau - \int_a^b\int_a^b K(s,\tau)K(\tau,\omega)K(\omega,t)\,d\tau d\omega + \dots.$$

Volterra also observed that the integral equation

$$f(s) = \int_a^b K(x,s)\Phi(x)\,dx$$

is a limiting form of a system of n *linear equations in* n *unknowns as* n *becomes infinite.* **Fredholm** *(1900–3) used this idea to solve*

$$u(x) = f(x) + \lambda\int_a^b K(x,\xi)u(\xi)\,d\xi.$$

Dividing the x*-interval* $[a,b]$ *into* n *equal parts* $(x_1, x_2, \dots, x_n)$*, he presented his solution in the form*

$$u(x,\lambda) = f(x) + \int_a^b \frac{D(x,y,\lambda)}{D(\lambda)} f(y)\,dy, \qquad D(\lambda) \neq 0, \qquad \text{where}$$

$$D(\lambda) = 1 - \lambda\int_a^b K(\xi_1,\xi_1)\,d\xi_1 + \frac{\lambda^2}{2!}\int_a^b\int_a^b \begin{vmatrix} K(\xi_1,\xi_1) & K(\xi_1,\xi_2) \\ K(\xi_2,\xi_1) & K(\xi_2,\xi_2) \end{vmatrix} d\xi_1 d\xi_2 + \dots$$

$$D(x,y,\lambda) = \lambda K(x,y) - \lambda^2\int_a^b \begin{vmatrix} K(x,y) & K(x,\xi_1) \\ K(\xi_1,y) & K(\xi_1,\xi_1) \end{vmatrix} d\xi_1 + \frac{\lambda^3}{2}\int_a^b\int_a^b \begin{vmatrix} K(x,y) & K(x,\xi_1) & K(x,\xi_2) \\ K(\xi_1,y) & K(\xi_1,\xi_1) & K(\xi_1,\xi_2) \\ K(\xi_2,y) & K(\xi_2,\xi_1) & K(\xi_2,\xi_2) \end{vmatrix} d\xi_1 d\xi_2 - \dots$$

***Hilbert** (1904–1912) completed Fredholm's solution by carrying out the limiting process for the infinite number of algebraic equations, dispensing with Fredholm's infinite determinants. He then applied his research to a variety of problems in geometry and physics. Hilbert's work was simplified by **Schmidt** (1907), completed by **Fischer** (1907) and **Riesz** (1907), and extended to nonlinear integral equations. Moreover, the theory was extended to non-continuous functions $f(x)$ and $K(x, \xi)$ and to infinite limits of integration (singular integral equations) by **Weyl** (1908).*

1887–1907 CE **Emil Hermann Fischer** (1852–1919, Germany). Distinguished organic chemist. Analyzed the structure of sugars (1887). First to promote the idea of an encoding of genetic specificity in a spatial arrangements of subunits (1907): proposed the theory of 'lock and key' to explain stereo-specific interaction of enzyme with substrate. Synthesized *polypeptide* (1907), a small protein consisting of 18 amino acids, and showed that it could be broken by digestive juices just as natural proteins are.

His studies on the structures of *purines* (1882–1901) and *polypeptides* (1900–1906), opened the way for an understanding of *nitrogen metabolism*, which was essential before the biochemistry of these substances could be developed.

Fischer was born in Euskirchen, Rhenish Prussia. Studied at Bonn and Strasbourg. Professor at Wirzburg (1885) and Berlin (1892). Awarded the Nobel prize for chemistry (1902).

By the turn of the century, with a dozen amino acids isolated from proteins[626], the time was ripe to try to reverse the process and to form a protein out of amino acids: Fischer, using the technique of organic chemistry as developed over the previous half-century, painstakingly treated amino acid mixtures under such conditions as would encourage combination. By 1907 he had managed to build up a molecule made up of 18 amino acid units, consisting of 15 *glycines* and 3 *leucines*[627].

[626] The major units of the protein molecule were discovered in the following order: *Glycine* (1820), *Leucine* (1820), *Tyrosine* (1849), *Serine* (1865), *Glutamic Acid* (1866), *Asparic Acid* (1868), *Phenylalanine* (1881), *Alanine* (1888), *Lysine* (1889), *Arginine* (1895), *Histidine* (1896), *Cystine* (1899).

[627] Such relatively small strings of amino acids are called *peptides* (Greek for "digestion") because they are produced in the process of digestion.

In doing this, Fischer demonstrated unequivocally that amino acids are indeed the building blocks of proteins, for his 18-unit product had all the properties that might be expected of a very simple protein molecule.

He was further able to show that, in forming a chain, the carboxyl group of one amino acid combines with the amino group of another via a *peptide bond*[628].

1887–1910 CE **Gaston Jean Darboux** (1842–1917, France). Notable mathematician. Made important contributions to analysis and differential geometry: Founded the general theory of algebraic manifolds and coined the phrase '*repermobile*' (today's '*soldering form*', known in GTR as '*tetrad*' or '*vierbein*'). Applied infinitesimal calculus to geometry in the study of the differential geometry of curves and surfaces.

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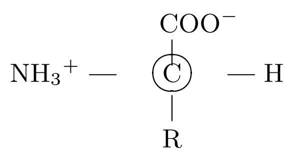
[628] The general structure of amino acids is



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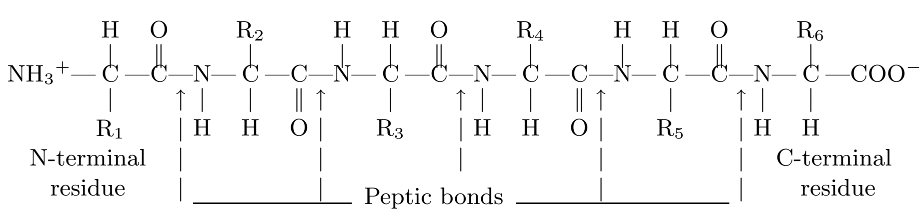
It involves an *amino group* (H_3N^+) and a *carboxyl group* (COO^-), both of which are bonded to the α-carbon Ⓒ, which is also bonded to a hydrogen and to the *side chain group* (R).

The R group determines the identity of the particular amino acid. The two-dimensional formula shown here can only partially convey the common structure of amino acids, because one of the most important properties of these compounds is their 3-dimensional shape (*Stereochemistry*).

Individual amino acids can be linked together by formation of *covalent bonds*. The bond is between the α-carboxyl group of one amino acid and the α-amino group of the next one. Water is eliminated in the process, and the linked amino acid *residues* remain after the elimination of water: e.g.



In a protein, many amino acids (usually more than a hundred) are linked by peptide bonds to form *polypeptide chain*. The peptide bond can be written as a *resonance hybrid* of a *single* bond between carbon and nitrogen and another with a *double* bond between the carbon and nitrogen. The peptide bond is stronger than an ordinary single bond because of this resonance stabilization.

He solved Euler's equations for the rotation of a rigid body about a point, relative to a fixed inertial frame, when the angular velocity vector is known in the rotating frame.

Darboux left his mark on several fields of pure and applied mathematics: We have *Darboux surfaces*, *Darboux vector*, *Darboux theorem*[629] and *Darboux integral* in the infinitesimal calculus, the *Darboux transformation* in the theory of linear differential equations and the *Darboux equation* in modern gas dynamics[630].

In his many papers and books he combined geometrical intuition with mastery of algebra and analysis. His treatises "*Lessons on the General Theory of Surfaces and the Geometrical Applications of Infinitesimal Calculus*" and "*Lessons on Orthogonal Systems and Curvilinear Coordinates*" (originally in French) are a vast source of information, and among the best written mathematical books of the 19^{th} century.

Darboux was born in Nimes, France. He was a professor of mathematics at the Sorbonne during 1873–1890.

1887–1896 CE **Gregorio Ricci-Curbastro** (1853–1925, Italy). Outstanding mathematician. Distilled and perfected the tensor calculus as an independent discipline. He was instrumental in bringing to fruition the ideas of **Christoffel**, **Beltrami** and **Lipschitz**. In his studies of surfaces, Ricci encountered several interesting metric attributes of hyperspaces. One of them was the *Ricci tensor.*

This new invariant symbolism, originally constructed to deal with the transformation theory of partial differential equations and quadratic differential forms, turned into what he now called the *theory of tensors.* He elaborated on the theory and worked out an elegant and comprehensive notation. With the aid of his pupil **Tullio Levi-Civita** (1873–1941) he showed that tensors could provide a unification of many invariant symbolisms, and deal with a wide variety of problems in analysis, geometry and the physical disciplines of elasticity, hydrodynamics, electromagnetism and relativity.

Thus, the mathematical machinery demanded by the theory of general relativity was available a year after the Michelson-Morley experiment, which was partly responsible for the special theory of relativity in 1905. Without the tensor calculus, the general relativity theory of 1915–1916 would have been

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[629] If $f(x)$ is differentiable for $a \leq x \leq b$, $f'(a) = \alpha$, $f'(b) = \beta$, and γ lies between α and β, then there is a ξ between a and b for which $f'(\xi) = \gamma$.

[630] $(x+y)\frac{\partial^2\Phi}{\partial x \partial y} + k\left(\frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial y}\right) = f(x,y), \quad (k > 1).$

impossible. With later modifications and generalizations, tensor methods quickly induced a vast development of modern differential geometry.

Ricci was a professor at the University of Padua during 1880–1925.

From Vectors to Tensors[631]*; the Principle of Covariance*

Vector analysis was born in the middle of the 19^{th} *century in the minds of* ***W.R. Hamilton*** *(1844) and* ***H. Grassmann*** *(1844). The 'pregnancy' of this idea lasted for about 22 centuries, since its 'conception' during the era of Greek science.* ***Aristotle*** *(ca 350 BCE) was aware of the parallelogram of composition of forces.* ***Stevin*** *(ca 1583) employed the same principle in his studies of static mechanics and* ***Galileo*** *(ca 1583) recognized the concepts of*

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631 For further reading, see:

- Sokolnikoff, I. S., *Tensor Analysis* (Theory and Applications to Geometry and Mechanics of Continua), John Wiley & Sons: New York, 1964, 361 pp.
- Lass, H., *Vector and Tensor Analysis*, McGraw-Hill Book Company: New York, 1950, 347 pp.
- Crowe, M. J., *A History of Vector Analysis*: *Evolution of the Idea of a Vectorial System*, University of Notre Dame Press: South Bend, 1967, 270 pp.
- Schwartz, M., S. Green, and A. W. Ruthledge, *Vector Analysis*, Harper and Brothers: New York, 1960.
- Brand, L., *Vector and Tensor Analysis*, John Wiley & Sons: New York, 1948, 439 pp.
- Marsden, J. E., and A. J. Tromba, *Vector Calculus*, W. H. Freeman and Company: New York, 1988, 655 pp.
- Danielson, D. A., *Vectors and Tensors in Engineering and Physics*, Addison-Wesley: Redwood City CA, 1992, 280 pp.
- Lovelock, D., and H. Rund, *Tensors, Differential Forms and Variational Principles*, John Wiley & Sons: New York, 1975, 364 pp.

parallelogram of forces and velocities. However, no scientist until 1844 *ever comprehended the full scope* of the vector concept and its latent potentialities.*

From the mathematical point of view, vector and tensor analysis is a study of geometric entities and algebraic forms independent of the coordinate system. This creates a link between the nascence of vectors and the algebraization of geometry through the invention of analytic geometry by ***Fermat*** *and* ***Descartes*** *during* 1629–1637. *These mathematicians combined the notation and problem-solving ability of the algebraist (which originated with the Babylonians) with the geometry of the plane and space developed by the Greeks. [****Apollonios of Perga*** *(ca 230 BCE) produced a characterization of conic sections in terms of what we now call coordinates.]*

The systematic transition from one to another is achieved by means of a system of coordinates.

With the idea of coordinate system established in the first half of the 17th century, there came the first strides taken in the geometric representation of complex numbers[632] *[****Wallis*** *(1673),* ***Wessel*** *(ca 1785),* ***Argand*** *(1806)]. This was the common geometric and algebraic background against which* ***Hamilton*** and ***Grassmann*** *operated in* 1884. *Yet their concepts were introduced from quite divergent modes of thought, and in significantly different frameworks. Hamilton seems to have been inspired mainly by a necessity for appropriate mathematical tools with which he could apply Newtonian mechanics to various aspects of astronomy and physics.*

From the strictly mathematical standpoint he was perhaps stimulated by the desire to introduce a binary operation that could be interpreted physically by means of rotation in space. On the other hand, Grassmann's motivations were of a more philosophical nature. His chief desire seems to have been that of developing a theoretical algebraic structure on which a geometry of any number of dimensions could be based. It was Grassmann who introduced for the first time the concept of indeterminate product, *a special case of which was the* 2^{nd} *rank tensor,* the dyadic[633].

In addition to the geometrical and algebraic ingredients of the vector concept, the advent of the infinitesimal calculus added the analytical dimension to its development. The idea of the arithmetical and geometrical limit appeared

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[632] Complex numbers are important in the historical background of vectors because of the analogy between these entities in two dimensions. The term "*complex number*" was introduced by **Gauss**.

[633] Grassmann was apparently *not* aware of the fact that his 2^{nd} rank indeterminate product could serve as a mathematical representation of the *inertia tensor*, that had appeared already in 1785 in **Euler**'s equations of rotation of a rigid body about a point.

already in embryonic form in both Babylonian and Greek mathematics. It was explicitly introduced by **Newton** *and* **Leibniz** *during* 1665–1679 *via the concepts of the derivative and integral. This marked the advent of geometrical analysis and in particular differential geometry, the basic ideas of which were introduced by* ***Gauss*** *(1827).*

Thus, the triple merger of Newtonian analysis, Euclidean geometry and Cartesian coordinate systems produced the ultimate mathematical vehicle for the development of tensor analysis. Nevertheless, in 1844 *the time was not yet ripe for the exposition of the full theory. In spite of the great merit of Grassmann's work, it made little impression on the scientific world and because of a lack of pressing need, the tensor theory was slow in coming into formal being.*

However, the first realization of the need for tensors arose with the doctoral thesis of **Riemann** *in* 1854, *in which he based the metric properties of n-dimensional space on a fundamental quadratic form: $ds^2 = \sum_{\alpha,\beta=1}^{n} g_{\alpha\beta} dx^{\alpha} dx^{\beta}$. He generalized the concept of curvature on a surface to n dimensional space, in terms of the metric coefficients $g_{\alpha\beta}$. His work was followed by* ***E. Beltrami*** *(1864),* ***E.B. Christoffel*** *(1869) and* ***R. Lipschitz*** *(1869), who introduced further concepts into the algebra and calculus of n-dimensional manifolds, including the concept of covariant differentiation. In other veins,* ***Cayley*** *(1857) created the theory of matrices,* ***Aronhold*** *and* ***Clebsch*** *(1858–1861) and* ***Gordan*** *(1868–1870) developed the theory of algebraic invariants and covariants, and* ***Clifford*** *(1873–1878) invented his algebra.*

At the close of the 19^{th} *century all these ideas were compiled and integrated by* ***Gregorio Ricci-Curbastro*** *(1887) into what is known today as the algebra and calculus of tensors. His pupil* ***Tullio Levi-Civita*** *(1901), generalized the concept of parallelism to Riemannian spaces.*

In spite of these developments and the many applications of tensor analysis to both mathematics and physics, the subject was, at the beginning of the 20^{th} *century, little more than a plaything of a small group of mathematicians. Only since* 1916, *with the advent of Einstein's theory of general relativity, did tensors come of age. Wide areas of applications in theoretical physics, applied mathematics and differential geometry have been found. Due to the remarkable effectiveness of the tensor apparatus in the study of nature, it is serving as the universal language which* **Hamilton** *and* **Grassmann** *envisioned in their original theories.*

In 1915, ***Einstein*** *said:*

"The magic of this theory will hardly fail to impose itself on anybody who has truly understood it; it represents a genuine triumph of the method

of absolute differential calculus, founded by ***Gauss****,* ***Riemann****,* ***Christoffel****,* ***Ricci*** *and* ***Levi-Civita****".*

Non-relativistic physical laws are written in terms of scalars, vectors and tensors. Scalars are entities such as time, volume and mass that are specified by a single number, the magnitude.

Vectors are entities having a direction as well as a magnitude (examples are position, velocity and force). They require more than one number for their specification.

A tensor[634] *is a more complex entity, specified at each point by an array of numbers. A rank-2 tensor is a matrix — such as the state of stress or strain at a given point in an isotropic elastic solid, or the moment of inertia of a rotating rigid body. The tensor that relates stress and strain in a general elastic medium, is a tensor of rank 4.*

Physical laws are usually mathematical statements that establish algebraic, differential and integral relations between tensors. As such they must assume the same mathematical form, irrespective of the position, orientation and state of uniform motion of the observer. This is known as the principle of covariance of physical laws. Had it been otherwise, these laws would be only of limited local value and loose their universality.

To see how the principle of covariance manifest itself in the mathematical properties of tensors, we imagine two observers that view a given physical relation from two different coordinate systems, one being rotated with respect to the other about the common origin O*. This rotation is specified mathematically by a* 3×3 *orthogonal matrix* $\mathfrak{R}$*. The components* (V_x, V_y, V_z) *of any vector* $\boldsymbol{V}$ *are transformed by the rotation into* $\boldsymbol{V}' = (V'_x, V'_y, V'_z)$ *such that* $\boldsymbol{V} = \mathfrak{R} \cdot \boldsymbol{V}'$*. The inverse relation*[635] $\boldsymbol{V}' = \mathfrak{R}^T \cdot \boldsymbol{V}$ *is the law of vector covariance. A triplet of numbers that transforms in this way under rotation of the axes is defined to be a physical vector.*

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[634] Of rank 2 or higher; technically, scalars and vectors are tensors of ranks 0 and 1, respectively.

[635] The axes $O(x, y, z)$ rotate into their new positions $O'(x', y', z')$. Through this transformation the vector $\boldsymbol{V}$ remains intact, but its *components* relative to the new axes are different. In the inverse relation $\boldsymbol{V}' = \mathfrak{R}^T \cdot \boldsymbol{V}$, $\mathfrak{R}^T$ is the transpose (= inverse) of $\mathfrak{R}$. For a rotation by an angle θ about the z-axis,

$$\mathfrak{R} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now, consider the law in O, $W = \boldsymbol{F} \cdot \boldsymbol{s}$, which measures the work W (scalar) done by a force $\boldsymbol{F}$ that displaces a body over a finite straight-line path $\boldsymbol{s}$. How will the same law be observed in O'? To see this we simply substitute therein the relations $\boldsymbol{F} = \mathfrak{R} \cdot \boldsymbol{F}'$, $\boldsymbol{s} = \mathfrak{R} \cdot \boldsymbol{s}'$ and $W = W'$ (a scalar is invariant *under rotation of the axes). The result is*

$$W' = (\mathfrak{R} \cdot \boldsymbol{F'}) \cdot (\mathfrak{R} \cdot \boldsymbol{s}') = \boldsymbol{F}' \cdot \{\mathfrak{R}^T \cdot \mathfrak{R}\} \cdot \boldsymbol{s}' = \boldsymbol{F}' \cdot \boldsymbol{s}'.$$

The law thus has the same form *in O' as in O.*

The mathematical form of the law is the same for any pair of observers, and therefore to all *inertial-frame observers in space. The law of vector* covariance *therefore secures the* invariance *of the physical law under rotation of the coordinates.*

A tensor of rank 2 in Cartesian 3-dimensional space is an array of 3×3 components (with respect to a given coordinate system) that obey a law of tensor covariance. *The tensor is represented by the symbol σ_{ij}, $i, j = 1, 2, 3$ (or σ), and the law of covariance can be shown to have the mathematical form*

$$\sigma' = \mathfrak{R}^T \cdot \sigma \cdot \mathfrak{R}$$

(or $\sigma = \mathfrak{R} \cdot \sigma' \cdot \mathfrak{R}^T$ in the reciprocal form).

Let σ have the physical meaning of the state of stress *at a point O in an elastic solid and let the physical law associated with it be given in O as*

$$\boldsymbol{F} = \sigma \cdot \boldsymbol{n},$$

where $\boldsymbol{n}$ is the normal vector to a plane passing through O and $\boldsymbol{F}(\boldsymbol{n})$ the force vector across this plane at O.

The observer at O' will read the same law as

$$\mathfrak{R} \cdot \boldsymbol{F}' = (\mathfrak{R} \cdot \sigma' \cdot \mathfrak{R}^T) \cdot (\mathfrak{R} \cdot \boldsymbol{n}') = \mathfrak{R} \cdot \{\sigma' \cdot \boldsymbol{n}'\},$$

or

$$\boldsymbol{F}' = \sigma' \cdot \boldsymbol{n}'.$$

Again, the physical law is of the same form *in both coordinate systems.*

Tensor analysis deals with abstract objects (entities) that are independent of the choice of the reference frames used to describe them. A tensor is represented in a particular reference frame by a set of functions called components. *As we learned in the above discussion, a given set of functions representing a tensor depends on the* law of transformation *of these functions from one coordinate system to another. But the independence of the form of the laws obeyed by the tensor upon the choice of the reference frame, provides an ideal tool for the study of* natural laws.

Indeed, whether a logical deduction based upon a conglomerate of observational facts deserves the name of natural law is often determined by the generality of such a deduction including its validity in a sufficiently wide class of reference systems. This is intimately bound up with the possibility of formulating the deduction in the form of tensor equations. The concept of covariance *of mathematical objects under coordinate transformations, is thus of prime importance in tensor analysis.*

A fundamental concept that permeates the entire calculus of tensors is that of covariant differentiation *(***Ricci***, 1884). It constitutes a generalization of partial differentiation that is covariant under general coordinate transformations.*

The basic idea behind the covariant derivative is as follows: consider a vector field in a 3D Cartesian coordinate system. The physical vectors may vary from point to point, but the Cartesian unit vectors are the same *at each point. Hence, when we come to* compare *two field-vectors at two points $P(\boldsymbol{r})$ and $Q(\boldsymbol{r} + d\boldsymbol{r})$ that are infinitesimally close to each other (differentiation is basically an operation of comparison!), the variation of the vector between these points is naturally measured in the same coordinate system, and therefore reflects the observed* physical *change of the field, as given by the ordinary partial derivative of its Cartesian components w.r.t. to the coordinates.*

Now, suppose that the same physical vector field *is quantified in a curvilinear orthogonal system in which, say, spherical coordinate are used. Since the orientation of the coordinate axes at* any *two neighboring points is now* different *(rotated), the change in the field vector between these points reflects both a true* physical *change and a superposed, artificial, change which arises from the fact that the two measurements are performed in two differently oriented local Cartesian systems.*

This last superfluous effect can be eliminated, simply by displacing the vector at P parallel to itself to the point Q and making the comparison there! *The result of this process is the so-called* covariant derivative *which expresses the rate of change of physical quantities (vectors and higher tensors) in a way that is independent of the coordinate system used.*

Let us put the above idea into quantitative form and apply it to a covariant[636] *vector field, say, in an affine space, with components $A_i(x^j)$ at a point $P(x^j)$. At a neighboring point $Q(x^j + dx^j)$, the value of the field is $A_i(x^j + dx^j) = A_i(x^j) + dA_i$. To make the comparison, we transplant A_i*

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[636] A vector is said to be *covariant* if its component transform from A_i to B_i with $A_i = \frac{\partial y^k}{\partial x^i} B_k$ upon the transformation $y^k(x^i)$. It is said to be *contravariant* if its components transform instead as $A^i = \frac{\partial x^i}{\partial y_k} B^k$.

to Q parallel to itself. However, at Q the local coordinate axes are different then those at P and therefore the new components, $\widehat{A}_i$, are not the same as they were at P. Rather, $\widehat{A}_i(x^j + dx^j) = A_i(x^j) + \delta A_i$. The difference $dA_i - \delta A_i$ will yield the true physical change of A_i. Therefore we write $dA_i - \delta A_i = A_{i\|j}dx$, where $A_{i\|j}$ are the components of a rank-2 covariant tensor known as the covariant derivative of A_i.

Explicitly

$$\begin{aligned} dA_i - \delta A_i &= A_i(x^j + dx^j) - \widehat{A}_i(x^j + dx^j) \\ &= \left[A_i(x^j + dx^j) - A_i(x^j)\right] - \left[\widehat{A}_i(x^j + dx^j) - A_i(x^j)\right]. \end{aligned}$$

The first term on the r.h.s. is simply $\frac{\partial A_i}{\partial x^j}dx^j$. The second term involves the a priori information of the extent to which the Cartesian axes have rotated from P to Q. But this is calculable through the covariance law $A_i = \frac{\partial y^j}{\partial x^i}B_j$, where $y^j(x^i)$ are the transformation functions. If $\{y\}$ is a Cartesian coordinate system, then $\delta A_i = \delta\left[\frac{\partial y^j}{\partial x^i}B_j\right] = \delta\left[\frac{\partial y^j}{\partial x^i}\right]B_j$ since $\delta B_j = 0$ in parallel displacement of Cartesian axes.

However,

$$B_i = \frac{\partial x^j}{\partial y^i}A_j, \qquad \delta\left[\frac{\partial y^j}{\partial x^i}\right] = \frac{\partial^2 y^j}{\partial x^i \partial x^k}dx^k.$$

Therefore

$$\delta A_i = \Gamma^m_{ik}A_m dx^k,$$

where the entity

$$\Gamma^m_{ik} = \frac{\partial^2 y^j}{\partial x^i \partial x^k}\frac{\partial x^m}{\partial y^j}$$

is known as the affine connection[637] *between the points of the space [a space which is affinely connected or an affine space possesses sufficient structure*

[637] The coefficients Γ^m_{ik} of the affine connection are *not* components of a tensor. A given affine connection can always be decomposed into its symmetric and skew-symmetric parts according to the usual rule

$$\Gamma^m_{ik} = \frac{1}{2}(\Gamma^m_{ik} + \Gamma^m_{ki}) + \frac{1}{2}(\Gamma^m_{ik} - \Gamma^m_{ki}).$$

The connection is said to be *symmetric* if $\Gamma^m_{ik} = \Gamma^m_{ki}$. However, $T^m_{ik} = \Gamma^m_{ik} - \Gamma^m_{ki}$ *is* a tensor, and is often referred to as the *torsion tensor* of the connection. Clearly, if the torsion tensor vanishes in some coordinate system, it will vanish in any other system, and accordingly, the symmetry condition is independent of the choice of the coordinate system.

In GTR, spacetime is both a *Riemannian* space (manifold) *and* an affine one,

to permit the operations of tensor calculus. An affine space is more general than a Riemannian space[638] *since it does not necessarily have a metric]. An*

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with the Christoffel connection determined by the metric (and vanishing torsion):

$$\Gamma^m_{ik} = \frac{1}{2}g^{m\sigma}\left(\frac{\partial g_{\sigma i}}{\partial x^k} + \frac{\partial g_{\sigma k}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^\sigma}\right).$$

In his efforts to unify the gravitational and electromagnetic field theories, **Einstein** (1928) hoped to be able to identify the *contracted* torsion tensor T^m_{im} with the electromagnetic potential. Previously, **Cartan** (1922), motivated by work of **Cassirer** on non-symmetric stress tensors in magnetic materials, developed an alternative to GTR in which the affine connection is not symmetric.

638 In a flat Riemannian space a Euclidean (Cartesian) vector $\boldsymbol{A}$ can be written as $\boldsymbol{A} = A^i\boldsymbol{g}_i$, where A^i are its *contravariant components in the given curvilinear coordinate system* and $\boldsymbol{g}_i = \frac{\partial \boldsymbol{r}}{\partial x^i}$ are the *base vectors*. Or it can be written as $\boldsymbol{A} = A_i\boldsymbol{g}^i$, where A_i are the *covariant components* and $\boldsymbol{g}^i$ is the *reciprocal base*. Now,

$$\frac{\partial \boldsymbol{A}}{\partial x^j} = \frac{\partial}{\partial x^j}(A^k\boldsymbol{g}_k) = \frac{\partial A^k}{\partial x^j}\boldsymbol{g}_k + A^k\frac{\partial \boldsymbol{g}_k}{\partial x^j}$$
$$= \frac{\partial A^k}{\partial x^j}\boldsymbol{g}_k + A^k\Gamma^m_{ki}\boldsymbol{g}_m = \left[\frac{\partial A^m}{\partial x^j} + \Gamma^m_{ki}A^k\right]\boldsymbol{g}_m \equiv A^m_{\cdot j}\boldsymbol{g}_m$$

Similarly,

$$\frac{\partial \boldsymbol{A}}{\partial x^j} = \frac{\partial}{\partial x^j}(A_k\boldsymbol{g}^k) = \left[\frac{\partial A_m}{\partial x^j} - \Gamma^k_{mj}A_k\right]\boldsymbol{g}^m = A_{m,j}\boldsymbol{g}^m.$$

The relation

$$\nabla \boldsymbol{A} = \left(\boldsymbol{g}^\alpha\frac{\partial}{\partial x^\alpha}\right)\boldsymbol{A} = A^\beta_\alpha\boldsymbol{g}^\alpha\boldsymbol{g}_\beta$$

shows that the curvilinear components of the gradient tensor are the covariant derivatives of the vector components.
Note that $\nabla\boldsymbol{g}^\alpha = \nabla\boldsymbol{g}_\alpha \equiv 0$. Moreover, the fundamental tensors g_{jk} and g^{jk} behave like constants w.r.t. covariant differentiation (*Ricci's Theorem*).

Let $\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{A}^{(0)}$ be a flat-space Cartesian vector field which does not vary from point to point, i.e. its magnitude and direction are constant. Since in this case $\frac{\partial \boldsymbol{A}}{\partial x^k} = A^m_{\cdot k}\boldsymbol{g}_m \equiv 0$, it follows that $A^m_{\cdot k} = 0$. Thus, the covariant derivative of a uniform vector field vanishes.
But a uniform field can be regarded as the result of displacing the vector $\boldsymbol{A}$ parallel to itself from some fiducial (origin) point to every point of the field. With this interpretation the equation

$$\frac{\partial A^m}{\partial x^j} + \Gamma^m_{kj}A^k = 0$$

affine space in which there exists, at each point, a coordinate system in which $\Gamma = 0$*, is said to be flat. Here the term "space" is synonymous with manifold.*

Altogether,

$$A_{i,k} = \frac{\partial A_i}{\partial x^k} - \Gamma^m_{ik} A_m$$

is the covariant derivative of a covariant vector. Likewise, the covariant derivative of a contravariant vector is

$$A^k_{\cdot j} = \frac{\partial A^k}{\partial x^j} + \Gamma^k_{mj} A^m.$$

The power of these definitions is that they, and the tensor calculus based upon them, hold for any affine space — even if it is not flat, i.e. cannot be locally reduced to a Euclidean space via coordinate transformations. In a Riemannian space, which is endowed with a (covariant, rank 2) metric tensor $g_{\mu\nu}$*, the Christoffel symbol* Γ *is that connection for which* $g_{\mu\nu,\alpha} = 0$*. Thus, with this connection, index contractions (via* $g_{\mu\nu}$*) commute with covariant differentiation.*

1888 CE Telescopic photographs reveal the spiral shape of the Andromeda Nebula. [In 1923, **Hubble** established its galactic nature.]

1888 CE **Henry Louis Le Châtelier** (1850–1936, France). Physicist. Established a principle, named after him, for the behavior of a thermodynamic system at equilibrium [its macroscopic parameters such as temperature, pressure, composition, entropy — do not depend on either time or space, i.e. the system is uniform and either isolated (closed) or in contact with uniform environment]. This principle states: "*An external influence disturbing the equilibrium of the system, induces in it processes tending to weaken the effects of this influence*".

In other words: any change in the equilibrium conditions, results in a shift of that equilibrium in the direction that will partially nullify the perturbation, and thus tend to restore the unperturbed state.

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becomes the *condition* for a vector to be its *own parallel displacement* (in any coordinate system). Such covariantly-constant vector fields can *only* exist (if A is not to be 0 everywhere) in *flat* (curvature-free) spaces.

This law, valid for *quasistatical thermodynamics*, is actually valid for a wider class of phenomena. Example: the law of electromagnetic induction introduced by **Heinrich Friedrich Emil Lenz** (1804–1865, Russia, 1834).

In 1947, **Ilya Prigogine** (1917–2003, Belgium), extended the thermodynamic principle to a wider class of stationary states, namely *open non-equilibrium* stationary states[639], where entropy-producing processes are sustained by a continual flux of energy (or matter and energy) between the system and its surrounding. In that case the stationary state is the configuration of minimum entropy. This generalization enables us to include processes with *thermal diffusion* under the umbrella of the Le Châtelier principle, where the entropy production is a *Lyapunov function* (1892).

The theory of open systems has been applied with success to many specific problems of biology, thus demonstrating that thermodynamic principles related to open systems lie at the core of central biological problems. Prigogine's theory accounts for many features of life, which can thus be treated as physical phenomena.

1888 CE **Wilhelm Hallwachs** (1859–1922, Germany). Physicist. Stimulated by Hertz's work, he showed that irradiation with ultraviolet light causes uncharged metallic bodies to acquire a positive charge (emit electrons). The earliest speculations on the nature of the effect predate the discovery of the electron in 1897.

Hallwachs demonstrated the possibility of using photoelectric cells in cameras. This property, called *photoemission*, was applied in the 20th century in the creation of the electronic television camera.

1888 CE **Frank Julian Sprague** (1857–1934, U.S.A.). Electrical engineer and inventor. Built the first large electric passenger railway system, in Richmond, VA (20 km long).

1888 CE The American Mathematical Society established. Lick Astronomical Observatory established on Mount Hamilton, California, equipped with a 36-inch refractor telescope.

1888–1891 CE **Pierre Paul Émile Roux** (1853–1933, France). Physician, bacteriologist and immunologist. Discoverer of the anti-diphtheria

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[639] Definitions:

Isolated system: Completely disconnected from its surroundings. No exchange of *energy or matter* possible.

Closed system: May exchange *energy* with its surroundings, but not matter.

Open system: May exchange *energy and matter* with its surroundings.

serum, the first effective therapy for this disease. One of the close collaborators of **Louis Pasteur**.

Roux joined Pasteur's laboratory as a research assistant (1878–1883) at the École Normale Supérieure in Paris. He worked with Pasteur in *Avian cholera* (1879–1880), *anthrax* (1879–1890) and *rabies* (1881–1883).

In 1888 he published with **Alexandre Yersin** (1863–1943) the first of his works on the causation of *diphtheria* by the *Klebs-Loeffler bacillus*. He then began (1891) to develop an effective serum to treat the disease, following the demonstration by **Emil von Behring** (1854–1917) and **Shibasaburo Kitasato**(1852–1931) that *antibodies* against the diphtheric toxin could be produced in animals. He demonstrated successfully this antitoxin in the Hospital des Enfants-Malades (1891).

In the following years, Roux dedicated himself to the immunology of *tetanus*, *tuberculosis*, *syphilis* and pneumonia. He became the director of the Pasteur Institute in 1916.

1888–1903 CE **Nikola Tesla**[640] (1856–1943, U.S.A.). An American inventor of Croatian origin. A key figure in the history of electrical technology. Invented the alternating current induction motor (*electric alternator*, known also as the electromagnetic motor) and polyphase power transmission. He also invented the *Tesla coil transformer* (produces high voltage at high frequencies), *arc lightning*, a system of *wireless transmission* (in 1893, two years ahead of Marconi (1874–1937)), a telephone repeater, rotating magnetic field principle, fluorescent light and more than 700 other patents.[641]

Tesla was born of Serbian parents in Smiljan Lika, Croatia and was raised and educated in the Austro-Hungarian kingdom. In 1882 he conceived the ideas that would form the foundation of his only truly successful inventions: the induction motor and polyphase power transmission.

In 1884, while working in Paris for the Continental Edison Company, he obtained a letter of introduction to Edison and immigrated to New York. He worked for Edison for about a year before having some kind of falling out[642].

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[640] For further reading, see:

- Cheney, M., *Tesla*: *Man Out of Time*, A Laurel Book, Dell Publishing: New York, 1981, 320 pp.

[641] Tesla patented (1903) the electrical logical circuits that become crucial to addition, subtraction, and multiplication in later computer machines.

[642] The standard story is that Edison told Tesla it would be worth $50,000 to him if he could improve upon Edison's electric generators significantly. Tesla did this and then asked Edison for his money. "*Tesla*", Edison replied, "*you don't*

Tesla then caught the attention of George Westinghouse (1846–1914), inventor of the air-brake (1868), who was looking to break into electrical technology and thought Tesla's ideas on electric power distribution had merit. Using the ideas of Tesla, Tesla and Westinghouse made commercial use of AC motors, generators and transmission lines (1891) and the polyphase AC power transmission (1893). At this time Tesla further developed his induction motor and his high voltage generator known as the *Tesla coil*[643]. In the next few years Tesla would install the world's first true commercial electric power station at Niagara Falls. He would continue to produce remarkable ideas for decades but would never again be able to finish what he started.

Tesla developed all the components needed to construct a practical radio system, but then seems to have lost interest — he never took his ideas beyond some very short-range demonstrations. This left the field to Marconi, who proved the feasibility of long-range wireless communication just a few years later. Although he anticipated Marconi and others in many ways, histories of early radio make only incidental mention of Tesla.

A U.S. supreme court decision (June 21, 1943) found that Tesla anticipated the four-circuit tuned combination of Marconi, and ruled that Tesla had anticipated all other contenders with his fundamental radio patents. [Yet the Nobel prize in physics for 1909 had gone to Marconi and K.F. Braun.]

Tesla was obsessed with the idea of wireless transmission of electric *power* (in contradistinction to wireless transmission of *information* via low-energy electromagnetic waves). He also talked of making the upper atmosphere fluoresce — abolishing the dark night forever. None of these ideas was ever realized.

Although a millionaire in the 1890's, Tesla had so indulged his appetite for expensive experiments that from the early 1920's until his death in 1943 he was nearly destitute.

In his honor, the physical mks (SI) unit of magnetic flux density, is named the 'tesla'.

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understand our American humor".

[643] *Tesla coil*: a specialized electrical transformer and spark gaps, used in circuits that produce high-voltage at high frequencies. Large Tesla coils can produce millions of volts and are used to make spectacular electrical displays, but have no important scientific or industrial applications. Today's scientists and engineers have far superior methods of producing high voltage — methods that do not derive from Tesla's work .

Science Progress Report No. 12

Tesla vs. Edison, or — the 'War of the Currents'

Alternating currents technology is rooted in the discovery of ***Joseph Henry*** *(1830, USA) and* ***Michael Faraday*** *(1831, England) that a changing magnetic field near an electric circuit, or a static one through which the circuit moves, is capable of inducing an electric current in the circuit. Earlier studies had been confined to* static *magnetic fields.*[644] *Faraday is usually given credit for the discovery since he published his results first.*

The principle of the voltage transformer was applied in 1851 by ***Heinrich Daniel Ruhmkorff*** *(1803–1877, Germany and France) in his* induction-coil *(also known as the Ruhmkorff-coil). Through this device he generated a train of unidirectional* high-voltage pulses *in an open secondary coil circuit induced by rapid mechanical make-and-break switching in a primary direct-current low resistance coil circuit. The alternations of the magnetic flux induce an emf between the ends of the secondary coil, and a high voltage is produced that tends to cause a spark or an arc to pass. If, e.g. an X-ray tube is connected between the secondary terminals, the magnetic-field energy is transformed partly into X-ray energy and partly into heat.*

Exploitation of the discoveries of Henry and Faraday began in 1887, with the construction of the first commercial alternating current (AC) power transformer *by the engineers* ***Lucien Gaulard*** *(1850–1888, France) and* ***John Gibbs*** *(England). Improvements were introduced in Budapest by* ***Otto Blathy*** *(1860–1939, Hungary),* ***Max Deri*** *(1854–1938, Hungary) and* ***Karl Zipernowsky*** *(1853–1942, Hungary) during 1881–1885.*

The American electrical engineer and inventor ***William Stanley*** *(1858–1916), using the patents of Gaulard and Gibbs built a transformer system to form an integral part of the first multiple-voltage AC power system in Great Barrington (Massachusetts, U.S.A., 1886). The network was driven by a hydropower generator producing 500 Volts AC. It was* stepped up *to 3 kV for transmission, then* stepped down *to 100 V to power electric lights. Stanley also invented two-phase motors and patented a carbonized filament incandescent lamp.*

George Westinghouse *(1846–1914, USA), Stanley's employer (an adventurous Pittsburgh industrialist and the inventor of railroad air breaks), was an early advocate of AC with great plans for the electrification of America. He bought the American rights to the Gaulard and Gibbs' patents (1885).*

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[644] Whether generated *by* naturally-occurring magnets or electric currents, and whether acting *upon* magnetic/magnetizable materials or electric currents.

*Three-phase currents were introduced into electrical engineering by **Nikola Tesla** (1887) and the Italian engineer **Gallileo Ferraris** (1847–1897), (1888). A decisive factor in bringing about the subsequent almost universal adoption of three-phase currents for the transmission of power over large distances was the successful transmission of electric power between Lauffen-on-the-Neckar and Frankfurt-on-the-Main on the occasion of the important exhibition at Frankfurt (1891), a distance of 175 km. It was accomplished by the Berlin engineer **M. Von Dolivo-Dobrowolski**.*

***Elihu Thomson** (1853–1937, USA), electrical engineer and inventor, invented the standard three-phase alternating current generator, the high-frequency transformer, the high-frequency generator(1890), the centrifugal cream-separator, the common Watt-meter, the street arc lamp (fed by alternating currents, 1878–9) and 700 other patented inventions. He became one of the great pioneers of the electrical manufacturing industry in the USA. Thomson and **Edwin James Houston** founded the Thomson-Houston Electric Company (1883), which merged with **Edison's** firm (1892) to form the General Electric company.*

Electricity was first introduced to New York in the late 1870s. Edison's incandescent lamp had created an astonishing demand for electric power, and his DC power station on Pearl Street in lower Manhattan was quickly becoming a monopoly. Edison knew little of alternating current and did not care to learn more about it. In short, AC power sounded like competition to Edison.

*In November and December of 1887, Tesla filed for seven U.S.patents in the field of polyphase AC motors and power transmission. These comprise a complete system of generators, transformers, transmissions, motors and lighting. **George Westinghouse** heard about Tesla's invention and thought it could be the missing link in long-distance power transmission.*[645]

[645] The main advantage of AC over DC (direct current) is the ease and efficiency with which AC voltages can be raised or lowered. When electric power is transmitted over long distances it is economical to use high voltage and low current to minimize the I^2R heating losses (R = resistance, I = current) in the transmission lines for the same amount of power transmitted. The voltages are stepped up or down by passive devices called *transformers* which usually operate with an efficiency of 99 percent.

In practice, the generator's voltage is stepped up to around 230,000 V at the generating station, then stepped down to around 20,000 V at a distributing station, and finally stepped down to 110–120 V at the customers utility poles. DC power is useful only in the vicinity of the generator. Its use over larger distances would require very thick wires to decrease resistance to energy flow.

He came to Tesla's laboratory and purchased his patents for $60,000.With the breakthrough provided by Tesla's patents, a full scale industrial war erupted. At stake, in effect, was the path of industrial development in the United States, and whether the Tesla-Westinghouse alternating current or Edison's direct current would be the chosen technology.

It was at this time that Edison launched a propaganda war against alternating current.[646] *He even hired a professor who went around talking to audiences and electrocuting dogs and old horses right on stage, to show how dangerous alternating current was.*

In spite of bad press, good things were happening for Westinghouse and Tesla. The Westinghouse Corporation won the bid for illuminating The Chicago World's Fair, the first all-electric fair in history. The fair was also called the Columbian Exposition — in celebration of the 400th Anniversary of Columbus discovering America. Up against the newly formed General Electric Company (the company that had taken over the Edison Company), Westinghouse undercut GE's million-dollar bid by half. Much of GE's proposed expenses were tied to the amount of copper wire necessary to utilize DC power. Westinghouse's winning bid proposed a more efficient, cost-effective AC system.

The Columbian Exposition opened on May 1, 1893. That evening, President Grover Cleveland pushed a button and a hundred thousand incandescent lamps illuminated the fairground's neoclassical buildings. This "City of Light" was the work of Tesla, Westinghouse and twelve new thousand-horsepower AC generation units located in the Hall of Machinery.

In the Great Hall of Electricity, the Tesla polyphase system of alternating current power generation and transmission was proudly displayed. For the twenty-seven million people who attended that fair, it was dramatically clear that the power of the future was AC. From that point forward more than 80 percent of all the electrical devices ordered in the United States were for alternating current.

[646] This must be attributed to Edison's lack of education and ignorance of some basic principles of physics. Tesla had a formal European education and knew better.

1888–1906 CE **Friedrich Wilhelm Ostwald** (1853–1932, Germany). Chemist. With **Arrhenius** and **van't Hoff** he established physical chemistry as a separate discipline of science. Developed new methods for measuring the rate of chemical reactions. Rediscovered catalysis, pointing out that its essence lay in its accelerating the rate of the reaction, but not creating it. Was awarded the Nobel prize for chemistry (1909). Ostwald was born in Riga, and was a professor at Leipzig University (1888–1906).

1888–1906 CE **Fridtjof Nansen** (1861–1930, Norway). Arctic explorer, marine zoologist, pioneer oceanographer, and statesman. Began the first scientific study of the Arctic Ocean (1893–1896), obtaining information about the ocean's bed, current, ice, weather, and wildlife.

In the summer of 1888, he and five other men crossed Greenland by land from east to west, a feat that experts had declared impossible. This expedition confirmed that Greenland is nearly completely covered with ice. Detailed meteorological conditions compiled during the winter of 1889 led to a better understanding of weather conditions in Northern Europe.

In 1893 he led the *Fram* expedition to the North pole. To this end he had a ship specially built to withstand the grinding ice floes[647]. The *Fram* sailed from Christiania (Oslo) in June 1893, provisioned for 5 years with a crew of 13, sailing along the coast of Siberia. On Sept. 27, upon encountering an impassable ice barrier, its engine was dismantled, a windmill set up to work the dynamo, and the *Fram* froze in and began to drift through the ice, while the crew carried on their various scientific tasks. They took meteorological, astronomical, electrical, magnetic, and hydrodynamical observations, and collected wildlife and underwater specimens.

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[647] In 1881, the steam yacht *Jeannette*, of the De-Long expedition, was crushed by the ice of the Arctic Ocean, and sank 240 km off the New Siberian Islands. Nansen had planned a ship that, skillfully reinforced, would ride up under the pressure of ice and rest on its surface until a thaw released it to float again — "*that the whole craft should be able to slip like an eel out of the embraces of the ice*".

Nansen noted that 1100 days after the sinking of the *Jeannette*, some of its objects were found by Eskimoes in drift-ice near Julianehab, on the southwest coast of Greenland, some 4600 km from where it sunk. He saw this as an evidence of the existence of a slow steady current across the polar basin. It convinced him that it was possible to drift across it in a vessel, traveling with the ice instead of fighting against it, and possibly, at the same time, reach the pole, providing that the right sort of vessel could be constructed. Nansen's plan was greeted with skepticism, if not derision, by most Arctic experts. However, the *Fram* (forward) *did drift* for 35 months, carrying Nansen and his crew to within about 640 km south of the Pole.

In March 1895, at 84°N, Nansen and Hjalmar Johansen left the *Fram*, taking with them 2 kayaks, three sledges and 28 dogs. On April 8 Nansen hoisted the Norwegian flag in 86.13°N, 95°E, 438 km of the North Pole, nearer than anyone before him. They could go no further. On June 17, 1896 they reached Cape Flora and met some of their friends.

Nansen used his fame to facilitate his entry into Norwegian and international politics, as organizer of the Leage of Nations Refugee Work, and inventor of the 'Nansen Passport for Stateless Persons' (resulting from the collapse of the European empires and the revolution of 1917–1922). For that he was awarded the Nobel Peace prize in 1922.

1888–1906 CE **Georges Fernand Isidore Widal** (1862–1929, France). Distinguished physician. Laid the foundations to *citodiagnosis* and contributed to *pathological physiology.* Known for his pioneering work on bacteria agglutination and its applications ('*Widal reaction*') to the serological diagnosis of typhoid fever (1896). Recognized (1906) the value of salt-deprivation in nephritis and cardiac edema.

Widal was born to Jewish parents in Alger, studied medicine in Paris and served as a professor at the University of Paris (1911–1929).

1888–1910 CE **Salvatore Pincherle** (1853–1936, Italy). Mathematician. Founded (together with Volterra) *functional analysis.* Contributed to functional equations, the theory of functions, the expansion of functions in infinite series,[648] and to abstract linear spaces.

Pincherle was born in Trieste of a Jewish family. A student of **Betti**. Professor at the University of Bologna (1881–1928). Pincherle worked on a formal theory of linear operators on an infinite dimensional vector spaces, basing his work on the abstract operator theory of **Leibniz** and **d'Alembert**, but not on that of Peano. His work had little immediate impact. Axiomatic infinite dimensional vector spaces were not studied again until **Banach** and his associates took up the subject in the 1920's.

1889 CE **Sophia (Sonya) Vasilyevna Kovalevsky**[649] (1850–1891, Russia). Outstanding woman mathematician of the 19^{th} century. A favorite pupil

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[648] *Pincherle's expansion* (1896): for every $\Phi(z)$, analytic near $z = 0$, the series

$$f(z) = \sum_{n=0}^{\infty}[1 + \lambda_n e^z]\frac{d^n\Phi(z)}{dz^n},$$

where $\lambda_n(z) = -1 + z - \frac{z^2}{2!} + \frac{z^3}{3!} - \cdots + (-)^{n+1}\frac{z^n}{n!}$, is convergent near $z = 0$, and $f'(z) = f(z) - \Phi(z)$.

[649] For further reading, see:

of **Weierstrass**. She is remembered today mainly because of her solution of Euler's equations for the motion of a spinning symmetrical top under gravity. She was able to find a third integral for the special case $A = B = 2C$ where the center of gravity lies in the equatorial plane of the body. The solution can be made to depend on integrals of the form $\int \frac{dx}{f(x)}$, where $f(x)$ is an rational function of the fifth degree. She also contributed to the theory of partial differential equations, where the '*Cauchy-Kovalevsky theorem*' bears her name.

Recently her name was assigned to a crater on the moon. She is thus one of less than a dozen women from all of history to be so honored.

1889 CE **Alexandre Gustave Eiffel** (1832–1923, France). Structural and aeronautical engineer. Designed the *Eiffel Tower* in Paris for the World's Fair of 1889. The tower rises 300 meters from a base 101 m^2. Elevators and stairways lead to the top. It contains about 6400 tons of iron and steel and cost over one million dollars[650].

1889 CE **Otto Ludwig Hölder** (1859–1937, Germany). Mathematician. Discovered one of the most useful *inequalities* of analysis, the *Hölder Inequality.* This states that if x and y are positive, if $x + y = 1$ and if the numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are nonnegative, then

$$\sum_1^n a_i^x b_i^y \le (\sum_1^n a_i)^x \cdot (\sum_1^n b_i)^y$$

- Kennedy, D.H., *Little Sparrow: A Portrait of Sophia Kovalevsky*, Ohio University Press: Athens, OH, 1983, 341 pp.
- Cooke, R., *The Mathematics of Sonya Kovalevskaya*, Springer-Verlag: New York, 1984, 234 pp.

[650] In order to minimize construction materials cost (steel, iron), the compressive strength of the structure was fully exploited. To this end, it must be required that the gravitational compressive stress at any horizontal cross-section is made independent of the height of this section above the ground. Mathematically: $\rho g \int_x^\infty A(x)dx/A(x) = K$, where $A(x)$ is the area of the cross-section at level x, ρ is the density of steel, and g is the acceleration of gravity.
Differentiation yield a differential equation for $A(x)$, the solution of which is $A(x) = A_0 e^{-\lambda x}$, where A_0 is the base area, and $\lambda = \rho g/K$.
The shape of the tower's profile is therefore:

$$y(x) = y_0 \exp\left[-\frac{\rho g x}{2K}\right]$$

.

or equivalently

$$\sum_1^n a_i b_i \leq (\sum_1^n a_i^{\frac{1}{x}})^x (\sum_1^n b_i^{\frac{1}{y}})^y.$$

Equality holds iff $a_i^{\frac{1}{x}} = K b_i^{\frac{1}{y}}$, K constant. The special case $x = y = \frac{1}{2}$ is known as Cauchy–Schwarz Inequality (1821) and has a simple geometrical interpretation (i.e. the cosine of the angle between two vectors may not exceed 1).

Hölder's Inequality holds for complex numbers: If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ then $|\sum_1^n a_i b_i| \leq (\sum_1^n |a_i|^p)^{\frac{1}{p}} (\sum_1^n |b_i|^q)^{\frac{1}{q}}$. Moreover, it holds also for *integrals*, where integration takes the role of summation: if f and g are continuous real-valued functions defined on $[a, b]$, if $p > 1$, and if $\frac{1}{p} + \frac{1}{q} = 1$, then

$$|\int_a^b f(t)g(t)\, dt| \leq \int_a^b |f(t)g(t)|\, dt \leq (\int_a^b |f(s)|^p\, ds)^{\frac{1}{p}} (\int_a^b |g(t)|^q\, dt)^{\frac{1}{q}}.$$

Hölder was born in Stuttgart. He studied at Berlin under Weierstrass, Kronecker and Kummer (1877–1882), and became a professor at Tübingen (from 1889).

1889 CE **Oskar Minkowski** (1858–1931, Germany). Distinguished physician and endocrinologist. With **Joseph von Mering** discovered the direct connection between the pancreas and diabetes which led to the discovery of *insulin* (they found that the pancreas supplies a hormone essential to glucose metabolism).

Oskar was born in Lithuania, to Jewish parents. He was professor at Strasbourg (1891–1904), Cologne (1904), Greifswald (1905–1909), Breslau (1909–1926). He was the brother of the physicist **Hermann Minkowski**. Both converted to Christianity to be able to pursue their academic careers.

1889–1890 CE Great influenza epidemic afflicted 40 percent of the world population. Millions died.

1889 CE **Otto Lilienthal** (1848–1896, Germany). Inventor and aeronaut. Designed (with the assistance of his brother Gustav) and flew the first gliders that can soar above the height of takeoff. Their observation of the takeoff of storks *against the wind*, brought to aviation one of its first breakthroughs at the end of the 19th century.

Born in Anklam, Pomerania, Lilienthal and his brother studied the flight of birds and while still at school succeeded in constructing s glider. Lilienthal's

theory was that artificial flight must follow the principles of bird-flight. His experiments extended over a period of 20 years — building many gliders and executing over 2000 flights. He demonstrated (1891) the superiority of curved wings over flat-surfaced type. Wrote pioneering book on aeronautics (*Des Vogelflug als Grundlage der Fliegekunst*, 1889; *Die Flugapparate*, 1894). While on flight on Aug 9, 1896, near Rhinow, Germany, his machine was upset by a sudden gust of wind and he was killed. His work was continued by the Wright brothers (1903) who inherited his tenacity and perseverance.

1889–1899 CE **Rudolf Christian Karl Diesel** (1858–1913, Germany). Mechanical engineer, inventor, industrialist. Invented the *'compression-ignition'* engine. In a paper *'The theory and construction of an economical heat engine'* (1889) he proposed a more efficient engine than the petrol engine in which no carburetor or ignition system would be required since spontaneous ignition would occur as the fresh-air mixture was compressed at constant pressure.[651]

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[651] The ordinary petrol engine draws its heat supply from the combustion of petrol vapor in the pressure of air: vapor and air are mixed in the carburetor. A suitable mechanism causes the inlet and exhaust valves to open and close at the appropriate times, and a spark to pass through the compressed charge at the right moment.

In the Diesel cycle, the working substance (air) is raised to a very high temperature by adiabatic compression. The fuel is injected in a liquid form into the cylinder during the first part of the outward motion of the piston. The rate of injection is carefully controlled so that the pressure on the piston during the supply of the fuel is maintained constant. Thus the air is heated at *constant pressure*, instead of at constant volume as in the petrol engine.

The Diesel cycle, while less efficient than the *Carnot cycle*, is more efficient than the *Otto cycle* working between the same temperatures.

The *thermal efficiency* (ability to convert stored chemical energy in the fuel into mechanical energy) of the *Otto cycle*, assuming the air-fuel mixture to be an ideal gas, is $e = 1 - \left(\frac{v_2}{v_1}\right)^{\gamma-1}$. For a typical compression ratio of $\frac{v_2}{v_1} = \frac{1}{8}$ and $\gamma = 1.4$ a theoretical efficiency of 56% is predicted for an engine operating in the idealized Otto cycle. This is much higher than what is achieved in real engines (15% or 20%) because of such effects as friction, heat loss to the cylinder walls and incomplete combustion of the air-fuel mixture.

The efficiency of an idealized Diesel cycle is given by $e \approx 1 - \frac{1}{\gamma}\left(\frac{v_2}{v_1}\right)^{\gamma-1}$. With $\frac{v_2}{v_1} = \frac{1}{16}$, the theoretical limit is 76.4%. The difference is due to both a higher compression ratio and a higher combustion temperature. The realizable efficiency is about 30 or 35 percent.

A two-stroke gas engine was patented (1881) by the Scottish engineer and

Applying the idea of **Sadi Carnot** (1824), according to which the output from a reversible movement motor depends on the temperature at which it operates, Diesel sought to modify the *piston cycle*, for that was what caused the heat loss. In a first patent (1892), revised in 1893, Diesel proposed to obtain the necessary heat for the air-fuel mixture in the classical combustion engine to burn, not by using a spark but by a very *high compression of air alone* which would be enough to bring the air to the required temperature. The injected fuel then burns to vapor in the cylinder and the pressure of the hot gases pushes the piston.

Up to this point the principle of the diesel engine is not much different from the 4-stroke engine, except that the spark is removed. But in other respects Diesel modified the design of his engine in such a way that the compression of the air took place *outside* the cylinder; thus it was the compressed air which injected directly into the cylinder. Moreover, the injection of compressed air pushes the gases from the burnt fuel through openings at the base of the cylinders; the track of the piston is reduced for it no longer has the space to descend to the base of the cylinder. Thus heat loss was reduced by further reduction of the piston track. In this respect, Diesel was following an idea of **James Joule** (1885) who was first to try to design Carnot's ideal engine, which he did by using a *porous piston* through which the exhaust escaped.

Diesel also introduced a significant factor of *economy*: since the high volatility of high grade petrol (such as gasoline) is not required, heavier and less refined fuel oil are sufficient. Indeed, in his first engine, Diesel had used *coal dust* as a fuel, but he later discarded this along with several other types in a favor of a form of refined mineral oil.

The Diesel motor rapidly proved its *reliability* and the superiority of its output: it is easy to manufacture, strong and hardly ever breaks down and it costs less to operate since unrefined oil is decidedly cheaper. On the other hand, the Diesel engines are heavier than petrol engines of comparable horsepower, for the cylinders must withstand the high pressure and the engine must also accommodate a separate fuel pump to inject the oil into the cylinder at high pressure. This engine thwarted the intellectual habits of the engineers of the time and it took 20 years to become widely used. It was criticized firstly for its weight and for the noise it made when working and the particularly unpleasant smell of its exhaust.

Diesel himself was one of the main reasons why industry took so long to adopt his engine: until his death the engineer actually demanded that the engines built under license fitted his rigorous specifications and, in particular,

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inventor **Dugald Clerk** (1854–1932), known as the *Clerk Cycle engine*. It was used for large gas and small petrol engines.

that they were designed to function at a constant temperature, for he wanted to keep strictly faithful to Carnot's theory. The problem was that the engine functioned much too slowly when kept at constant temperature; in order to attain the desired output it had to be much more powerful (higher running speeds).

The diesel engine fulfilled its true potential when it was improved after its inventor's death. The *automotive* diesel was first built in the US (1923), and became popular among farmers during the Depression.

The diesel engines has greatly increased the efficiency of industry and transportation. They are used chiefly for heavy-duty work: they drive high freight trucks, large buses, tractors, and heavy road-building equipment. They are also used to power submarines and ships, and the generating of electric-power stations in small cities.

Diesel was born in Paris, France of German parents. They moved to Germany after the outbreak of the Franco-Prussian War (1870) and Rudolf studied at Munich Polytechnic. He was trained as a refrigeration engineer and the idea of the compression-ignition first occurred to him at the age of 19 (1878). When he first built his engine (1893), it exploded and almost killed him, but it proved that fuel could be ignited without a spark. At the same year he had taken his first patent. **Friedrich Krupp** backed the project and the engine bearing Diesel's name was created (1897). In 1899 he founded his own manufacturing company in Augsburg.

License fees on Diesel engine soon made him a millionaire. Diesel was a proverbial success for 15 years. He combined his inventive talents with the social skill of a modern executive, being competent, widely traveled, and fluent in various languages. He apparently committed suicide when he vanished without trace from a cross-Channel steamer (29 April, 1913).

1889–1907 CE **Henri Louis Bergson** (1859–1941, France). A philosopher, who at the end of the 19^{th} century undertook a search for an acceptable alternative to the science of his time. His philosophical system represents the revolt against the 19^{th} century materialism and the reduction of psychology to physics.

The primacy of mathematics and mechanics in the development of modern science, and the reciprocal stimulation of industry and physics under the common pressure of expanding needs, lent to speculation a materialistic flavor; and the most successful of the sciences became the models of philosophy. Despite **Descartes**' insistence that philosophy should begin with the self and travel outward, the industrialization of Western Europe drove thought in the direction of material things. It was **Schopenhauer** who first emphasized in modern thought the possibility of making the concept of life more fundamental

and inclusive than that of force; it is Bergson who has taken up this idea, and has almost converted a skeptical world to it by the impact of his sincerity and eloquence.

Bergson was born in Paris of Jewish parentage[652]. He specialized first in mathematics and physics, but in 1881 turned spontaneously to philosophy. He was a professor of philosophy at the École Normale Superieure (1897–1900) and the College de France (1900–1921). Awarded the Nobel prize for literature in 1927. His influence extended far beyond the realm of philosophy into such areas as literature, the social sciences and religion (e.g., the various attempts of writers such as Virginia Woolf, Luigi Pirandello, Marcel Proust to penetrate beneath the static images and facsimiles of the self and to render the flux of consciousness, owe much to him). Moreover, Bergson's philosophy originated a new philosophical attitude, revolutionary in its impact on thought. It was a great liberating force from over-intellectualized modes of thought.

Bergson recognized the three weak cleavage planes of modern knowledge: between *matter and life*, between *body and mind*, and between *determinism and choice*. On the first issue, after a hundred years of theory (since **Pasteur**), and many vain experiments, the materialists were no nearer than before to solving the problem of the origin of life. On the second issue, the mode of connection of thought and brain was as mysterious as it had ever been; consciousness could not be yet explained in terms of an electromechanical neural model. Finally, he rejected any materialistic mechanism that would claim that a sonnet of Shakespeare 'evolved' from the primeval nebula of the solar system.

In his three major works: *Time and the Free Will* (1889), *Matter and Memory* (1896), and *Creative Evolution* (1907), Bergson advanced his basic psycho-physical credo, which he believed capable of tackling the above three tasks:

- *Time*: One must take a sharp distinction between "mathematical time" (objective Newtonian time) and lived time (duration). The former is just a succession of instantaneous states linked by a deterministic law, a quantity without quality, a form of space.

  Duration, on the other hand, is the essence of life, and perhaps all of reality. It exhibits itself in *memory*. Lived time (duration) means that

[652] The name Bergson stems from Berkson (the son of Behr), an illustrious Jewish family of Warsaw, Poland, that descended from Samuel Zbitkower, the financial advisor of the last Polish King Stanislas Poniatowski (king: 1764–1795). In 1891, Bergson married a cousin of the novelist **Marcel Proust** (1871–1922), whose own writings were influenced by the philosophy of Bergson.

the past endures and nothing of it is quite lost. Life is a matter of time rather than of space. It is not *being*, it is *becoming* and change. It is not redistribution of matter and motion, it is fluid and persistent creation, a constant flow from the past into the future.

- *Intellect versus intuition* (instinct): Pure *perception*, which is the lowest degree of the mind (mind without memory) is really part of matter.

  The brain is a system of images and reaction-patterns. The part of our minds which we call the *intellect* was developed, in the process of evolution, to understand and deal with material, spatial objects; from this field it derives all its concepts and its "laws", and its notion of a fatalistic and predictive regularity everywhere. Our intellect is intended to secure the perfect fitting of our body to its environment, to represent the relations of external things among themselves. It is at home with solid, inert things; it sees all becoming as being, as series of states; *it misses* the connective tissue of things.

  In other words, the intellect, for practical purposes, introduces measurements and substitutes for qualitative processes as abstract, spatialized representations of reality; whereas the intellect is connected with space, *intuition* is associated with *time*. It is a way of thinking in duration. Intuition apprehends the true nature of things. It is essentially the most trustworthy guide to understanding. It does not falsify things by analyzing them. *Consciousness* is the recall of images and the choice of reactions.

- Strict *determinism* is unacceptable. To break the chain of deterministic evolution one must relate time to life, to mind, to choice and free-will.

- The concepts of *physics* are inappropriate in the world of the mind. The essence of life is mind, not matter; time, not space; action, not passivity; choice, not mechanism. Intuition, as a form of speculative knowledge, is the only means through which we can restore primary flexibility into scientific methods. Geometrical predictability, which is the ultimate goal of a mechanical science, is only an intellectual delusion[653].

At the end of his life, Bergson leaned toward Catholicism, but the persecution of the Jews by the Nazis caused him to identify with the Jewish

[653] Bergson had obviously misunderstood Einstein's theory of relativity. An historic scene took place on April 6, 1922, when Henri Bergson attempted to defend the cause of multiplicity of coexisting "lived" times against Einstein. Einstein's reply was absolute: he categorically rejected "philosopher's time", stating that *lived experience cannot save what has been denied by science*.

cause. After the collapse of France (1940), the Vichy government offered him exemption from the Jewish laws, patterned after the Nuremberg Laws. Bergson declined the offer and resigned his professorship from the College de France. Sick and enfeebled he stood for hours in que lines for food and daily commodities, with his coreligionists, loyal to the end to his brethren.

Over a century has passed since Bergson published his first book. With hindsight perspective we can say that

"*his grand attempt to limit the scope of modern science, as well as to open new avenues alien to those of science — has failed*[654]*. He has failed insofar as the methaphysics based on intuition he wished to create has not materialized, although the problems which he identified are still our problems. The limitation of the science of his day* (*which he erroneously attributed to science in general*) *are beginning to be overcome, not by abandoning the scientific approach or abstract thinking but by perceiving the limitations of the concepts of classical dynamics and by discovering new formulations valid in more general situations.*

Bergson's case convinces us that only an opening, a widening of science can end the dichotomy between science and philosophy. This widening of science is possible only if we revise our conception of time. To deny time — that is, to reduce it to a mere deployment of a reversible law — is to abandon the possibility of defining a conception of nature coherent with the hypothesis that nature produced living beings, particularly man. It dooms us to choosing between an antiscientific philosophy and an alienating science".

1889–1928 CE **Santiago Ramon y Cajal** (1852–1934, Spain). Histologist. A pioneer of modern neurophysiology. First to formulate the *neuronal theory* (based on the individuality of the nerve cell) which replaced the older view of a reticular system of nerve channels through which impulses were distributed. In his research he was able for the first time, to display the structure of individual cells and the contact of dendrites with adjacent cells by modifying a hitherto unreliable method of *staining*. This new staining technique

[654] Quoted from *Order Out of Chaos* by Ilya Prigogine and Isabelle Stengers, Bantam Books, New York, 1984.

Bergson failed in this respect because he was too deeply versed into the physical doctrine of his time: The equilibrium thermodynamics of the 19^{th} century was based on the second law, which predicted a gradual disorganization of the system. It could not account for the daily observations which showed the reverse phenomena. Consequently, *vitalistic theories* were invoked whereby it was suggested that biological organisms obey laws that are not part of ordinary physics and chemistry.

also provided for long-distance *tracing of axons* to other parts of the brain or junction with other nerve bundles.

Cajal was born in Petilla de Aragon, Navarra. After taking his degree in medicine at Zaragoza University in 1873, he joined the Spanish army as a medical officer, serving in Cuba during the Spanish-American war. From 1892 he held the chair of histology at Madrid University, and in 1906 he received the Nobel prize for medicine, shared with **Camillo Golgi**.

1889–1930 CE **Herbert Henry Dow** (1866–1930, USA). Chemist and manufacturer. Discovered electrolytic method for extracting bromine from brine (1889); organized chlorine-extracting firm (1895); founded Dow Chemical Co. (1897). Developed and patented over 100 chemical processes.

Dow was born in Belleville in Ontario, Canada. He graduated from Case School of Applied Science (1888) with a B.S. degree.

During Dow's lifetime, the company obtained its bromine, chlorine, sodium, calcium, and magnesium from the brine (sea water) of ancient seas under Midland, Ohio. But Dow, like **Fritz Haber**, in Germany, developed experimental processes to mine modern seas.

Three years after his death, his company opened its first seawater plant in North Carolina. By WWII, Dow plants on the Gulf Coast were in position to supply magnesium for firebombs and to make lightweight parts for airplanes.

1890 CE **Alfred Marshall** (1842–1924, England). Economist. A founder of the school of *neoclassical economics*. Professor at Cambridge University (1845–1908).

Previously, the mechanism of supply and demand was considered only in a single market that is assumed to be an infinitesimally small but representative fraction of the whole economic system (*microeconomics*). Marshall's analysis covered markets for factors of production (labor, land, etc) as well as commodities; and it made pioneering contributions to the study of adjustment processes and stability, notably in applying the concept of *elasticity*[655] and in distinguishing among time periods required for different types of adjustment (capital costs and the like being fixed in the short run but variable in the long run).

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[655] A quantitative method to measure the public's responsiveness to a price change. Such a measure is given by the ratio of the percentage of change in *demand* to the percentage of change in price. The ratio when x units are sold is known as the *price elasticity*:
If the price p is regarded as a function of the demand x, and a change in demand Δx corresponds to a change of price Δp, then the elasticity $E(x)$ of the price

1890 CE **Herman Hollerith** (1860–1929, U.S.A.). Statistician. Invented the electromechanical *punched-card* calculating machine. It was the first major advance of automatic computing since **Babbage**.

In 1886, the returns of the 1880 U.S. census were still being counted and sorted and it was clear that, with the methods then existing, the job would still be unfinished in 1890, when the next census was due. Hollerith, on the staff of the U.S. Bureau of the Census, saw that the solution lay in some measure of mechanization, and set about the task of devising suitable equipment. He was familiar with the punched-card system of control used on the *Jacquard* looms (1805), and realized that the answer to many census questions, which are of the 'yes' or 'no' type, could be represented by the presence or absence of a hole in a particular position on a Jacquard type card. The answers to more complex questions could be represented in coded form by the presence or absence of holes in a *group of positions*. He also realized that the positions of holes in a card could be detected by electrical means: the presence of a hole would allow a current to flow through; the absence of a hole would stop it.

Hollerith experimented with devices based on this principle for sorting and counting — the main census operations — and some of his machines were used for analyzing the U.S. Census in 1890. Thereafter progress was rapid: the range of 'Hollerith' machines was extended to deal with most of the operations of office arithmetic.

During the first half of the 20^{th} century, punched-card equipment has been extensively applied to the ever increasing mass of clerical work in commerce, industry, and administration — and to a lesser extent, to scientific and technical calculations.

1890–1901 CE **Emil Adolf von Behring** (1854–1917, Germany). Microbe-hunter, bacteriologist, physiologist. Pioneer in immunology. Discovered *antibodies*. Explained that both tetanus and diphtheria immunity depend

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with respect to the demand is defined by

$$E(x) = -\frac{p(x)}{xp'(x)}$$

where $p(x)$ is the price per unit of an item when x units are demanded, or sold. The price function is said to be *elastic* when $E(x) > 1$ and *inelastic* for $E(x) < 1$. The second case indicates that a decrease in price is accompanied by a decrease in total revenue, while in the first case a decrease in price will increase the total revenue. The concept of 'elasticity of demand' was previously introduced by **Cournot** (1838).

on the capacity of the cell-free blood serum to neutralize the toxic substance produced by the tetanus/diphtheria bacilli. Developed vaccine against tetanus and introduced the concepts of passive immunization and antitoxins[656] (1890).

Behring was born at Deutsch-Eylau. Worked at the Koch Institute of Hygiene, Berlin (1889–1994); professor at Hale University (1894–1851), Marburg (1895 ff). Won the Nobel prize for physiology or medicine (1901).

Seismology[657] — Birth of a New Science (1889–1936)

Early historical records contain references to earthquakes as far back as 2000 BCE. ***Aristotle*** *(ca 340 BCE) gave a classification of earthquakes into six types, according to the nature of the earth movement observed; for example, those which caused an upward earth movement, those which shook the ground from side to side, etc.*

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656 *Antitoxin*: A substance with the ability to counteract the effect of toxin or poison; the specific antibody capable of neutralizing the pathogenic toxin.

Toxin: The Greek word for bow is *toxon*. The Greeks used *toxikon* for the poison in which the arrow was dipped; hence the English *toxin*, *toxic*, *antitoxin*.

Poison: Was originally a harmless draught which the Old French borrowed from the Latin *potionem* from *potare*, *potum* = to drink; but with the medieval practice of lethal beverages it took on its fatal sense.

This etymology has yet another twist: the word tocsin is composed of two parts *toc* (knock on a door) + the Latin *signum* which together implies: alarm, bell!

657 For further reading, see:

- Ben-Menahem, A. and S.J. Singh, *Seismic Waves and Sources*, Dover: New York, 2000, 1102 pp.

The earliest instrument made to respond to earthquake ground motion, known to us, is the seismoscope, invented in 132 *CE by the Chinese scholar* ***Chang Heng****. It consisted of a column so suspended that it could move in one of* 8 *directions; a ball was held lightly along each of these lines and, when thrown down by the rod, was caught in a cup below and so revealed the direction of motion. [Later seismoscopes were designed to give the time of occurrence of a shock: They were equipped with horizontal rod lightly pivoted at one end and provided with teeth below so that, when the rod fell, the teeth caught a pin projecting from the pendulum of a clock.] This instrument is reputed to have detected some earthquakes not felt locally.*

The ancients attributed earthquakes to supernatural powers; indeed, a writer in the Philosophic Transactions of the Royal Society of London, as late as 1750 *CE, deemed it expedient to apologize to 'those who are apt to be offended at any attempts to give a natural account of earthquakes'. Notwithstanding, stubborn facts of earthquake effects continued to accumulate, especially in the wake of the disastrous Lisbon earthquake of* 1755.

Finally it was firmly established in 1760 *by* ***John Michell*** *(England) that earthquakes originate within the earth. He declared that "earthquakes were waves set up by the shifting masses of rock miles below the surface… the motion of the earth in earthquakes is partly tremulous and partly propagated by waves which succeed each another", and he estimated that the earthquake waves after the Lisbon earthquake had traveled outward at* 530 *m/sec.*

Most of the work on earthquakes during 1760–1840 *was concerned with appraisals of geological effects of earthquakes, and of effects on buildings. Early in the* 19^{th} *century, earthquakes lists were being regularly published, and in* 1840 *there appeared the first earthquake catalogue for the whole world.*

Meanwhile a great deal of progress had been taking place on the theoretical front, namely the theory of elasticity. In 1638, ***Galileo*** *investigated the behavior of a loaded beam attached at one end to a wall. He found that with increasing load the beam bends around an axis perpendicular to its length and situated in the plane of the wall. Even though he did not give any mathematical relations between load and deformation, his works were pioneering in elasticity theory.*

In 1660 ***Robert Hooke*** *established the linear relationship between stress and strain in one dimension, which forms the basis for the mathematical theory of elasticity, and still serves as a good first approximation to the elastic conditions in the earth.*

During 1821–1830, *the French mathematicians* ***Navier****,* ***Cauchy*** *and* ***Poisson*** *laid the foundation to the mathematical theory of dynamic elasticity relevant to seismology. In particular, Poisson (1828) predicted the existence*

of longitudinal and transverse waves, moving with different speeds in the interior of perfectly elastic substances (known in seismology as P and S waves, respectively). In 1845 ***Stokes*** *defined the moduli of compressibility* and *rigidity* for *isotropic elastic bodies, and in* 1849 *he conceived the first mathematical model of an earthquake point-source.*

In 1857, *the first true seismologist (as we would now recognize the term in hindsight), appeared on the scene: He was* ***Robert Mallet***[658] *(1810–1881, Ireland), the engineer who laid the foundation of instrumental seismology.*

The first seismometer[659], *worthy of the name, was designed in* 1841 *by the physicist* ***James David Forbes*** *(1809–1868, Scotland). It consisted of an inverted pendulum, hinged below by a cylindrical steel wire. A pencil attached to the top of the pendulum rod, recorded the motion on paper.*

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[658] He was born in Dublin, and after taking his degree at Trinity College in that city, he went into his father's small engineering factory. After building a lighthouse and a number of bridges, he became interested in global seismicity and earthquake engineering problems. His detailed study of the damage caused by the Napolitan earthquake of 1857 led him to suggest the setting up of a network of observatories over the earth's surface. He published the first world seismicity map (including material from many books) and made the first systematic attempt to apply physical principles to earthquake effects (1860–1862). He made estimates of the epicentral depth and also carried out a number of experiments to determine the velocity of earth waves, by setting off charges of explosives in different soils and by measuring the effects on *bowls of mercury* set at varying distances up to 800 meters away.

[659] The name was coined by **David Milne Home** in 1841. A few years later, the name *seismograph* was given to an instrument built by **Luigi Palmieri** (1855) in the observatory on Vesuvius.
The word derives from the Greek $\sigma\epsilon\iota\sigma\mu\acute{o}\delta$ = earthquake. A *seismometer* is an instrument that amplifies and records small movements of the ground. Most sensitive seismographs magnify ground motion by as much as ten million times. It consists of a weight suspended from a frame by a spring. The frame moves with the ground, but the mass, due to its inertia, tends to remain stationary (evidently, any instrument containing a pendulum can be considered as a kind of seismograph). The *relative* motion between the mass and the frame is magnified by using an electromagnetic transducer and an electronic amplifier. The amplified signal controls a recording device that displays the ground motion in analog or digital form. Seismographs can detect ground movements of the order of an Angström (10^{-8} cm). Most seismographs are designed to measure ground *velocity*. Others are capable of monitoring ground *displacements*, *accelerations*, and *strains* (extensions, tilts, rotations).

The first useful seismograph system was constructed in Japan in 1880 *by* ***John Milne*** *and his assistants* ***James Alfred Ewing*** *and* ***Thomas Gray****. But this instrument had insufficient magnification and could record only Japanese earthquakes. However, on April 1889,* ***Ernst von Rebeur-Paschwitz*** *(1861–1895, Germany) was experimenting in Potsdam with a modified form of* ***Zöllner****'s horizontal pendulum* $(V_0 = 50,\ T_0 = 18\ sec,\ no\ damping)$ *when an earthquake from Japan was recorded. This event marks the birth of instrumental seismology in its world-wide sense.*

Stimulated by these observations, Milne was able by 1893 *to design, construct and test the now famous seismograph which bears his name. It was capable of detecting earthquake waves which had traveled many thousands of kilometers from their origin. Moreover, it was sufficiently compact and simple in operation to enable it to be installed and used in many parts of the world. It could record all three components of the ground motion (up-down, east-west, north-south). From this time onwards, precise instrumental data on earthquakes began to accumulate, and seismology has developed from the qualitative towards the quantitative side.*

The seismograph is to the earth scientist what the telescope is to the astronomer — a tool for peering into inaccessible regions. For that reasons one may consider the year of the deployment of the Milne seismographs as an important milestone in the history of seismology. Indeed, since 1893*, the number of instrumentally recorded earthquakes steady increased; the earliest known list of earthquakes with computed origin-times and epicenters is that for the period* 1899–1903*. Further improvement in the design of seismographs was due to* ***Emil Wiechert*** *(1861–1928, Germany) who gave a detailed account of his mechanical seismograph*[660] *(1900) and* ***Boris Borisovich Golitzin*** *(1862–1916, Russia) who designed the first electromagnetic seismograph with*

[660] **Wiechert** designed a seismograph in which the pendulum is vertical and inverted, being maintained by small springs pressing against supports rigidly attached to the ground. The mass of the pendulum is large (up to several tons), and the seismograph records both horizontal components at once. A cardinal development took place when **Golitzin** introduced the idea of recording ground motion by means of a ray of light reflected from the moving mirror of a galvanometer: the motion of the mirror is excited by an electric current generated by electromagnetic induction when the pendulum of the seismometer moves. The strain seismometer measures the variation in the distance between two points, some 30 meters apart, caused by the passage of seismic waves. **Benioff**'s recording was electromagnetic, the original galvanometer period being 40 sec, subsequently increased to 480 sec. His strain seismograph was the first to record earth motions with periods up to the order of one hour, such as the gravest mode of the *free oscillations of the earth* (1952).

photographic recording (1906). The next development came in 1935, *when* ***Hugo Benioff*** *(1899–1968, U.S.A.) designed and constructed an instrument to measure a component of ground strain, instead of the usual ground displacement.*

The science of seismology aims simultaneously to obtain the infrastructure of the earth's interior with the aid of seismic wave phenomena, and to study the nature of earthquake sources with the ultimate goal of mitigating and eventually controlling the phenomenon. This double feature is apparent from the early days of the science.

The achievements toward the first goal began in 1799, *when* ***Cavendish*** *employed Newton's law of universal gravitation to estimate the earth's mean density* $\left[\langle\rho\rangle = \frac{3}{4\pi G}\frac{g(R)}{R} \simeq 5.5\ \frac{g}{\text{cm}^3}\right]$. *As this density exceeded the density of surface rocks, the conclusion was that the density must increase with depth in the earth. By means of observations of the tidal effect in the solid earth,* ***Lord Kelvin*** *claimed in* 1863 *that the earth as a whole is more rigid than glass. [This opinion has been confirmed later, when it was found that steel offers a better comparison, where the gravest mode of the earth's free oscillation is concerned.]*

In 1897, ***Wiechert*** *conjectured from theoretical calculations that the earth's interior consists of a mantle of silicates, surrounded a core of iron. The existence of the earth's core was established by* ***Richard Dixon Oldham*** *(1858–1936,* India and England*) in* 1906, *from observations of earthquake waves.*

In 1909, ***Andrija Mohorovičic*** *(1857–1936,* Croatia*) discovered*[661] *a sharp material discontinuity at some level below the earth's surface (known*

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[661] This was known to **John Milne** already in or prior to 1906! In his *Bakerian Lecture* delivered March 22, 1906, and published in the *Proceedings of the Royal Society of London A* **77**, 365–376, he reported an outcome of recent seismological research in the following words: "*Preceding the large waves of a teleseismic disturbance we find preliminary tremors… for (ray paths) which lie within a depth of* 30 *miles, the recorded speeds do not exceed those which we would expect for waves of compression in rocky material. This, therefore, is the maximum depth at which we should look for materials having similar physical properties to those we see on the earth's surface. Beneath this limit, the materials of the outer part of this planet appear rapidly to merge into a fairly homogeneous nucleus with high rigidity*".

In the same lecture Milne was also the first to observe (1906) that breaks in the trajectory of the secular motion of the earth's North Pole (relative to its mean position) could be correlated with the occurrence of major earthquakes during 1892–1904. A quantitative theory of this effect was only given in 1970.

today as the Moho), which could explain the travel-times of seismic rays from a local earthquake. It was subsequently found to demarcate the base of the earth's crust. This discovery demonstrated that the structure of the earth's outer layers could be deduced from travel-times of reflected and refracted seismic signals.

In 1914, ***Beno Gutenberg*** *(1889–1960,* Germany and U.S.A.*), published his accurate determination of the depth of the boundary of the earth's core at* 2900 *km below the surface. He speculated that this discontinuity divides a liquid core of radius* 3500 *km from a solid mantle*[662]*. [In* 1955 *he discovered a global low velocity zone at depth* 70–250 *km in the earth's mantle*[663]*]. In 1936,* ***Inge Lehmann*** *(1888–1993,* Denmark*) produced the first evidence of the existence of the earth's solid inner core with a radius of ca* 1400 *km.*

The advent of elastodynamics began with the discovery of longitudinal and transverse waves by ***Poisson*** *in* 1828*, and their physical interpretation by* ***Stokes*** *in* 1845*. In* 1885*,* ***Lord Rayleigh*** *discovered, ahead of observations, another type of elastic waves (to be known later as the Rayleigh wave) that is associated with material discontinuities such as a free surface of a body.*

In 1897, ***Oldham***[664] *identified on earthquake recordings (seismograms) the three main types of waves predicted by Poisson and Rayleigh, thus confirming that, at least for short period wave-motion (dominating periods: 0.1–1 sec), the earth indeed behaves like an elastic body for which Hooke's law may*

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[662] In his treatise *Principles of philosophy* (1644), **Descartes** made one of the first attempts to speculate about the earth's interior. He wrote that the earth had a central nucleus made of primordial, sun-like fluid surrounded by a solid, opaque layer. Succeeding concentric layers of rock, metal, water and air made up the rest of the planet. In the current view, the earth possesses a solid inner core and a molten outer core. Both consist of iron-rich alloys. The earth's composition changes abruptly about 2900 km below the surface, where the core gives way to a mantle made of solid magnesium-iron silicate minerals. Another significant discontinuity, located 670 km below the surface marks the boundary between the upper and lower mantle (the lattice structure of the mantle minerals changes across that boundary because of high pressure).

[663] Known as the *Astenosphere*. Now believed to be due to partial melting (1–10%) of basaltic magma. The major mineral in the earth's mantle is *Olivine* (Mg_2SiO_4 with Fe_2SiO_4). In the Astenosphere, shear-wave velocities take low value and seismic waves are more strongly attenuated.

[664] Joined the Geological Survey of India in 1879. Retired in 1903.

apply. In 1899, ***Cargill Gilston Knott*** *(1856–1922, Scotland) derived the general equations for reflection and refraction of plane elastic waves at plane boundaries. This was needed to relate the amplitudes of the waves activating the seismometer to the corresponding seismogram traces, modified by the presence of the free surface of the earth.*

In 1904, ***Horace Lamb*** *(1849–1934, England) came forth with the first mathematical theory of a point-source earthquake in a half-space earth model. He thus laid the theoretical foundation for the propagation of seismic waves in layered media.*

The first inverse problem in geophysics was formulated and solved in 1907 *by* ***Gustav (Ferdinand Joseph) Herglotz*** *(1881–1953, Germany), enabling the intrinsic compressional and shear velocities to be determined from travel-time data*[665]*. By* 1909 ***E. Wiechert****,* ***K. Zoeppritz****, and* ***L. Geiger*** *ex-*

[665] In seismology, observations are mostly made at seismograph stations on the earth's surface. Rays emitted from an earthquake source (*focus*), eventually reach the stations located at various distances from the point of the earth's surface above the source (*epicenter*). The distance from the epicenter to the observing point is the *epicentral distance*. For the case when both the source and the receiver are on the earth's surface, we have the relation:

$$\Delta(p) = 2p\int_{r_m}^{a} \frac{d(\ln r)}{\sqrt{r^2/V^2 - p^2}},$$

where Δ is the angle subtended by the seismic *ray* at the earth's center (equal in this case to the angular source-receiver distance), and $p = \frac{dT}{d\Delta}$ is the ray-parameter. [This relation was discovered by **Hans Benndorf** (1870–1953, Germany) in 1905.] T is the travel-time along the curved ray, r_m is the distance from the earth's center to the lowest point of the ray, and $V(r)$ is the intrinsic wave velocity at radial coordinate r and a is the earth's radius.
Knowing $p(\Delta)$ (travel-time data) for a sufficiently dense grid of points in some interval $0 \le \Delta \le \Delta_1$, the above equation turns into an integral equation for $V(r)$. It leads to the *Abel integral equation*

$$f(x) = \int_x^b \frac{u(y)dy}{(y-x)^k} \qquad (0 < k < 1)$$

for the unknown $u(y) = \frac{d}{dy}\ln r$ with $y = \left(\frac{r}{aV}\right)^2$, $x = \left(\frac{r_m}{aV_m}\right)^2$, and $f(x) = \frac{1}{2\sqrt{x}}\Delta(x)$.

Its explicit solution:

$$u(y) = -\frac{\sin \pi k}{\pi}\frac{d}{dy}\int_y^b \frac{f(x)dx}{(x-y)^{1-k}},$$

ploited this method to obtain for the first time a profile of compressional wave velocity in the earth's mantle.

A significant contribution to theoretical seismology was made in 1911 *by* ***A.E.H. Love***[666] *(1863–1940, England) with his discovery of a horizontally-polarized surface-wave (now known as the Love-wave), from the analysis of which seismologists could derive estimates of the thickness of the earth's crust and its rigidity.*

Further advance during 1915–1936 *was made by* **Harold Jeffreys** *(1891–1989, England), who brought to bear mathematical and statistical methods and a great knowledge of wider geodynamical problems. His attention to scientific method and statistical detail has been one of the main forces through which pre-WWII seismology has attained its level of precision.*

Significant progress in seismology has been made through the first four decades of the 20^{th} *century: In* 1901, *the first Geophysical Institute was founded in Göttingen (Germany), and the number of seismic observatories capable of teleseismic recording did not exceed 25 (compared to* 8 *in 1894). By* 1940, *there were about* 10 *major seismic research centers and* 250 *seismic stations around the globe.*

An international Association of Seismology was founded in 1905 *at a meeting of representative of* 23 *countries in Berlin, and met in Rome in* 1906 *where it was decided to establish an international center at Strasbourg. The year* 1919 *saw the appearance of a bulletin for global recordings of earthquakes, published at Oxford, under the name International Seismological Summary (I.S.S.).*

Following the catastrophic San-Francisco earthquake of April 18, 1906, ***Harry Fielding Reid*** *(1859–1944, U.S.A.), advanced his elastic rebound theory (1910). Earthquakes are associated with large fractures, or faults, in*

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can be recast in the form:

$$V(r_1) = \frac{a}{p(\Delta_1)} \exp\left[-\frac{1}{\pi}\int_0^{\Delta_1} \mathrm{ch}^{-1}\left\{\frac{p(\Delta)}{p(\Delta_1)}\right\} d\Delta\right],$$

where a is the radius of the earth and Δ_1 is the epicentral distance for a ray that bottoms at $r = r_1$. The integration extends over *a family of rays* for *each* specific depth.

666 **Augustus Edward Hough Love** was a Sedleian professor of natural philosophy at Oxford University during 1899–1940. He discovered a horizontally-polarized guided shear wave that propagates in the earth's crust (1911). It was subsequently named after him ('*Love wave*'). His name is also associated with a dimensionless number in the theory of earth tides ('*Love number*').

the earth's crust and upper mantle. As the rock is strained, elastic energy is stored in the same way that it is stored in a wound-up watch spring. The strain builds up until the frictional bond that locks the fault can no longer hold at some point on the fault, and it breaks. Consequently, the blocks suddenly slip at this point, which is the focus of the earthquake.

Once the rupture is initiated it will travel at a speed of about 3.5 *km/sec*[667]*, continuing as much as* 1000 *kilometers. In great earthquakes, the slip, or offset, of the two blocks can be as large as* 15 *meters. Once the frictional bond is broken, the elastic strain energy, which had been slowly stored over tens or hundreds of years, is suddenly released in the form of intense seismic vibrations — which constitute the earthquake*[668]*. The process through which the frictional bond is 'lubricated' to enable the commencement of the slip is yet not understood.*

The time between great earthquakes is about 50–100 *years in California and somewhat less in more active seismic regions, such as Japan or the Aleutians. Thus the time required to build up the elastic strain energy in the rocks adjacent to a fault is enormous compared with the time that elapses during the release of stored energy.*

The present state of knowledge of earthquake phenomena precludes the reliable prediction of the time of occurrence of the next major earthquake in any given location. Perhaps the most adequate answer to such questions was given long ago by Mark Twain: "I was gratified to be able to answer promptly, and I did. I said I did not know".

Since 1556*, an estimated* 3.5 *million persons were killed by earthquakes.*

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[667] This was first discovered, both experimentally and theoretically, by **Ari Ben-Menahem** (Ph.D thesis, CALTECH, 1960). It led to establishing of a novel intrinsic magnitude scale of earthquakes based on the physical concept of '*earthquake moment*' (**A. Ben-Menahem** and **D.G. Harkrider**, *Journal of Geophysical Research* **69** 2605–2620, 1964).

[668] About 10^9 erg of strain-energy is released from each cubic meter of the earthquake source volume. The greatest earthquakes release such energy from a strained volume of $1000\text{ km} \times 100\text{ km} \times 100\text{ km} = 10^{16}\text{ m}^3$, which gives a total of 10^{25} erg. This is about the equivalent of 1000 nuclear explosions, each with strength of 1 megaton (1 million tons) of TNT.
It is of interest to note that the few large earthquakes each year release more energy than hundred of thousands of small shocks combined. About 10^{26} erg of seismic energy are released each year. This is about 1 percent of yearly amount of the heat energy reaching the earth's surface from the interior.

The Primeval 'Seismologist'

Homo sapiens invented the seismograph and discovered Rayleigh waves some hundred years ago. Nature, however, produced 60 million years ago an arthropod, devoid of visual, auditory or olfactory senses, but equipped (in modern terminology) with a mobile array of 8 seismometers, amplifiers, and a minicomputer that enables it to locate its subsurface prey from amplitudes and travel-times of P waves and Rayleigh waves in the sand.[669]

The sand scorpion Paruroctonus mesaentis, a nocturnal hunter of the Mojave Desert, has receptors on its legs that are extraordinary sensitive to subtle disturbances of the sand. With this unusual prey-detection mechanism it derives information needed to locate its prey. It essentially locates the source of a signal by detecting and interpreting minute differences in the time and amplitude of mechanical waves through the sand by means of its spatially separated sensors.

The sand scorpion is one of the largest dune arthropods, growing to a length of 8 centimeters and a weight of 4 grams over the course of its 5- to 6-year lifetime. It can detect disturbances as far away as 30 centimeters. At a distance of 10 centimeters or less their estimates of target angle and distance are virtually perfect. It determined the turning angle towards its prey by integrating the input from all its legs.

As a granular disaggregated medium, sand acts as a reasonably good conductor of mechanical waves up to a distance of several decimeters in the 1 to 5 kilohertz bandwidth; lower frequencies are damped and higher frequencies are scattered. Of the four types of elastic waves that can propagate in solids, sand conducts only P waves and Rayleigh waves. Typical group velocities of P waves are 150 m/sec at 5 kilohertz and those of Rayleigh waves are 50 m/sec at the same frequency. The wavelengths corresponding to these frequencies commensurate with the size of the scorpion.

Two types of mechanoreceptors on the tarsal (terminal) leg segment of the scorpion are sensitive to subtle vibrations of the substrate: Hairs protruding

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669 Philip H. Brownell, *Compressional and surface waves in sand: used by desert scorpions to located prey*, *Science* **197**, 479–482, 1977. (Also in *Scientific American*, December 1984).

from the sides and bottom of the tarsus rest on and between sand grains. The basitarsal slit sensillum consists of regions where the cuticle folds in on itself. The slit sensillum is particularly sensitive to vibrations that compress the slits in a direction perpendicular to their long axis; it is capable of detecting movements in the substrate that have amplitudes of about one Angström unit (10^{-8} *cm). Experiments have shown the hairs detect the compressional waves, and the slit sensila register the arrival of Rayleigh waves. The adult scorpion's eight legs form a roughly circular sensor "array" about* 4 *to* 6 *centimeters across.*

When the sand is disturbed, the first signals to arrive at the tarsus are compressional waves, which stimulate the tarsal hairs, causing large amplitude action potentials to ascend the leg nerve. A few milliseconds later, the vertical ground particle motion associated with the slower-traveling Rayleigh wave compresses the slit sensillum, triggering smaller-amplitude signals.

Rayleigh-wave stimulation of the slit sensillum appears to be the basis of the scorpion's perception of target direction*. It may also exploit the* time-delays *of both P and Rayleigh waves across his "array"; assuming a sensory field of* 5 *centimeters in diameter, this time delay would be about one millisecond for a Rayleigh wave, and* 0.3 *millisecond for a compressional wave. It might then determine the direction of the source from the time delay between stimulation of sensors close to the source and those further away; that is, the scorpion might simply turn in the direction of the sensors that are stimulated first (many animals use smaller time delays to locate the source of compressional waves propagated in the air; humans, for example, can easily judge the direction of a sound source on the basis of a time delay between the two ears of less than* 10 *microseconds).*

Alternatively, a scorpion might gauge the direction of a wave source from differences in the intensity with which the wave stimulates different sensors; as a wave propagates, its amplitude decreases, partly because the wave front expands geometrically, spreading out the energy of the wave, and partly because the signal is absorbed by the medium. Sensors nearest to the source should thus be stimulated most intensely.

Experiments have shown that the scorpion can detect time delays as small as 0.2 *milliseconds, but they respond most consistently to delays of one to two milliseconds — roughly the time it takes for a Rayleigh wave to traverse the span of their legs. There remains only the question of how the scorpion perceives the distance to its prey, i.e., how does it translate time delays into distance.*

Field observations showed that it rarely missed at 10 *centimeters or less. One possibility is that the animal times the delay between the arrival of the fast-moving compressional wave and the slow-moving Rayleigh waves.*

The delay would be proportional to the distance of the source. The second possibility would involve sensing the gradient of the amplitude of the wave across the "array", which increases with the decrease of its distance from the prey.

In any case, the evolutionary process endowed this creature with a suitable "computer" to achieve this task since its mere existence must rely so exclusively on information transmitted through the ground.

1890–1897 CE **David Schwarz** (1845–1897, Germany). Invented, designed and built the first *airship* (*metal* dirigible balloon). It was made of aluminum, filled with gas and driven by a Daimler benzine motor [length = 48 m, diameter = 14 m, volume = 3700 m^3, weight = 3100 kg, speed = 27 km/h].

It was tested in Berlin in 1897: after flying for 4 hours, a driving belt slipped, and in descent the balloon was damaged beyond repair. Among the spectators was **Ferdinand von Zeppelin** (1838–1917), of the German army, who foresaw the potentialities of the airship for the military. He bought all plans and models of Schwarz's airship from his widow, and developed it further.

Schwarz was born in Hungary to Jewish parents and started as a successful lumber merchant in Zagreb. He then studied mechanical engineering on his own. Impressed by the special properties of the aluminum metal (its large-scale industrial production began during 1886–7 in America, England and France), he set forth to harness it to the construction of light airships. In 1890 he flew his first model in Austria and Russia, but failed to interest the respective governments. Finally, when in 1897 the Germans were ready to support his invention, the excitement caused his untimely death.

1890–1903 CE **Samuel Pierpont Langley** (1834–1906, USA). Astronomer, physicist, pioneer in aerodynamics and inventor. His steam-driven aeroplane flew for 90 seconds (1896) — the first flight by an heavier-then-air, engine-equipped, aircraft (uncrewed).

Langley was born in Roxbury, MA and attended Boston Latin School. He spent several years studying architecture and engineering before turning to astronomy. His several inventions included an instrument called *bolometer*, which measures the sun's radiation. He was a professor of physics and astronomy at the Western University of Pennsylvania (1866–1887), studying

the infrared portions of the sun's spectrum. In 1890 he turned to pioneering work in aerodynamics, contributing greatly to the design of early aircraft wing shape.

The United States government gave Langley 50,000 dollars to build a man-carrying "aerodrome". After two failed attempts (1903) to get his flying machine off the ground, Samuel Pierpont Langley was criticized by the *new York Times* for wasting government funds on and idle dream.

A third attempt using a smaller model succeeded. The subsequent catapult-launched flights of the Wright brothers at Kitty Hawk owed much to Langley's principles as well as to the more powerful engines available by the early 1900's. The Langley design was tested in later years by using a model with a modern engine; it flew successfully with a pilot aboard.

1890–1908 CE **Edouard-Eugene Branly** (1844–1940, France). Physicist, physician and inventor. Invented the *coherer* (1890), a primitive form of radio detector that made wireless telegraphy possible. He thus established the principles later developed by Marconi. He also evolved the forerunner of the receiving antennae.

Branly was born in Amiens. He obtained a doctorate from the Sorbonne and a medical degree from the University of Paris. By 1908[670], he developed the *remote-controlled* torpedo, fired from a torpedo-boat and operated by electromagnetic waves via a relay system.

1890–1911 CE **Sebastian Ziani de Ferranti** (1864–1930, England). Engineer and inventor. Innovator in the development of electrical engineering who led the application of power generation and distribution.

Ferranti was born in Liverpool, where his father had a photographic art studio. At the age of 22 he became Chief Engineer of the London Electric Supply Corporation, and was deeply involved in the planning, generation and distribution of electricity. He was one of the first people to advocate large power generating stations sited outside of population centers and established the principle of the national grid, using alternating current transmission.

1891 CE *California Institute of Technology* (*Caltech*) founded.

1891 CE **Seth Carlo Chandler** (1846–1913, U.S.A.). Astronomer. Discovered a periodicity of 428 mean solar days in the spectrum of the latitude

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[670] By 1868, **Robert Whitehead** (1823–1905, England), engineer, had developed the first real torpedo. Powered by compressed air, it was completely self-propelled.

variation. This value exceeds Euler's (1765) theoretical value for the free precession of a rigid ellipsoid of revolution, by about 4 months. In 1892 **Simon Newcomb** (1835–1909, U.S.A.) explained this period lengthening as being due to the elastic yield of the earth.

The Chandler Wobble (1765–1909)

The equations of rigid gyroscopic motion were given by ***Euler*** *in* 1758. *On the basis of this theory, he suggested in* 1765 *that the earth might undergo a free precession with period* $A/(C-A)$ *sidereal days. Assuming this to be true, a spectator, partaking in the earth's motion, should observe periodic changes in latitude relative to the fixed stars. Indeed,* ***Lord Kelvin*** *urged astronomers in* 1876 *to look for a period of 10 months, as predicted by Euler. However, no such period could be found.*

Instead, ***S.C. Chandler*** *established in* 1891 *the existence of a* 428*-days period in the spectrum of the latitude variation.*

The lengthening of the period was explained by ***S. Newcomb*** *(1892) to be the result of the earth's elasticity. A theoretical verification was given by* ***A.E.H. Love*** *(1863–1940, England) and* ***J. Larmor*** *(1857–1942, England), based on first order theory of the figure of the earth.*

This 14*-month precessional motion of the instantaneous axis of rotation about the earth's axis of figure is known today as the Chandler Wobble. The source of excitation of this motion has not been fully accounted for.*

The Chandler Wobble is accompanied by two additional observed phenomena:

(1) Secular (transient polar shifts resulting from impulses and jumps in the source of excitation);

(2) Changes in the length of day due to such excitations.

The French novelist ***Jules Verne*** *(1828–1905) cleverly used realistic, believable explanations to support incredible tales of adventure. In his book "Sens dessous dessous" (1890) he concocted a plot in which a colossal missile*

of mass $180,000$ *tons is launched at latitude* 45°, *in order to displace the pole by* 23.5° *and so remove the obliquity of the ecliptic. Working out the physics of this problem for an earth model with no equatorial bulge* $(A = B = C)$, *one discovers a little fact which Verne did not bother to tell his readers — the earth will require 'only'* 10^8 *years to creep to the desired state!*

1891–1892 CE **Arthur Moritz Schönflies** (1853–1928, Germany). Mathematician and crystallographer. Classified the complete list of 230 *space groups*. Introduced the known *Schönflies notation* for *point groups*.[671]

Schönflies was born to Jewish parents in Landberg an der Warthe. Was a student at Berlin and did his doctorate (1877) under **Kummer** and **Weierstrass**. He taught in Berlin, Colmar, Göttingen, Königsberg and Frankfurt a. M. He worked mainly on set theory and crystallography.

1891–1892 CE **Almon Brown Strowger** (1839–1902, USA). Undertaker and inventor. Invented, patented and installed the first *automatic telephone exchange system*, known at that time as 'Strowger's switch'. It replaced the switchboard operator for placing local cells.

The first automatic exchange began operating in La Porte, Indiana (1892); the central office switch worked in concert with a similar switch at the subscribers home, operated by push buttons. The contact electromechanical switch, which operated the telephone, could select a line of a wanted subscriber. Later (1894) **A.E. Keith**, **J. Erickson** and **C.J. Erickson** invented the rotating finger-wheel needed for a dial which first began operating in Milwaukee's City Hall (1896).

[671] The most important type of group in crystallography is the one which consists of the symmetry operations pertaining to molecular structure. For such a group the combining rule is one operation followed by another. Since the application of any symmetry operation leaves a molecule physically unchanged and with the same orientation in space, its center of mass must also remain fixed in space under all symmetry operations. From this it follows that all the axes and planes of symmetry of a molecule must intersect in at least one common point. Such groups are called *point groups*. For a crystal of infinite size we can have symmetry operations, e.g. translation, that leaves *no* point fixed in space; these give rise to *space groups*.

The automatic dial system changed telephony forever — it became “girl-less, cuss-less, out-of-order-less, and wait-less”, and expedited the extension of the telephone network.

Strowger was born in Penfield, New York, a suburb of Rochester. He went to a New York state university, served in the Civil War (1861–1865), ending as a lieutenant. He then taught school in Kansas and Ohio and wound up first in Topeka and then in Kansas City as an undertaker (1886), an unlikely profession for an inspired inventor.[672]

1891–1896 CE **Edward Goodrich Acheson** (1856–1931, USA). Engineer and inventor. Produced silicon carbide, or *carborundum*[673] (1891). Invented a process for manufacturing *graphite* by heating a mixture of coke and clay (1896).

The discovery of carborundum, which is the hardest surface made by man and second only to diamond in hardness, ended the search for a highly effective and durable abrasive needed by industry to manufacture precision-ground interchangeable metal parts. One of the byproducts of the carborundum manufacturing process was graphite, which proved useful as a lubricant.

Acheson was born in Washington, Pennsylvania. In 1880 he had secured a position with Thomas Edison in his Menlo Park, N.J. laboratories and was involved in the development and installation of electrical lighting, including working on the lamp exhibit at the Paris Exhibition (1881).

1891–1913 CE **Alfred Werner** (1866–1919, Switzerland). Distinguished chemist. Father of *coordination chemistry.* First to put forward ideas on bonding which were eventually to revolutionize inorganic chemistry. His theory led to the discovery of many cases of *isomerism.* The importance of his ideas was amplified in modern times since it was discovered that the mode of action of many *enzymes catalysts*, that are essential for life processes, depends on the formation of *metal ion coordination complexes.*

The idea of a *privileged* central metal atom or ion surrounded by a group of tightly bound molecules or ions (e.g., as Mg in chlorophyll or Fe in hemoglobin) puts Werner in a class with **Kekulé** and **van’t Hoff** before him, and **Pauling** after him as far as our present day understanding of molecular architecture is concerned.

[672] The story surrounding his motivation to invent the automatic switch is odder still: the wife of his competitor, working as a switchboard operator, gave busy signals to customers calling Strowger, thus stealing his business.

[673] By mixing clay with carbon and fusing it electrically

Werner argued that the factor determining the structure of coordination compounds was not the primary valency of the central atom but the number of ions, atoms, radicals or molecules directly bonded to the metal[674], now known collectively as *ligands*. The ligands were postulated to be arranged in simple, *spatially geometric structures*, with the *octahedron* as the commonest arrangement. A corollary of this theory was that some coordination complexes should exist as *optically active isomers*.

Werner was awarded the Nobel Prize in chemistry (1913).

1891–1914 CE **Paul Painlevé** (1863–1933, France). Mathematician and statesman. Developed the theory of functions defined by non-linear differ-

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[674] Werner was faced with a need to explain a perplexing experimental fact; cobalt chloride can bind itself to ammonia in 4 different ways: $CoCl_3{\cdot}6NH_3$, $CoCl_3{\cdot}5NH_3$, $CoCl_3{\cdot}4NH_3$ and $CoCl_3{\cdot}3NH_3$. There were two questions involved here: first, why was there such arbitrariness about the number of ammonia molecules. Second, when these cobalt complexes were dissolved in water and $AgNO_3$ added, one obtained strikingly different quantities of *insoluble* silver chloride precipitating from one mole of the complex:

$CoCl_3{\cdot}3NH_3 \longrightarrow$ no precipitate at all;
$CoCl_3{\cdot}4NH_3 \longrightarrow$ 1 mole AgCl precipitate;
$CoCl_3{\cdot}5NH_3 \longrightarrow$ 2 mole AgCl precipitate;
$CoCl_3{\cdot}6NH_3 \longrightarrow$ 3 mole AgCl precipitate.

Why not 3 moles of AgCl in each case? After all, aren't there 3 moles of chlorine available?

Werner suggested correctly that the *cobalt ions* form *octahedral* complexes with 6 surrounding groups (octahedron = a regular polyhedron with 6 vertices and 8 equilateral triangles faces). For $CoCl_3{\cdot}6NH_3$, all three chlorines are *loosely held* in an ionic bond $[Co(NH_3)_6]^{+++}\ 3Cl^-$ (like NaCl); for $CoCl_3{\cdot}3NH_3$, all three chlorine atoms are *tightly held* as $[Co(NH_3)_3Cl_3]$ such that Ag could not pull the chlorine atoms from this complex. The other two cases fall in between.

Another important aspect of coordination theory concerns the possible alternate spatial arrangements of the six different ligand groups coordinated about the metal atom. For example, in the case of $[Co(NH_3)_4Cl_2]^+$, the two chlorines can be on the same or opposite sides of the octahedron. That makes for electronic differences, and results in slightly different properties. The two *isomers*, as such compound are called, differ only in the spatial geometry or arrangements of atoms. Most evident immediately are their different colored solutions: the *trans* form, in which the chlorine are opposite, is green while the *cis* form, in which the coordination complex has the chlorines on the same side, is violet. Such spatial isomers are common in nature and of great importance in *living systems*.

ential equations of the first and second order. (*Painlevé property*, *Painlevé transcendents*[675], *Painlevé function*).

Painlevé was born in Paris, the son of a lithographic draughtsman, and was educated at the École Normale Superiere. He became a professor at the Sorbonne and the École Polytechnique in 1898.

His interest in dynamics led Painlevé to take part in the early development of aeronautics, on the practical as well as on the theoretical side: he was, in fact, one of the first passengers of **Wilbur Wright**.

From his 40^{th} year on, public affairs occupied a greater and greater portion of his time. In 1917 he became Minister of War in the government of M. Ribot; on Sept. 1917 he became Prime Minister. In 1925 he became Prime Minister for the second time. In 1930–1931 and 1932–1933 he was Air Minister. He died suddenly of heart failure, and was accorded public funeral in the Panthéon.

1891–1916 CE **Charles Proteus (Karl August Rudolf) Steinmetz** (1865–1923, US). Electrical engineer and inventor. Developed the theory of alternating-current (AC) phenomena (using complex numbers a la Heaviside). This enabled the design of AC machines to be made more efficient and consolidated the victory of AC over DC gained by Tesla in fierce competition with Edison.

Worked on the design of AC transmission, developing lightning arresters for high-power transmission lines; patented over 200 inventions, including improvements on generators and motors. Formulated the *Steinmetz hysteresis law* (1891), which describes the dissipation of energy that occurs when a system is subject to an alternating magnetic field. This made it possible to reduce loss of efficiency in electromagnetic systems.

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[675] In the course of classifying nonlinear differential equations, he considered all equations of the form $w'' = R(z,w)(w')^2 + S(z,w)w' + T(z,w)$ where R, S, T are rational functions of w (but have arbitrary dependence on z). The solutions may have various kinds of *fixed singularities* (poles, branch points, essential singularities), but may not have *movable singularities* (its location depending on the initial conditions) except for poles. There are 50 distinct types of equations having these properties. Of those, 44 types are soluble in terms of elementary transcendents (sines, cosines, exponentials), functions defined by linear second-order equations (Bessel functions, Legendre functions, and so on) or elliptic functions. The remaining 6 equations define the 6 Painlevé transcendents, one of which is $y'' = 6y^2 + x$. Painlevé transcendents have recently found an application in the theory of random surfaces and two-dimensional quantum gravity

Steinmetz was born in Breslau (now Wroclaw, Poland) to Jewish parents. Educated there and at the Technical High School, Berlin. Forced to flee Germany (1888) because of his socialist activities just before receiving his Ph.D. from Breslau University, he completed his studies in Zürich. Migrating to the US (1889), he worked for an electrical firm in Yonkers until his monographs attracted wide attention. In 1893 he became chief consulting engineer in General Electric's Schenectady plant, where he spent the rest of his life experimenting with electrical appliances and machinery. He was a Professor of Electrophysics at Union College, Schenectady (from 1902) and authored several books on electrical theory.

Throughout his life, Steinmetz retained his belief in socialism and in later years favored Zionism.

1891–1917 CE **Roland von Eötvös** (1848–1919, Hungary). Experimental physicist. Established through his torsion-balance experiments that inertial and gravitational mass are equivalent to accuracy of 1 part in 10^9.

Eötvös was born in Budapest. In 1872 he was appointed professor of physics at the University of Budapest. During 1894–1895 he was Minister of Education.

1891–1921 CE **Eugene Dubois** (1858–1940, Holland). Physician, anatomist and paleontologist. The man who found the Missing Link in the Darwinian evolutionary trail from ape to human.

While serving as military surgeon in the Dutch East Indies (1887–1895), he discovered in Java the bones of a *hominid*, apparently intermediate between man and simian ancestors, which he named (1891) *Pithecanthropus erectus* (now *Hominid erectus*).

Dubois gave up a promising post at the University of Amsterdam to go to Java with the aim of finding a fossil of a prehuman that would be demonstrably the "Missing Link". After finding what he believed to be such a fossil he had to spend some thirty years defending his claim. He has been an underestimated scientist.

1891–1923 CE **George Ellery Hale** (1868–1938, U.S.A.). Astronomer. Advanced solar and stellar spectroscopy, discovered the existence of magnetic fields in sunspots[676] and founded three large observatories in the United States: Yerkes (1895), Mount Wilson (1904) and Palomar (1948).

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[676] *Sunspots* are one of many phenomena associated with the 22-year solar cycle; they are irregularly-shaped dark regions in the photosphere of the sun. Although they vary greatly in size, typical sunspots measure a few tens of thousands of kilometers across. On very rare occasions, a sunspot is so large that it can be seen with the naked eye (using special dark filters!). Ancient Chinese

The last two were formally known as the *Hale observatory*, The Palomar Observatory's Hale telescope has a diameter of 508 cm (200 inch[677]).

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astronomers recorded such sightings 2000 years ago. **Galileo** (1612) was the first person to examine sunspots in detail and **Schwabe** (1843) discovered that the number of sunspots varies in a periodic fashion (*sunspot cycle* of about 11 years). **Maunder** (1904) discovered also a spatial periodicity, i.e. that the location of sunspots varies in a regular fashion over the sunspot-cycle: the first sunspots of a cycle appear at large distance from the solar equator, whereas the last spots of a cycle are formed very near the equator. At *sunspot maximum* in the middle of the cycle, most sunspots occur at latitude of 10° to 15° north and south of the equator.

In 1908, **Hale** observed the splitting of the Fe I spectral line into three lines corresponding to a very intense magnetic field of 4130 Gauss (compared to the terrestrial dipolar field of 0.7 Gauss). Hale also discovered that sunspot groups are *bipolar* and that the polarity pattern reverses itself every 11 years, making a complete cycle of 22 years through which the *solar surface features* vary (the *average number* of sunspots still increases and decreases in a regular 11-year-cycle). Hale's discovery demonstrates that sunspots are places where a powerful, concentrated magnetic field protrudes through the hot gases of the photosphere. Because of the temperature, many of the atoms in the photosphere are ionized, so that the photosphere is a mixture of electric charges. This *plasma* is an extremely good conductor of electricity, and it interacts vigorously with magnetic fields, which in turn, *restricts and contains* the motions of a plasma and *inhibits* the natural convective motions. Since energy cannot flow freely upward from the sun's convective zone, the plasma within sunspot *cools off*. This is why temperatures in a sunspot are typically 4000–4500 K, i.e. more than 1000 K cooler than the surrounding undisturbed photosphere. Because of this lowered temperature, sunspots look *dark* in contrast to their brighter surroundings.

A host of exotic phenomena occur around and above sunspots as a direct result of their intense magnetic fields. One of them – the *solar flare* – is a brief eruption of very hot ionized gases from a sunspot group; vast quantities of particles and radiation are blasted into space. When the resulting UV and X-rays, and solar wind surges, arrive at the earth a day or so later, they produce aurorae and interact with the gases of the upper atmosphere.

In 1960, the astronomer **Horace Babcock** put forward a *magnetic-dynamo* model which makes use of the sun's *differential rotation* and its *convective envelope* to explain the sunspot cycle as the result of the wrapping of a magnetic field around the sun: sunspots appear where the concentrated magnetic field has broken through the solar surface.

[677] The Mount Wilson 150 cm reflecting telescope began observations in 1908, and the second, Hooker telescope (250 cm; 1917) was used to revolutionize astronomy, astrophysics, and cosmology in the 1920's and beyond. The 508 cm

Hale was born in Chicago, studied at M.I.T. and was professor at the University of Chicago (1897–1904). He invented the *spectroheliograph* [1891, with **Henri Deslandres** (1853–1948, France)], an instrument used to photograph the sun at a single wavelength. Founded the *Astrophysical Journal* (1895).

In 1908, Hale examined solar magnetic storms and determined that the *Zeeman effect* (1896) is apparent in the spectra of *sunspots*, namely, the splitting of spectral lines due to the strong magnetic fields associated with these sunspots. This led to his discovery (1919) of the periodic *reversal* in the polarization of their magnetic fields.

1891–1933 CE **Sven Andreas Hedin** (1865–1952, Sweden). Central Asia explorer. Drew the first maps and gathered information about areas in Persia, Turkestan, Tibet, China, and Mongolia. During his early travels, Hedin unearthed ancient cities in Turkestan deserts, and the Lop Nor basin of Western China. In 1893, he began a three-year trip over the Pamir, a mountainous plateau in Central Asia and the plateaus of Tibet. In the early 1900's Hedin explored the high sources of the Brahmaputra River, locating mountains and waters never before known. In 1933 he mapped the ancient silk trade route that extended 16,000 km from Asia to Europe.

Hedin was born in Stockholm to Jewish ancestry. By the time he was 22 he had already crossed the Elburz Mountains, traveled through Persia on horseback, crossed the Kara Kum, visited Bokhara and Samarkand, and crossed Tien Shan from Andizhan, in Ferghana, to Kashzar. Even an outline map of the routes he followed looks as if it was the work of a centipede whose feet had been dipped in ink. He described his travels in many books. During WWI, Hedin was a Nazi sympathizer.

1892 CE **Dmitri Iosifovich Ivanowski** (1864–1920, Russia). Microbiologist. Discovered a disease-causing agent smaller than bacteria — the *virus*.

Explained the infectiousness of tobacco mosaic disease (1892) by showing it can be transmitted via cell-free filtrates[678] of diseased plants to leaves of healthy plants.

1892–1894 CE **Richard Friedrich Johannes Pfeiffer** (1858–1945, Germany). Bacteriologist. First to observe a complex *immune reaction* (1894) of the body to an invading microbe. He injected live cholera vibrios (bacteria)

Palomar (Hale) telescope was completed in 1908; It was the world's largest until 1976, and retired in 1987 because of air and light pollution.

[678] An agent in the sap of leaves is not filtered out of the sap even with the so-called chamberlands' bacteriological filter. The term *filterable virus* was thus coined. Later, 'filterable' was dropped and *virus* took its modern meaning.

into guinea pigs which had already been immunized, then extracted some of the germs. Examining the extract under a microscope, he observed the germs becoming motionless, then swelling and finally disintegrating (a process he named *bacteriolysis*).

He showed that the same process occurred in vitro, and that the reaction would cease when heated over 60°C [This let **J. Bordet** to study the immune system and discover the *complement* (1898).] During the influenza epidemic of 1889–1892 he discovered the bacillus *Haemophilus influenzae* (1892), later found to be responsible for many of the complications of the influenza viral infection.

Pfeiffer was born near Posen and educated in Berlin as a military surgeon. He worked under **Koch** at the institute for Hygiene (1894) and became a professor of hygiene at Koenigsberg (1899) and Breslau (1909).

1892–1899 CE **Henri Eugene Padé** (1863–1953, France). Mathematician. Developed an important analytical method through which a function with singularities can be approximated as a ratio of two polynomials.

Padé was educated at the Ecole Normale Superieure in Paris and at Leipzig and Göttingen under **Klein** and **Schwarz** (1889–1890). He returned to France and obtained his doctorate under **Hermite's** supervision. He held positions at Besancon, Dijon and Aix-Marseilles.

Historically, Padé was motivated by the work of **Stieltjes** on the analytic theory of continued fractions (1889), which he came to know on his visit to Göttingen in 1890. His starting point, however, was the work of **Frobenius** (1881) who made a systematic study of those rational fractions.

1892–1905 CE **James Dewar** (1842–1923, Scotland). Chemist and physicist. First to produce liquid hydrogen (1898), later (1899) obtaining it as a solid. Studied the properties of matter at low temperatures. Invented the *Dewar vessel* (1892), forerunner of the vacuum bottle.

Dewar demonstrated (1898) that hydrogen, a gas which at normal temperatures tends to *warm upon expansion*, exhibits the normal Joule-Thomson cooling (1852) at temperatures below −80 °C. Hence, below −80 °C the Joule-Thomson effect allows a mechanism for further cooling of hydrogen to below its critical temperature for liquefaction.

Dewar was born at Kincardine-on-Forth, Scotland. He was educated at the universities of Edinburgh (under **Playfair**) and Ghent (under **Kekulé**). In 1877 he became Fullerian professor of chemistry in the Royal Institution, London. In 1904, he was the first British subject to receive the Lavoisier medal of the French Academy of Sciences. He was also a professor of natural experimental philosophy at Cambridge (1875–1923).

1892–1923 CE **Michael (Mihailo) Idvorsky Pupin** (1858–1935, U.S.A.). Physicist and inventor. His inventions led to great advances in long-distance telephone systems, telegraphy and radio transmission networks.

His main contributions:

- Multiplex telegraphy accomplished by electrical tunning (1892–1894)
- Extending the range of long-distance telephony by amplifying the signal at intervals along the line without distortion (1894)
- A rapid method for X-ray photography, shortening the time of exposure from about an hour to a few seconds (1896)
- Discovered the Secondary X-ray Radiation (1896)

Pupin was born in Idvor, Austria-Hungary (now Yugoslavia), a son of illiterate parents who encouraged his education. He arrived in America, a penniless immigrant, in 1874, and set out to understand the Maxwell theory like a knight in quest of the Holy Grail. First he went to Columbia University, but found nobody there who could explain Maxwell. Then he went to Cambridge, England, where Maxwell had worked; but Maxwell was dead, and Pupin's tutors were mainly interested in getting him good marks in the mathematical tripos.

Finally he went to Berlin, and there he found **Ludwig Boltzmann**, who taught Pupin what he knew about Maxwell's equations. Pupin was amazed to find out how few were the physicists who had caught the meaning of the theory, even 20 years after it was stated by Maxwell in 1865. He obtained his PhD degree at the University of Berlin (1888) and returned to the US in 1889.

After various adventures he became a professor of electromechanics at Columbia University in 1892. In 1923 he published his autobiography *From Immigrant to Inventor*, which won the 1924 Pulitzer prize.[679]

1892–1924 CE **Maxim Gorky** (**Aleksei Maksimovich Peshkov** 1868–1936, Russia). Novelist, humanist, social reformer and pioneer social-democratic thinker. The Socrates of modern times.

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[679] It was estimated that Pupin's invention of the 'Pupin's coils' (loading a telephone wire with inductance coils) had saved over 100 million dollars in the first 22 years. Pupin asked: "Where are those one hundred million dollars which the invention has saved? I know that not even a microscopic part of them is in the pockets of the inventor".

Gorky was born in Nizhny Novgorod. He became orphan at age nine and was raised by his grandparents. At age 19 he traveled on foot across the Russian Empire, changing jobs and accumulating impressions used later in his novels, stories and plays.

In 1887 Gorky witnessed a *Pogrom* in Nizhny Novgorod. Deeply shocked by what he saw, Gorky became a life-long opponent of racism. Gorky worked with the *Liberation of Labor* group and in October, 1889 was arrested and accused of spreading revolutionary propaganda. He was later released because they did not have enough evidence to gain a conviction. However, the *Okhrana* decided to keep him under police surveillance.

In 1891 Gorky moved to Tiflis where he found employment as a painter in a railway yard. The following year his first short-story, *Makar Chudra*, appeared in the Tiflis newspaper, *Kavkaz*. The story appeared under the name Maxim Gorky (Maxim the Bitter). The story was popular with the readers and soon others began appearing in other journals such as the successful *Russian Wealth*.

Gorky also began writing articles on politics and literature for newspapers. In 1895 he began writing a daily column under the heading, *By the Way*. In this articles he campaigned against the eviction of *peasants* from their land and the persecution of *trade unionists* in Russia. He also criticized the country's poor educational standards, the government's treatment of the *Jewish* community and the growth in foreign investment in Russia.

His short stories such as *Twenty-six Men and a Girl*, often showed Gorky's interest in social reform. In a letter to a friend, Gorky argued that "the aim of literature is to help man to understand himself, to strengthen the trust in himself, and to develop in him the striving toward truth; it is to fight meanness in people, to learn how to find the good in them, to awake in their souls shame, anger, courage; to do all in order that man become nobly strong."

In 1898 Gorky published his first collection of short-stories. The book was a great success and he was now one of the country's most read and discussed writers. His choice of heroes and themes helped him emerge as the champion of the poor and the oppressed. The *Okhrana* became greatly concerned with Gorky's outspoken views, especially his articles and stories about the police, but his increasing popularity with the public made it difficult for them to take action against him.

Gorky secretly began helping illegal organizations such as the *Socialist Revolutionaries* and the *Social Democratic Labor Party*. He donated money to party funds and helped with the distribution of radical newspapers such as *Iskra*.

On the 4th March, 1901, Gorky witnessed a police attack on a student demonstration in Kazan. After publishing a statement attacking the way the police treated the demonstrators, Gorky was arrested and imprisoned. Gorky's health deteriorated and afraid he would die, the authorities released him after a month. He was put under house arrest, his correspondence was monitored and restrictions were placed on his movement around the country. When he was allowed to travel to the Crimea, he was greeted on the route by large crowds bearing banners with the words: "Long live Gorky, the bard of Freedom exiled without investigation or trial."

After *Blood Sunday* Gorky was arrested and charged with inciting the people to revolt. Following a world-wide protest at Gorky's imprisonment in the Peter and Paul Fortress, *Nicholas II* agreed for him to be deported from Russia.

In 1906 Gorky toured Europe and the United States. He arrived in *New York* on 28th March, 1906 and the *New York Times* reported that "the reception given to Gorky rivaled that of Kossuth and Garibaldi." His campaign tour was organized by a group of writers that included **Ernest Poole**, **William Dean Howells**, **Jack London**, **Mark Twain**, **Charles Beard** and **Upton Sinclair**.

In 1907 Gorky attended the Fifth Congress of the *Social Democratic Labor Party*. While there he met **Vladimir Lenin**, **Julius Martov**, **George Plekhanov**, **Leon Trotsky** and other leaders of the party. Gorky preferred Martov and the *Mensheviks* and was highly critical of Lenin's attempts to create a small party of professional revolutionaries.

Gorky continued to write and his most successful novels include *Three of Them* (1900), *Mother* (1906), *A Confession* (1908), *Okurov City* (1909) and the *Life of Matvey Kozhemyakin* (1910).

Gorky was strongly opposed the *First World War* and he was attacked in the Russian press as being unpatriotic. In 1915 he established the political-literary journal, *Letopis* (Chronicle) and helped establish the Russian Society of the Life of the Jews, an organization that protested against the persecution of the *Jewish* community in Russia.

Gorky started a newspaper, *New Life*, in 1917, and used it to attack the idea that the *Bolsheviks* were planning to overthrow the government of Alexander Kerensky. On 16th October, 1917, he called on Vladimir Lenin to deny these rumors and show he was "capable of leading the masses, and not a weapon in the hands of shameless adventurers of fanatics gone mad."

In January, 1918, Gorky led the attack on Lenin's decision to close down the *Constituent Assembly*. Gorky wrote in the *New Life* that the *Bolsheviks* had betrayed the ideals of generations of reformers: "For a hundred years the

best people of Russia lived with the hope of a Constituent Assembly. In this struggle for this idea thousands of the intelligentsia perished along with tens of thousands of workers and peasants."

The Bolshevik government controlled the distribution of newsprint and in July, 1918, it cut off supplies to *New Life* and Gorky was forced to close his newspaper. The government also took action making it impossible for Gorky to get his work published in Russia.

In 1921 Gorky once again clashed with the Soviet government over the suppression of the *Kronstadt Uprising*. Gorky blamed **Gregory Zinoviev** for the way the sailors were treated after the rebellion. Gorky failed to save the life of the writer, **Nikolai Gumilev**, who was arrested and executed for his support for the Kronstadt sailors. He was also unsuccessful in obtaining an exit visa for the poet, **Alexander Blok**, who was dangerously ill. By the time Zinoviev gave permission for Blok to leave the country, he was dead.

During the terrible famine of 1921, Gorky used his world fame to appeal for funds to provide food for the people starving in Russia. One of those who responded was Herbert Hoover, head of the American Relief Administration (ARA).

Gorky continued to criticize the Soviet government and after coming under considerable pressure from Vladimir Lenin, he agreed to leave the country. In October, 1921, Gorky went to live in *Germany* where he joined a community of around 600,000 Russian émigrés. He continued to criticize Lenin and in one article wrote: "Russia is not of any concern to Lenin but as a charred log to set the bourgeois world on fire." In July, 1922, Gorky campaigned against the decision to sentence to death twelve leading members of the *Socialist Revolutionary Party*.

Gorky stayed in Germany for two and half years before moving to Sorrento in Italy.

Joseph Stalin attempted to bring an end to Gorky's exile by inviting him back to his homeland to celebrate the author's sixtieth birthday. Gorky accepted the invitation and returned on 20th May, 1928. Stalin wanted Gorky to write a biography of him. He refused but did take the opportunity to seek help for those writers being persecuted in the Soviet Union.

It is unlikely that Gorky ever discovered the full picture of what Joseph Stalin was doing in the Soviet Union. He was kept under close surveillance by the *NKVD* and his private correspondence reveals that he believed Stalin that Leon Trotsky and his followers were behind the assassination of Sergey Kirov.

Maxim Gorky died of a heart attack on 18th June, 1936. Rumors began circulating that Stalin had arranged for him to be murdered. This story was

given some support when Yagoda, the head of the *NKVD* at the time of his death, was convicted of Gorky's murder in 1938.

Asteroid *2768 Gorky*, was named after him.

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Worldview XXVI: Maxim Gorky

* * *

Happiness always looks small while you hold it in your hands, but let it go, and you learn at once how big and precious it is.

* * *

In the carriages of the past you can't go anywhere.

* * *

Be good, be kind, be humane, and charitable; love your fellows; console the afflicted; pardon those who have done you wrong.

* * *

Only mothers can think of the future — because they give birth to it in their children.

* * *

There is no one on earth more disgusting and repulsive than he who gives alms. Even as there is no one so miserable as he who accepts them.

* * *

When everything is easy one quickly gets stupid.

* * *

When one loves somebody everything is clear – where to go, what to do – it all takes care of itself and one doesn't have to ask anybody about anything.

* * *

You can't do without philosophy, since everything has its hidden meaning which we must know.

* * *

You must write for children in the same way as you do for adults, only better.

* * *

When work is a pleasure, life is a joy! When work is a duty, life is slavery.

* * *

A good man can be stupid and still be good. But a bad man must have brains.

* * *

Everybody, my friend, everybody lives for something better to come. That's why we want to be considerate of every man — Who knows what's in him, why he was born and what he can do?

(1902)

* * *

The aim of literature is to help man to understand himself, to strengthen the trust in himself, and to develop in him the striving toward truth; it is to fight meanness in people, to learn how to find the good in them, to awake in their souls shame, anger, courage; to do all in order that man become nobly strong.

(1901)

* * *

Lenin and Trotsky and all who follow them are dishonoring the Revolution, and the working-class. Imagining themselves Napoleons of socialism. The

proletariat is for Lenin the same as iron ore is for a metallurgist. Is it possible, taking into consideration the present conditions, to cast out of this ore a socialist state? Obviously this is impossible. Conscious workers who follow Lenin must understand that a pitiless experiment is being carried out with the Russian people which is going to destroy the best forces of the workers, and which will stop the normal growth of the Russian Revolution for a long time.

(1917)

* * *

Lenin and Trotsky don't have any idea about freedom or human rights. They are already corrupted by dirty poison of the power, this is visible by their shameful disrespect of freedom of speech and all other civil liberties for which the democracy was fighting.

(1917)

* * *

If the trial of the Socialist Revolutionaries will end with a death sentence, then this will be a premeditated murder, a foul murder. I beg of you to inform Leon Trotsky and the others that this is my contention. I hope this will not surprise you since I had told the Soviet authorities a thousand times that it is a senseless and criminal to decimate the ranks of our intelligentsia in our illiterate and lacking of culture country.

(1922)

* * *

1892–1934 CE **Hayyim Nachman Bialik** (1873–1934, Ukraine and Israel). Poet, essayist, scholar, linguist, translator and rejuvenator of the Hebrew language. In his scientific work, Bialik turned to the lore of Jewish antiquity (especially medieval Hebrew poetry and the Talmudic legendary literature) to bring about the renaissance of the classical Jewish heritage, giving life and verve to modern Hebrew. He restored to the almost defunct Hebrew language its elasticity and originality, showing that it is capable of expressing all the effects of light, sound and color.

Bialik was born in the small hamlet of Radi, in the Volhynia district of the Ukraine and educated by his paternal grandfather in Zhitomir. He studied in the famous Lithuanian Talmudic Academy ("Yeshiva") of Volozhin. He then worked as a timber trader, teacher and publisher. Settled in Tel-Aviv[680] (1923) and became the symbol and leader of Hebrew national and cultural revival in the old-new homeland of Israel. There are few examples in history of real poetry influencing a generation so deeply and so directly. His poem "In the City of Slaughter" prophetically depicts the 1903 Russian pogrom in Kishinev as a prelude to world tragedy. This poem caused thousands of youths in Russia to joint the underground to fight the Czar and tyranny.

With an exceptional mastery of every layer of the Hebrew language, Bialik confronted head-on the struggle of Judaism with other civilizations. Breaking away from the traditional upbringing of the Talmudic scholar, he came under the influence of the enlightenment. He elevated the *mathmid*, that perpetual student of Talmudic scholasticism, to the height of an extraordinary man to whom pure intellect, ascetism, self-sacrifice for 'learning for the sake of learning', had become unity in the highest degree[681]. In this poem Bialik portrayed the rapidly vanishing life of the traditional Jewish past.

To save this heritage and incorporate it into the values of the new age he foreshadowed a new beginning of a more complete life in which not all that is old will be cast away for the sake of the new, but only that part which has become obsolete, and in place of it a new Jewish life would absorb all that is best in the new age. Among his many translations into Hebrew are works of Shakespeare, Schiller and Cervantes.

1893 CE **Albert Londe** (1858–1917, France). Photographer. Published the first book on medical photography.

1893 CE **Wilhelm** (**Carl Werner Otto Fritz Franz**) **Wien** (1864–1928, Germany). Physicist discovered his displacement-law concerning the radiation

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[680] **Maxim Gorky** helped him obtain a permission to leave the Soviet Union.

[681] From these seeds sprang the great Jewish mathematicians, physicists and biochemists of the 19th and 20th centuries.

emitted by a perfectly efficient blackbody. This law states that the spectral peak wavelength is inversely proportional to the absolute temperature of the body. This law led Planck to discover his quantum theory of radiation.

Wien was born in Gaffken, East Prussia. He served as a professor of physics at the Universities of Giessen (1899) and Münich (1920). He received the Nobel Prize for physics in 1911.

1893–1894 CE Renewed worldwide outbreak of *cholera*. Millions perished.

1893–1896 CE **Mordecai Wolfe** (**Waldemar**) **Haffkine** (1857–1930, Russia, France and England). Bacteriologist, immunologist and microbe-hunter. Discovered and used an improved successful method of inoculation against cholera, plague, and typhoid, which reduced significantly the mortality rate of these diseases.

Haffkine was born to Jewish parents at Odessa, Russia, and graduated from the University of Odessa (D.Sc., 1884). Since he could not obtain a suitable position without conversion, he went to Paris to work under **Pasteur** (1888). Here he developed an attenuated strain of the cholera which he tested on himself (1892). The next year he used it on 45,000 people in India where it reduced the death rate by 70% among those inoculated. In 1896 he was deputed by the Indian government to inquire into the bacteriology of the plague. He discovered an effective method of inoculation, and succeeded in reducing the mortality by 80 percent. The same method was used successfully in Egypt (1947).

1893–1912 CE **Karl Pearson** (1857–1936, England). Mathematician. One of the founders of modern statistics. His work established statistics as a subject in its own right.

Pearson defined the *standard deviation* of a set of measurements and the *Chi-square* test of goodness of fit[682] (1900). Stimulated by the evolutionary

682 The χ^2 (*Chi-squared*) test determines the goodness of fit of a given model to noisy data. Let a sample consist of N trials and let $F(n)$ be the frequency of event n (i.e. the number of occurrences of the value n). If the *parent distribution* (model) which we are testing is $f(n)$, then the frequency predicted by the parent distribution is just $Nf(n)$. The difference $Nf(n) - F(n)$ for each n characterizes the difference in the two frequencies. A measure of *goodness of fit* is

$$\chi^2 = \sum_n \frac{[Nf(n) - F(n)]^2}{Nf(n)},$$

yielding the a weighted mean of the square of the fractional difference of the expected and observed frequency distributions. A mathematical table then

writings of **Francis Galton** (1822–1911, England), he became immersed in the application of statistics to biological problems of heredity and evolution[683]. Pearson's other discoveries included the *Pearson coefficient of correlation* (1892), the theory of *multiple and partial correlation* (1896), the *coefficient of variation* (1898), work on *errors of judgment* (1902), and the *theory of random walk* (1905).

Pearson was born in London and educated at University College, London and King's College, Cambridge. He began his career as a lawyer (1881–1884) and also published literary works (1880–1882). During 1884–1933 he taught at University College, under the varied titles of professor of mathematics and mechanics (1884), geometry (1891) and eugenics (1911). He wrote books on the philosophy of science, and on statistics.

1893–1920 CE **Max Weber** (1846–1920, Germany) and **Emile Durkheim** (1858–1917, France). Founding fathers of modern *sociology*.[684]

Durkheim produced the first major sociological work (1853) employing a rigorous scientific methodology and single-handedly established sociology as an independent academic discipline. His thinking derived from French rationalism through which he sought to develop elementary forms as building-blocks of a theory of society; e.g. endeavored to explain *religion* in terms of totemism as an elementary form.

The heart of his sociology is a rejection of the individual basis of society[685]and the notion that society was prior to the individual. Therefore, methodologically, the social could not be reduced to the psychological.

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converts the values of (χ^2) and the number of degrees of freedom (DoF) to the probability that the model is correct. The integer DoF is N – the number of parameters in $f(n)$.

683 He showed that a wide variety of frequency distribution functions (including the 'Gaussian') can evolve from a single differential equation

$$\left[\frac{1}{y}\frac{dy}{dx} = (x+a)/(b_0 + b_1x + b_2x^2)\right]$$

by suitable choices of its coefficients.

684 The science of sociology was invented at least twice; once during 1830–1842 by **Auguste Comte** (1898–1857, France), who gave it its name by combining the Latin term *societas* with the Greek *logos*.

685 The *utilitarian* idea of **Hobbes** and **Bentham** that the *individual* and his self-interest comprise the unit of society and that the community is a superstructure that to succeed must bribe individuals into cooperation based on their material self-interest (social contract, minimalist government).

Thus Durkheim attacked the notion that society was simply a 'contract' between individuals, for the norms which govern contracts are embedded in a broader context of *moral* understanding or social solidarity. To him, the *religious* bond was simply the symbolic representation of the social bond, expressed through ritual.

Durkheim was born in Espinal, in the Lorraine, France to Jewish parents. Taught philosophy at Bordeaux (1887–1902) and was professor of sociology and education at the Sorbonne (1902–1917). He lost a son in WWI, where half of the 1913 class of the Ecole Normale (the school of France intellectual elite) was killed.

His books: *Suicide* (1897); *The Elementary Forms of Religious Life* (1912).

Max Weber, economist and sociologist, was born at Erfurt, Germany. Professor at Berlin (1893), Freiburg (1894), Heidelberg (1897–1903) and Munich (1919). His theory derived from German historical thinking. Taking his cue from Nietzsche, he claimed that successful capitalism in the Protestant countries derived from the value-positing of their charismatic founders (Calvin, Luther). What mattered to him was not the truth of religious experiences but the values it instilles in people.

The Modern Bicycle (1893)

Bicycle, a light two-wheeled steerable vehicle, propelled by human muscular power, evolved in the 19^{th} *century into an important popular means of transportation and recreation all over the world.*

Suggestions of vehicles having two or more wheels and propelled by muscular effort of the rider (or riders) are to be found in very early times, even on the bas-reliefs of Egypt and Babylon and the frescoes of Pompeii. There is some evidence for the presence of such vehicles in medieval England.

A primitive version of the bicycle appeared in France in 1779*; it was known as a velocipede, and invented by* ***Blanchard*** *and* ***Magurier****. It differed little from a later version known as célérifère, proposed by* ***Mede de Sivrac*** *(1790). His model consisted of a wooden bar rigidly connecting two wheels placed one in front of the other, and was propelled by the rider, seated astride the bar, pushing against the ground with his feet.*

The next advance was made in 1816 *by the Baron* ***Karl Drais von Sauerbronn*** *(1785–1851, Germany). In his contraption, the front wheel was pivoted on the frame so that it could be turned sideways by a handle, thus serving to steer the machine. It was known as the* draisine. *A similar machine, the* celeripede, *also with a movable front wheel, is said to have been ridden by* ***Joseph Nicèphore Niépce*** *(1765–1833, France) in Paris in* 1816. *The Scot blacksmith* ***Kirkpatrick Macmillan*** *added in* 1839 *to the* draisine *connecting rods working on the rear axle. Thus fitted, the* draisine *had wooden wheels with iron tires, the leading one about* 75 *cm in diameter and the rear driving one about* 100 *cm. It formed a prototype, though not the ancestor, of the modern rear-driven safety bicycle.*

About 1865, ***Pierre Lallement*** *in Paris constructed a bicycle in which the* front *wheel was driven by pedals and cranks attached directly to its axle, but it is unclear whether the origin of this idea should be attributed to him or to* ***Ernest Michaux***, *the son of his employer, who was a carriage repairer. (Lallement took his machine to the United States, and in* 1866 *was granted a patent which had an important influence on the subsequent course of the cycle industry in that country.) This machine, consisting of a wooden frame supported on two wooden wheels, soon became popular in England as well as in France and America, and came to be called* bicycle (or bysicle) *by those who took it seriously and* bone-shaker *by those who did not.*

Improvements quickly followed, chiefly in England, for the popularity of the machine in America was short-lived, and in France the industry was checked by the Franco-German war. Rubber tires, in place of iron ones, appeared in 1868 *the chain-drive was invented by* ***J.F. Tretz*** *in Germany in* 1869, *and applied to bicycles by* ***Guilmet*** *(France) in* 1870. *Suspension wheels with wire spokes in tension were seen in London.*

During the 1870*'s, a new type of a bicycle appeared with a large driving wheel in front and a small trailing behind. The same type retained its supremacy until* 1885. *The same year saw the first commercially successful* safety bicycle *produced by* ***John Kemp Starley*** *(England); it had equal-sized wire-spoked wheels, "diamond shape" steel-tubing frames, cone (and then ball) bearings at points of friction, crank and pedals in the center with a chain and sprocket drive to the back wheel. The rider sat so far back that he could not be thrown forward over the handles. Finally, the machine was made more* stable *with a* curved front wheel fork[686]

[686] A stable bicycle is one whose *forkpoint* (the point of intersection of a projection along the front steering axis and a horizontal line through the wheel center) falls as the wheel turns into a lean when the bike is tilted. *Gyroscopic effects* have little to do with riding stability, although if the bike is pushed off riderless, then the gyroscopic effect from the wheels will help stabilize the bike for a while.

With the invention of air-filled rubber tires by ***John Boyd Dunlop*** *(1840–1921, Scotland) in* 1888*, and the addition of coaster brakes and adjustable handle bars — the modern version of the bicycle was ready by* 1893*. Early forms of the gear shift came into use soon after* 1900*.*

By 1897 *about* 4 *million Americans were riding bicycles regularly, more than at any previous time. During the early* 1900*'s, the rapid development of the automobile caused many people to lose interest in bicycles, but in the early* 1970*'s bicycle riding in the United States became more popular than ever before, and* 75 *million bike riders were on the roads. In* 1990 *this number climbed to* 100 *million in the United States alone. The Annual Tour de France (began 1903), the most famous bicycle road-race, covers* 4800 *km and takes* 21 *days. The cyclist with the shortest total riding time is the winner.*

About 100 million bicycles were produced worldwide in 2000: China (60 m); India (11 m), Taiwan (7.5 m); Japan (4.7 m); Italy (3.2 m); UK (1.2 m); USA (1.1 m).

The first motorcycle[687] *was invented by* ***Gottlieb Daimler*** *in* 1885*, who attached a 4-stroke piston engine to a wooden bicycle frame. During the* 1900*'s, with continual improvements, motorcycles developed into useful, dependable vehicle.*

[D.E.H. Jones, "*The Stability of the Bicycle*", *Phys. Today*, April 1970; S.S. Wilson, "*Bicycle Technology*", *Sci. Amer.*, March 1973; A.T. Jones, "*Physics and Bicycles*, *Am. J. Phys.* **10**, 332, 1942.]

687 Two-wheeled vehicles powered by internal combustion engines comprise *motorcycles*, *motor scooters* and *mopeds* [abbr. for 'motor-assisted pedal cycle']. These all are similar in principle. The motorcycle comprises four main sections: the frame, the engine with gearbox and drive components (chain or drive shaft), the road wheels, and the petrol tank. Two-stroke and four-stroke engines are used as power units for motorcycles. The power is transmitted to the rear wheel through a gearbox and thence through sprockets and chains or through a drive shaft. Motorcycles and mopeds have wire-spoked wheels, whereas scooters generally have solid wheels like those of a car.

The Last Great Naturalists[688]

"When white man first come to Canada, he shoot all big animals, haul off meat. Next trip he trap all small animals, haul off fur. Third time he cut down all big trees, haul off lumber. Fourth time, cut down all small trees, make paper. Now he haul off all rocks".

Indian Chief's lament

The accelerated advance of science and technology that followed in the wake of the Industrial Revolution was mostly accomplished in the laboratories and the institutions of European universities and industrial research centers. Yet, the 19th *century was still abundant with a different breed of natural scientists who sought to study nature in its own milieu and for its own sake. These were explorers, geographers, ornithologists, entomologists and other naturalists who went out of the cities and away from the centers of higher learning to rediscover nature and our place in it.*

John James Audubon *(1785–1851, U.S.A.). Naturalist and artist. Captured for posterity the images of contemporary birds and animals of North America.*

Henry Baker Tristram *(1822–1906, England). Naturalist, ornithologist, and the first scientific explorer of the Sahara (1855–1857) and the Lands of the Bible (1863–64, 1872, 1880–81, 1894–95). Among the first ardent Darwinists.*

Jean Henri (Casimir) Fabre *(1823–1915, France). One of the greatest entomologists ever. His keen observations, patience, extraordinary intuitive power and unsurpassed ability to transmit the mysteries of the insect world to his fellow men, made him a unique figure in the history of science.*

John Muir *(1838–1914, U.S.A.). Explorer, naturalist and writer. The first man to explain the glacial origin of the Yosemite Valley. Explored Alaska, the Arctic, Africa, Asia and the United States.*

Ernest Thompson-Seton *(1860–1946, Scotland and Canada). Naturalist, artist, animal observer and writer.*

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688 For further reading, see:

- Adams, A.B., *Eternal Quest*: *The Story of the Great Naturalists*, Isaac Putnam's Sons: New York, 1969, 509 pp.

Charles William Beebe *(1877–1962, U.S.A.). Naturalist, explorer and writer. Explored the tropical jungles of Borneo, Guyana and Trinidad (1916–1925). He was first to dive into the depths of the ocean in a diving chamber (bathysphere), reaching a depth of* 800 *m (1930).*

Beebe was born in Brooklyn, NY. He became curator of ornithology (bird studies) at the New York Zoological Society in 1899. *He helped found the Society's Tropical Research Department in* 1916, *and wrote numerous books about his adventures.*

1894 CE Bacteriologists **Shibasaburo Kitasato** (1852–1931, Japan) and **Alexandre Yersin** (1863–1943, Switzerland) discovered simultaneously and independently the causative organism, *Pasteurella pestis*, of bubonic plague, during an outbreak at Hong Kong.

Prevention was found to be possible by inoculation with a killed vaccine or by injection of a live avirulent organism i.e. a relatively harmless strain of the bacteria. Antibiotic drugs, give good results when administered to infected patients.

1894, Feb 15 A group of anarchists attempted to blow up the *Greenwich Observatory*.[689]

1894 CE **Thomas Jan Stieltjes** (1856–1894, Netherlands and France). Dutch-born French mathematician. Made notable contributions to the analytic theory of continued fractions and integration theory.

Stieltjes was born in Zwolle, Netherlands, and educated at the universities of Delft, Leyden and Groningen. He moved to France in 1885 and became a professor of mathematics at the University of Toulouse, where he remained for the rest of his life. Stieltjes contributed to the fields of divergent and conditionally convergent series, number theory and spherical harmonics. He proposed the *Riemann-Stieltjes integrals*[690] and the *Lebesgue-Stieltjes inte-*

[689] The event prompted **Joseph Conrad** (1857–1924) to write his masterpiece *The Secret Agent* (1907). In this political-detective novel Conrad expressed society's disillusionment from science, the late 19th century 'god-substitute' that failed.

[690] $\int_a^b f(x)dg(x) = \lim_{\max|x_i - x_{i-1}|\to 0} \sum_{i=1}^m f(\xi_i)\left[g(x_i) - g(x_{i-1})\right]$ for arbitrary sequence of partitions

$$a = x_0 < \xi_1 < x_1 < \xi_2 < x_2 < \cdots < \xi_m < x_m = b.$$

grals which have wide applications in probability, distribution and Laplace-transform theories.

1894–1897 CE **George Oliver** (1841–1915, England), physician and physiologist and **Edward Albert Sharpey-Schäfer** (1850–1935, England) first demonstrated the action of a *specific hormone*: the effect of an extract of the adrenal gland (*adrenaline* or *epinephrine*) on blood vessels and muscle contraction. Upon injection into normal animals it produced a striking elevation in blood pressure.

John Jacob Abel (1857–1938, U.S.A.), pharmacologist and physiological chemist first isolated *epinephrine* (1897). He also developed artificial kidney (1914) and crystallized insulin (1926).

1894–1914 CE **Jean Léon Jaures** (1859–1914, France). Social philosopher, father of social democracy and socialist leader. With his political instincts inspired by the French Revolution, Jaures opposed imperialism in all its forms, yet he believed in the rights of the individual over the state.

Jaures was born in Castres and attended the Ecole Normale Superieure in Paris. After graduating he lectured on philosophy at the University of Toulouse (1883–1885) and earned his doctorate in philosophy there (1891). During 1885–1889 and 1893–1914 he was a member of the *Chamber of Deputies* as an independent socialist.

Involved in the Dreyfus affair in 1894 as a supporter of Dreyfus, Jaures argued that Alfred Dreyfus' treason conviction was based upon forged evidence.

A co-founder in 1904 of the socialist newspaper *L'Humanite* (along with Rene Viviani and Aristide Briand, both future French Prime Ministers), Jaures was a man of numerous talents. A prolific writer, he proved himself as capable at giving a speech as penning it.

A firm advocate of the Second International socialist movement, he accepted their argument preventing its members from participating in so-called 'bourgeois' governments. As such he never accepted a position within the French cabinet; which meant, given his leadership of the party (since 1905), that the Socialist Party was also denied a role in government.

As the storm clouds of war approached, Jaures' popularity waned somewhat, as he continued to advocate closer relations with Germany. Indeed, at the height of the July Crisis of 1914 he traveled to Brussels to try to persuade German socialists to strike against potential war in Europe.

The limit exists whenever $g(x)$ is of bounded variation and $f(x)$ is continuous in $[a, b]$.

Shortly after his return from Brussels to Paris, on 31 July 1914, Jaures was murdered by a 29 year old nationalist fanatic, Raoul Villain; three days later Germany declared war with France.

1894–1925 CE **Schlomo Sigmund Freud** (1856–1939, Austria). Neurologist and founder of psychoanalysis. One of the most influential thinkers in modern times. Revolutionized our view of human nature and affected almost every department of our culture. His method of treatment led to the use of psychotherapy, and greatly extended our sensitivity about human relations in general, and between doctor and patient in particular. Freud's work on the origin and treatment of mental illness helped form the basis of modern *psychiatry*. He especially influenced the field of abnormal *psychology* and the study of personality.

Freud's theories on sexual development led to open discussion and treatment of sexual matters and problems. His stress on the importance of childhood helped teach the value of giving children an emotionally nourishing environment. His insight also influenced the fields of *anthropology* and *sociology*. In art and literature, Freud's theories encouraged understanding of *surrealism*, which like psychoanalysis explores the inner depths of the unconscious mind. Freudian concepts have provided subject matter for many authors and artists.

Some of the Freud's theories are controversial. Future science will have to settle these problems, and it will be probably a long time before the value of his achievement is ultimately determined. But at face value, Freud was the discoverer of a new humanistic discipline whose significance went beyond the boundaries of psychiatry. He brought into the world a new definition of human fate, because he placed in the hands of man the means with which to alter impediments which were previously considered irremediable.

Freud at times stated that the psychic apparatus was free from any *anatomical* implications, but it is certain that he hoped for an eventual integration of his theory with neurology and that he always considered the *biological* facts to be quite relevant to his decisions about his own model. The following five biological facts become familiar to us only since his death and they decisively refute the model of *passive* reflex mechanism:

1. The nervous system is perpetually *active*. Electroencephalographic (EEG) data have shown that even in the deepest sleep and in coma the brain does not cease its activity; at these times of minimal input and behavioral output, hypersynchrony seems to produce the most massive discharges, the resting nerve cell periodically fires (produces a spike potential), and its nontransmitted activity waxes and wanes, all without any outside stimulation.

2. Thus, the effect of stimulation is primarily to *modulate* the activity of the nervous system. It may step up the frequency of discharge but mainly imposes an order and pattering on it; that is to say, encodes it.

3. The nervous system does not *conduct* energy; the nervous impulse is rather propagated. An appropriate physical analogy is not current flowing along a wired circuit, but a signal traveling along the axon to the synapses which in turn pass signals on to the dendrites of other cells.

4. The energies of the nervous system, whether or not triggered by the sensory organs, are *different in kind* from the impinging external stimuli. The sensory surface acts as a *transducer*.

5. The tiny energies of the nerves bear encoded information and are quantitatively negligible; their amount bear no relation to the motivational state of the person. The electrical phenomena associated with the neuron are accessible to quantitative study today, but this work offers no basis for the economic point of new — the assumption that mental events might be meaningfully examined from the standpoint of the 'volumes of excitation' involved. Rather than this kind of 'power engineering', '*information engineering*' seems to be relevant discipline.

It stands to reason that most of Freuds provisional ideas in psychology will presumably some day be based upon an organic substructure.

Freud was born to Jewish parents (of Chassidic rabbinic stock on both sides) in the town of Freiberg, Moravia, which is today part of the Czech Republic, but was then part of the Austrio-Hungarian Empire. His father's family was settled for a long period at Cologne, but fled eastward as a result of the persecution of the Jews during the 14th century. In the course of the 19th century they migrated from Lithuania through Galicia (*Buczacz*) into the Habsburg Empire.

In 1859 his father Jacob Freud (1815–1896), moved to Vienna. Earlier (1855) he married, the second time, to Amalie Nathanson (1835–1930), the descendant of a famous Talmudic scholar, Nathan Halevi Charmaz of Brody, Poland. Sigmund was the eldest and favorite of her 8 children. From an early age, Freud dedicated himself to learning which remained a unique trait identified with the Jewish ethics. Yet due to the liberal spirit prevailing then among the Viennese Jews, he was subjected to non-Jewish upbringing.

A youthful interest in science and human personality[691] led him to enter the University of Vienna medical school (1873). He took his degree in medicine (1881) and married (1882) Martha Bernays (1861–1951), a granddaughter of the chief Rabi of Hamburg.

After serving as intern and resident physician in a hospital, he decided to specialize in *neurology* (the treatment of disorders of the nervous system) and went to Paris (1885) to study under **Jean-Martin Charcot**, a leading authority of hysteria.

He returned to Vienna[692] (1886) and began medical practice, specializing in nervous diseases. The case histories of his patients convinced him that *sexual causes* played a major role in many forms of *neurosis*. He gradually formed ideas about the origin of mental illness, using the term *psychoanalysis* for both his theory and his method of treatment.

When he first presented his ideas in the 1890's, other physicians rejected with hostility, but Freud eventually attracted a group of followers (1902), and by 1910, gained international recognition and acclaim. During the following

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[691] It is important to note that there is a link between psychoanalysis and *Jewish mysticism*: Freud himself mentioned the 16^{th} century Jewish physician **Solomon Almoli** (1490–1542, Turkey) whose book *The Solution of Dreams* (1516) gives a description of sexual symbolism, wish fulfillment and word-play as elements found in dreams. Many counterparts of Freudian theory were found in the *Zohar* (the mystical writings of **Moshe de Leon**, 1286 CE), such as the portrayal of primordial man where the divine act of creation was given an erotic character and where sex relations were treated as avenues of salvation.

Another great sage, who anticipated so many of Freud's views was **Baruch Spinoza** (1632–1677). Indeed the essence of his philosophy which was expressed in his dictum: *Humanus actiones non ridere, nec lugere, nec detestera, sed intelligere* (Human actions should not be mocked, should not be lamented, nor execrated, but should be understood), could be taken as the source and origin of Freud's whole system.

A more recent connection of psychoanalytic thought to *chassidism* was suggested by Freud, whose father Jacob, came from chassidic stock.

[692] Vienna in the 1890s was famous for its Blue Danube, its wit, sensuality, waltzes and cafés. But it had a darker side: the Empire was in deep economic trouble. The jobless were crowded in slums and flophouses. Karl Lueger, Mayor of Vienna, made anti-Semitism politically fashionable. Austria's first anti-Semitic party was formed in 1880 and during the next 60 years or so anti-Semitism was made the central issue of both municipal and state elections. Between 1880 and 1914, almost two million Jews came to the United States from Eastern Europe. Throughout his entire adult life Freud's Vienna continued to remain a virulently anti-Semitic city. In fact, anti-Semitism pursued him *all his life*.

decade, Freud's reputation continued to grow, but two of his early disciples, **Alfred Adler** and **Carl Jung**[693] split with him. By 1914–1915 Freud had developed his earlier theory of infantile sexuality to cover and explain the distinction between conscious and unconscious functioning by means of the concept of *repression*.

In 1919 Freud was finally made full professor at the University of Vienna. But his appointment did not allow him the privilege of a seat on the board of the faculty.[694]

By the beginning of 1920, the work of Freud had contributed gradually to establish the category of the 'neuroses' in contemporary psychiatry. He went on (1923–1927) to develop his earlier views by stressing the role of the *ego*[695]

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[693] Freud's former disciple (1906–1913). Many years later (ca 1935), Jung contrasted Freud's inferior "Jewish" psychology with Hitler's perfect scientific doctrines of Aryan superiority. After Freud's books were burned in public by the German Nazis (May 1933), Jung published books and articles asserting the negative foundations of Freud's psychology. His primary function was to show that as Jews, Freud and his followers were unable to understand the "*superior German psyche embodied in the powerful National Socialism, at which the whole world looks on in astonishment*" (1935). Freud not wishing to degrade himself in nonsential arguments, remarked slyly: "*What Jung contributed to psychoanalysis, we can dispense with...*".

[694] It took Freud 38 years to climb the academic ladder from MD (1881), through the ranks of *privatdocent* (1885–1902) and associate professor (1902–1919). The fact that he was Jewish was one reason for the delay. The other was that he established himself as a pioneer in a new field of research which was looked upon by the leading men in psychology and psychiatry as fantastic and even indecent! When, at 70 (1926) congratulation arrived from leading scientists all over the world, the University of Vienna did not send him even a letter of felicitation.

[695] The mind consists of three parts:

- *id*: mental representation of the biological instincts, such as the drive to satisfy hunger and the drive to satisfy sexual needs. It does not distinguish between the internal mind (e.g. mental image of food) and the outside environment (the food itself).
- *ego*: controls the behavior that bridges the gap between mental images and the outside world. It distinguishes between the internal mind and the external reality, e.g. the ego directs a hungry person to look for and eat real food.
- *superego*: governs moral behavior. It is the mental representation of society's moral code. It seeks to limit behavior based on the drives of the id.

In mentally healthy individuals, the three parts of the mind work in harmony.

and the *super-ego* and to apply his ideas (1927–1930) to account for religious belief, social discontent, and produce a range of new concepts to describe and explain human reactivity.

In 1923 Freud was operated for cancer of the jaw and palate, the first of 33 operations. For the last 16 years, Freud often suffered agonizing pain; his speech and hearing were affected and eating was difficult.

When the Nazis invaded Austria (1938) they burned his books and banned his theories. Friends got him out of Austria to England.[696] He left his home in Vienna in which he lived continuously for 42 years: in the same house, in the same street, in the same Jewish section. The British Medical Journal said (1938): "The medical profession of Great Britain will feel proud that their

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But in others the parts may conflict, resulting in psychological disturbances.

Freud observed that many patients behaved according to drives and experiences of which they were not consciously aware. He thus concluded that the unconscious plays a major role in shaping behavior. He also concluded that the unconscious is full of memories of events from early childhood — sometimes as far back as infancy; if these memories were especially painful, people kept them out of conscious awareness (defense mechanisms). Freud believed that patients used vast amounts of energy in forming defense mechanisms. This tied energy could affect a person's ability to lead a productive life, causing an illness that Freud called *neurosis*.

Freud also concluded that many childhood memories dealt with sex. He theorized that sexual functioning begins at birth, and that a person passes through several psychological stages of development from infant sexuality to adult sexuality. If for some reason, the normal pattern of sexual development is interrupted in some individuals, mental illness in adulthood could result.

[696] Shortly after the *Anschluss* (Mar. 11, 1939), Freud's home in Berggasse was invaded by a gang of German Storm Troopers who helped themselves to whatever money was in the house, including 6000 Austrian schillings (then about 840 dollars) which belonged to the Psychoanalytic Association. Freud reacted to this "house-call", saying: "I've been a doctor for fifty years, but I never got 6000 schillings for a visit to an old, sick man".

In June 1938, thanks to the intervention of **Marie Bonaparte** (who paid the Nazis a ransom of 35,000 dollars), the American ambassador to France, and the British Home Secretary, Freud and the members of his immediate family received permission to leave Vienna for London. Before his departure, the Gestapo forced him to sign a certificate declaring that he had been well treated by the authorities. Freud complied, but added a sentence of his own in an advertising copywriter style: "*Ich kann die Gestapo jedermann auf das beste empfehlen*" ("I can heartly recommend the Gestapo to anyone").

country has offered an asylum to professor Freud and that he chose it as his new home". He died of the jaw and palate cancer in London on Sept 23, 1939.

His most important writings include: *The Interpretation of Dreams* (1900); *Three Essays on the Theory of Sexuality* (1905); *Totem and Taboo* (1913); *General Introduction to Psychoanalysis* (1920); *The Ego and the Id* (1923); *Civilization and Its Discontents* (1930).

Worldview XXVII: Freud

* * *

"I have often felt as if I had inherited all the passions of our ancestors when they defended their Temple, as if I could joyfully cast away my life in a great cause."

(1886)

* * *

"If you do not let your son grow up as a Jew, you will deprive him of those sources of energy which cannot be replaced by anything else. He will have to struggle as a Jew and you ought to develop in him all the energy he will need for the struggle. Do not deprive him of that advantage."

(to Max Graf, 1895)

* * *

"I found the essential characteristic and most significant part of my dream theory — the reduction of dream-distortion to an inner conflict — later in a writer who was familiar with philosophy though not with medicine, the engineer ***Josef Popper-Lynkeus****. A special feeling of sympathy drew me to him, since he too had clearly painful experience of the bitterness of the life of a Jew and the hollowness of the ideals of present-day civilization."*

(1899)

* * *

"I am not really a man of science, not an observer, nor an experimenter, and not a philosopher. I am by temperament nothing but a conquistador…

with the curiosity, the boldness and the tenacity that belong to that type of person."

* * *

"Poets are masters of us ordinary men, in knowledge of the mind, because they drink at streams which we have not yet made accessible to science."

* * *

"My life and work has been aimed at one goal only: to infer or guess how the mental apparatus is constructed and what forces interplay and counteract in it."

* * *

"I have no concern with any economic criticism of the communistic system: I cannot inquire into whether the abolition of private property is advantageous and expedient. But I am able to recognize that psychologically it is founded on an untenable illusion. By abolishing private property one deprives the human love of aggression of one of its instruments... This instinct did not arise as a result of property; it reigned almost supreme in primitive times when possessions were still extremely scanty..."

* * *

"Hatred of Judaism is at bottom hatred of Christianity."

* * *

"Toward the person who has died we adopt a special attitude: something like admiration for someone who has accomplished a very difficult task."

* * *

"From error to error one discovers the entire truth."

* * *

"Analogies make one venture to regard obsessional neurosis as a private religious system and religion as a universal obsessional neurosis."

(1907)

* * *

"There are no such things as Aryan or Jewish science. Results in science must be identical, though the presentation of them may vary. If these differences mirror themselves in the apprehension of objective relationships in science, there must be something wrong."

(1913)

* * *

"God is nothing other than an exalted father."

(1913)

* * *

"Totemism, with its worship of a father substitute, may be regarded as the earliest appearance of religion in the history of mankind, and it illustrates the close connection existing from the very beginning of time between social institutions and moral obligations."

(1913)

* * *

"You may be sure that if my name were Oberhuber, my new ideas would, despite all the other factors, have met with far less resistance."

* * *

"Only to my Jewish nature did I owe the two qualities which had been indispensable to me on my hard road: because I was a Jew I found myself free from many prejudices which limited others in the use of their intellect, and, being a Jew, I was prepared to enter opposition and to renounce agreement with the 'compact majority'."

(1926)

* * *

"They will always throw stones at me. You see, I have troubled humanity's sleep."

* * *

"Religion is an attempt to get control over the sensory world, in which we are placed, by means of the wish-world, which we have developed within us as a result of biological and psychological necessities. But it cannot achieve its end. Its doctrines carry with them the stamp of the times in which they originated, the ignorant childhood days of the human race... The ethical commands, to which religion seek to lend its weight, require some other foundation instead, since human society cannot do without them, and it is dangerous to link up obedience to them with religion itself."

* * *

"You need not be the victim of your own past, or your own environment."

* * *

"Sometimes a cigar is just a cigar."

History of Biology and Medicine, IV – The 19^th century

During this epoch, biology progressed along four major avenues: cytology *(cell theory),* genetics*,* bacteriology *and* physiological chemistry*.*

Into the 19^th century, explorer-naturalists such as ***Alexander von Humboldt*** *tried to elucidate the interactions between organisms and their environment, and the ways these relationships depend on geography-creating the foundations for* biogeography*,* ecology *and* ethology*. Many naturalists began to reject essentialism and seriously consider the possibilities of extinction and the mutability of species. These developments, as well as the results of new fields such as* embryology *and* paleontology*, were synthesized in Darwin's theory of evolution by natural selection. The end of the 19^th century saw debates over spontaneous generation and the rise of the germ theory of disease and the fields of* cytology*,* bacteriology *and* physiological chemistry*, though the problem of inheritance was still a mystery.*

Wöhler *showed In 1828 that organic molecules, such as* urea*, can be created by synthetic means that do not involve life, and thus provided a powerful argument against* vitalism*. The first enzyme,* diastase*, was described in 1833, and the science of* biochemistry *may be said to have begun.*

By the mid 1850's the miasma theory of disease was largely superseded by the germ theory of disease*, and* antisepsis *became a medically important invention. Surgery and medicine was advanced in 1858 when Gray's* Anatomy *was first published.*

In about the 1880's the science of bacteriology *began to be formed, especially through the work of* ***Robert Koch****, who introduced methods for growing pure cultures on* agar gels *containing specific nutrients in Petri dishes. He also introduced the "Koch's postulates" for the reliable determination of when a proposed microorganism caused a specific disease. The long-held idea that living organisms could easily originate from nonliving matter (spontaneous generation) was finally discredited in a series of experiments carried out by* ***Louis Pasteur****.*

Schleiden *and* ***Schwann*** *proposed the* cell theory *in 1839: the basic unit of organisms is the cell and all cells come from preexisting cells.*

The British naturalist ***Charles Darwin****'s seminal work* On the Origin of Species *(1859) described* natural selection*, the primary mechanism for* evolution*.*

In 1866 genetics *had its beginnings in the work of the Austrian monk* ***Gregor Mendel*** *who formulated his* laws of inheritance*. However, his work*

was not recognized until 35 years afterward. Three years after his publication, in 1869 ***Friedrich Miescher*** *discovered what he called nuclein, which was later realized to be a crude preparation of DNA.*

The cytologist ***Walther Flemming*** *in 1882 was the first to demonstrate that the discrete stages of mitosis were not an artifact of staining, but occurred in living cells, and moreover, that chromosomes doubled in number just before the cell divided and a daughter cell was produced. In 1887* ***August Weismann*** *proposed that the chromosome number must then be halved in the case of the sexual cells, the gametes. This was shortly proved to be the case and the process of meiosis began to be understood.*

In medicine, the rapid advancement of physical and chemical theories together with their corresponding industrial technologies induced important medical inventions, Thus we witness the stethoscope *(1816,* ***T. Laennec****);* dental plate *(1817,* ***A. Plantson****);* endoscope *(1827,* ***P. Segalas****);* anesthetics *(1846,* ***W. Morton****);* ophthalmoscope *(1851,* ***H. von Helmholtz****);* hypodermic syringe *(1893,* A. Wood*);* barbiturate *(1863,* ***A. Von Bayer****);* antiseptic *(1865,* ***J. Lister****);* rabies vaccination *(1885,* ***L. Pasteur****);* contact lens *(1887,* ***A. Frick****).*

Cytology

It is the study of the internal structure and organization of cells. Microscopic studies of the structure of the cell provided an explanation of cell division *and served as a foundation for* genetics*. These studies also showed that each structure has some function, and that each cell activity is related to changes in chemicals that make up the cell. Structures now recognizable as* cell nuclei *were described by many early microscopists. The term, however, was coined by* ***Robert Brown*** *(1833).*

*The term '*cell theory*' was introduced by* ***T. Schwann*** *(1839) to include the principle of construction of all organic products. The studies of* ***Hugo von Mohl*** *and* ***Max Schultze*** *clarified that plant cells alone possessed walls; what they shared with animals was the material within their walls, the primordial* protoplasm*. Evidence also accumulated that cells were formed by division of existing cells. This new model of cell formation, developed for animals by* ***Robert Remak****, was generalized and popularized by the anatomist* ***Rudolf Virchow*** *(1858).*

Embryonic development appeared as successive cell divisions from an egg cell *formed in the mother. By 1900 it was generally accepted that plants and animals were made up of discrete masses of nucleated protoplasm propagated by division.*

Bacteriology

Bacteria, one-celled organisms, were first seen by ***Leeuvenhoek****. Originally confused with* protozoa*, bacteria were variously called* animalcules*,* microbes *or* vibrionia*.*

During the 18th century bacteria contributed to the spontaneous generation controversy, as ***Spallanzani*** *(1729–99) refuted the assertion that microbes appeared in sealed flasks of boiled broth. Spallanzani demonstrated microbes appeared only after inadequate heating or the admission of air into the vessel. Bacterial studies outside medicine remained superficial until 1872 when* ***F.J. Cohn*** *defined and named bacteria, distinguishing four groups on the basis of external form and specific fermentative activity. He recognized bacteria take nitrogen from simple ammonia compounds, elucidated their life-cycles, identified spores and suggested bacteria were motile cells devoid of walls. Determining bacterial temperature limits,* ***Cohn****,* ***Pasteur*** *and* ***Tyndall*** *effectively ended the spontaneous generation controversy with their studies on sterilization.*

Some bacteria were suggested to be pathogenic by ***Casimir Davaine****'s experiments (1850) indicating anthrax was caused by rod-shaped organisms, 'bacteridia', found in the blood of diseased animals.* ***Robert Koch****'s classic experiments confirmed these suggestions in 1876. Koch also developed techniques for handling bacteria, introducing solid nutrient media (agar-agar) to grow pure cultures, and devising methods for fixing bacteria.*

Dimitri Ivanovski *using* ***Chamberland****'s bacteriological filter (1884) explained the infectiousness of tobacco mosaic disease (1892) by showing it can be transmitted via cell-free filtrates from leaves of diseased plants to leaves of healthy plants. Thus the term 'filterable virus' was coined; later filterable was dropped and 'virus' took on its modern meaning.* ***F. Loeffler*** *and* ***P.Frosch****'s work on foot-and-mouth disease first demonstrated (1898) an animal disease in which a virus was the causative agent. Yellow fever was the first human disease proved (1901) to be caused by a filterable virus by* ***W. Reed****.*

During the 1890's increased knowledge of soil and water bacteria was responsible for completion of the nitrogen, sulfur and carbon cycles. Nodule-forming bacteria living in the roots of leguminous plants were found to fix atmospheric nitrogen. As a result of ***Winogradsky****'s and* ***Beijerinck****'s work on anaerobic bacteria, knowledge of a whole world of organisms able to live on elementary nitrogen, iron or sulfur has emerged.*

Evolution Theory, Darwinism and Mendelian Genetics

*The first broad theory of the transmutation of organic forms was by **Jean-Baptiste Lamarck** (1800–1809). He advanced the idea that the simplest forms of life had been* spontaneously *generated and that from there all other forms of life had been successively produced.*

Lamarck explained organic change as the result of two factors: the 'power of life', which was responsible for the general scale of increasing complexity formed by the different animal classes; and the influence of particular environments, accounting for the fact that species and genera could not be aligned in a single series [natural order]. Explaining how animals change in response to different environments Lamarck affiliated himself with the idea of the inheritance of acquired characters. *In his view, animals responded to environmental changes by developing new habits, leading to changes in the animals' structures which were then passed on to offspring. It took many generations for the effects of this to become appreciable.*

*Lamarck had relatively little evidence for his theory beyond the structural similarities among living things. He believed the earth's age to be immeasurably greater than his predecessors had supposed, a prerequisite for any theory of the gradual change of living things over time. In Lamarck's day, however, the study of fossils, a primary impetus for enlarged views of the earth's antiquity, could not confirm the reality of evolution. Opponents such as **Georges Cuvier** argued the fossil record did not reveal the translation between forms that theories such as Lamarck's demanded. Also Cuvier's system of classification, identifying four fundamentally different types of animal organization, denied a chain of being and hence the idea of linear progression central to Lamarck's thinking.*

*In 1813, **William Charles Wells** produced essays assuming that there had been evolution of humans, and recognized the principle of* natural selection. ***Charles Darwin** and **Alfred Russel Wallace** were unaware of this work when they jointly published the theory in 1858, but Darwin later acknowledged that Wells had recognized the principle before them. **Augustin de Candolle**'s natural system of classification laid emphasis on the "war" between competing species.*

*By 1833 the geologist **Charles Lyell** in the second volume of his* Principles of Geology *had set out a gradualist variation of creation beliefs in which each species had its "center of creation" and was designed for the habitat, but would go extinct when the habitat changed.*

*Lamarckism became discredited as experiments simply did not support the concept that purely "*acquired traits*" were inherited. The mechanisms of* inheritance *were not elucidated until later in the 19^{th} century, after Lamarck's*

death. Lamarckism in toto has largely been discredited as a mechanism in evolution.

Although paleontologists *and* embryologists' *evidence appeared to confirm the reality of evolution, its mechanism remained unresolved at the end of the century, being vigorously debated by proponents of* neo-Darwinism*,* neo-Lamarckism*, orthogenesis and other views. Darwin had not explained the causes of variation or the means by which characters are passed on from one generation to the next. Without an adequate theory of heredity it was unclear how important natural selection was in the evolutionary process. For example, inheritance of acquired characters might account for the creative side of evolution, leaving natural selection with merely the negative function of weeding out the unfit.*

Though in retrospect it appears that what Darwin lacked was the theory of particulate inheritance proposed in the 1860's by ***Gregor Mendel****, when Mendel's work first came to be appreciated in 1900 people saw it as an alternative rather than complementing Darwin's theory. The three Mendelians most interested in evolution,* ***Hugo de Vries****,* ***William Bateson*** *and* ***Wilhelm Johannsen****, were all highly critical of the theory of evolution by natural selection.*

Thus, while the scientific community generally accepted that evolution had occurred, many disagreed that it had happened under the conditions or mechanisms provided by Darwin. In the years immediately following Darwin's death, evolutionary thought fractured into a number of interpretations, include neo-Darwinism, neo-Lamarckism, orthogenesis*,* Mendelism*, the* biometric *approach, and* mutation theory*. Eventually this boiled down to a debate between two camps. The Mendelians, advocating discrete variation, were led by* ***William Bateson*** *(who coined the word* genetics*) and* ***Hugo de Vries*** *(who coined the word* mutation*). Their opponents were the biometricians, advocating continuous variation; their leaders* ***Karl Pearson*** *and* ***Walter Frank Raphael Weldon****, following in the tradition of* ***Francis Galton****.*

An important issue in the debate between the Mendelians and the biometricians was the nature of variation in species. Darwin and Wallace believed that small variations were more important than large ones, since small variations hewed closely to an already-proven model. The biometricians agreed with this position, while the Mendelians insisted that discontinuous species were unlikely to arise from a continuous process of change. While the immediate issue of speciation *was resolved in large part by the clear definition of a species as a reproductively isolated population, the rate of evolution would arise again as a point of contention in the late 20^{th} century with the proposal*

of punctuated equilibrium. Most other questions resolving variation were resolved with the recognition that the size of a genotypic change did not always correspond with the size of the resulting phenotypic change.

Another source of clashes between Mendelians and biometricians was the debate over the origins of variation. Mendelians argued for intrinsic variations originating from genetic transmission; biometricians, observing primarily the phenotype of the organism, were not yet prepared to abandon Lamarckian views on the heritability of acquired characteristics. ***August Weismann*** *was among those who demonstrated that acquired characteristics were not always inherited, pointing out the existence of worker ants and worker bees, and the importance of 'germ plasm' or gametes in the biology of reproduction. The recognition of means of postnatal adaptation as inherited traits did much to explain acquired characteristics.*

Heredity

It was known from the 1840's that the organic cell reproduced asexually by fission, the nucleus dividing first. ***Mendel****'s hybridization experiments (1865) showed that independently transmitted characters separated and recombined in hybrid progeny.*

From the 1870's a number of technical advances were made in the field of experimental biology which allowed the processes occurring in the asexual reproduction of cells and in the union of sexual cells to be observed more closely: The achromatic microscope was further improved by the introduction of the high-power immersion lens and substage illumination, while the newly discovered aniline dyes, together with natural dyes and some inorganic salts, were found to stain selectively certain parts of the organic cell, particularly the nucleus.

For the next four decades, biologists succeeded to close the gap between cytology and heredity: In the 1870's it was shown by ***Hertwig****, at Berlin, and* ***Fol****, at Geneva, working on animals, and* ***Strassburger****, at Bonn, working on plants, that sexual reproduction involved the union of the nuclei of the male and female cells, from which they suggested (1884) that the nucleus of the cell was the physical basis for heredity.* ***Walter Flemming*** *(1879) coined the name mitosis and made first accurate accounts of chromosome numbers and figured their longitudinal splitting (1882). He then determined chromosome number (24) in man (1898).*

Edoúard van Beneden*, zoologist, first studied meiosis (1883) and* ***Wilhelm von Valdeyer-Hartz*** *coined the name chromosome (1888).* ***August Weismann****, biologist, proposed a germ-plasm theory of heredity and described the process of meiosis, whereby the number of chromosomes is halved.*

Oscar Hertwig *and* ***Theodore Bovery*** *(1889–1892) showed independently that pairs of chromosome split, replicating each member before dispersing into four separate nuclei.*

Finally, ***Walter Sutton*** *and Theodore Bovery pointed out (1902) the parallelism between chromosome behavior and Mendelism. Sutton coined the name* gene *(1902) and proposed that chromosomes carry genes.*

Anatomy and physiology

Form (anatomy) and function (physiology) were traditionally conceived as a single integrated subject, but experimental techniques, particularly in the 19th century, gradually divorced the two: ***Francois Magandie*** *(1783–1855),* ***Claude Bernard*** *(1813–1878),* ***Johannes Müller*** *(1801–1858),* ***Carl Ludwig*** *(1816–1895),* ***Emil Du Bois-Reymond*** *(1818–1896),* ***William Sharpey*** *(1802–1880),* ***Michael Foster*** *(1836–1907) and* ***H. Bowditch*** *(1840–1911) — helped create an autonomous discipline of physiology, with its research schools, professional societies and specialized journals.*

Organic chemistry

Using inorganic chemistry *as its paradigm, 19th century chemists created* organic chemistry*, whence emerged the powerful ideas of* valence *and* structure*. The advent of the* periodic law *in the 1870s finally provided chemists with a comprehensive classificatory system of elements.*

By the 1880s physics and chemistry were drawing closer together in the sub-discipline of physical chemistry. Finally, the discovery of the electron *enabled the chemists to solve the fundamental problem of* chemical affinity*.*

The dawn of biochemistry

Chemists of the 19th century were so busy with their own science that for a long time they did not attempt to systemize the chemistry of biological processes. Most of their biochemical discoveries were incidental to their major chemical work. The most important result of the development of organic chemistry *at first, from the viewpoint of biochemistry, was the demonstration that natural organic compounds were responsive to the same laws as inorganic substances.*

The urea synthesis of ***Wöhler*** *(1828) and the subsequent advances in organic synthesis struck telling blows at the vitalistic hypothesis that a special force controlled living matter. Toward the middle of the century a few*

chemists (chief among them was ***Liebig****) really did begin to integrate their work with that of biological investigators.*

Meanwhile physiology was developing as a science in its own right, much as was chemistry. Physiologists were chiefly concerned with the mechanics of bodily organs and with studies of the nervous system. Nevertheless, it was from the physiologists that most of the advances in biochemistry came until the end of the century. The approach of these men was usually related to their studies of special systems and organs, and so an overall view of the biochemical functioning of the body was not obtained.

Many important discoveries were made in this century, but they were like isolated pieces of a jigsaw puzzle. The science was properly called physiological chemistry *at this period, since it was used mostly to help understand specific physiological problems.*

It was only at the end of the 19th century and in the 20th that the pieces began to fit together so that a unified picture of the chemical changes in the cells and their significance for the body as a whole could be obtained.

The borderline between chemistry and physiology then became a science in its own right, and to this the name biochemistry*, the chemistry of life, can more properly be applied.*

By about 1920 biochemistry possessed the basic principles upon which it is still developing. The chemical nature of the body constituents was fairly well understood, the nutritional requirements could be seen, and the enzymatic *and* hormonal *mechanisms by which metabolic processes occurred were at least known to exist.*[697]

Modern medicine

Medicine was revolutionized in the 19th century by advances in chemistry and laboratory techniques and equipment, old ideas of infectious disease epidemiology were replaced with bacteriology.

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[697] *Chlorophyll* was isolated by **Pelletier** and **Caventou** in 1817, though at first its importance was not appreciated because the full significance of the photosynthetic process could not be realized until the concept of *energy* was better understood. Indeed, **J.R. Mayer**, who propounded the law of conservation of energy pointed out in 1845 that plants supplied sunlight energy as a source on which humans depended. In the meantime, the mechanism by which animals released the stored energy of plants, was clarified. By the middle of the 19th century many of the important principles of nutrition has been established, but the nature of “ferments”, as *enzymes* were called during the first three quarters of the century, was the subject of much discussion.

Ignaz Semmelweis (1818–1865) in 1847 dramatically reduced the death rate of new mothers from childbed fever by the simple expedient of requiring physicians to clean their hands before attending to women in childbirth. His discovery predated the germ theory of disease. However, his discoveries were not appreciated by his contemporaries and came into general use only with discoveries of British surgeon **Joseph Lister**, who in 1865 proved the principles of antisepsis; However, medical conservatism on new breakthroughs in pre-existing science prevented them from being generally well received during the 19^{th} century.

After **Charles Darwin**'s 1859 publication of The Origin of Species, **Gregor Mendel** (1822–1884) published in 1865 his books on pea plants, which would be later known as Mendel's laws. Re-discovered at the turn of the century, they would form the basis of classical genetics. The 1953 discovery of the structure of DNA by **Watson** and **Crick** would open the door to **molecular biology** and modern genetics. During the late 19^{th} century and the first part of the 20^{th} century, several physicians, such as Nobel prize winner **Alexis Carrel**, supported eugenics, a theory first formulated in 1865 by **Francis Galton**. Eugenics was discredited as a science after the Nazis' experiments in World War II became known; however, compulsory sterilization programs continued to be used in modern countries (including the US, Sweden or Peru) until much later.

Semmelweis' work was supported by the discoveries made by **Louis Pasteur**, who produced in 1880 the vaccine against rabies. Linking microorganisms with disease, Pasteur brought about a revolution in medicine. He also invented with **Claude Bernard** (1813–1878) the process of pasteurization still in use today. His experiments confirmed the germ theory. Claude Bernard aimed at establishing scientific method in medicine; he published An Introduction to the Study of Experimental Medicine in 1865. Beside this, Pasteur, along with **Robert Koch** (who was awarded the Nobel Prize in 1905), founded bacteriology. Koch was also famous for the discovery of the tubercle bacillus (1882) and the cholera bacillus (1883).

For the first time actual cures were developed for certain endemic infectious diseases. However the decline in many of the most lethal diseases was more due to improvements in public health and nutrition than to medicine.

Table 4.14: NOTABLE BIOLOGISTS AND MEN OF MEDICINE (1800–1900)

Key:

A = Anatomy	E = Ecology	EN = Entomology
BI = Biochemistry	EB = Evolutionary Biology	MI = Microbiology
BO = Botany	B = Biology	AN = Anthropology
H = Heredity	M = Marine Biology	OL = Origin of Life
PL = Paleontology	P = Physiology	T = Taxonomy
ZO = Zoology	CL = Chemistry of Life	EM = Embryology
MY = Mycology	CY = Cytology	PA = Pathology
S = Surgery	BG = Biogeography	BA = Bacteriology
IM = Immunology	NA = Naturalist	

Name	*fl.*	*Specialization*
Jean-Baptiste Lamarck	*1800–1829*	*EB*
G.R. Treviranus	*1802–1837*	*P*
K.F. Burdach	*1802*	*A, P, M*
P-J. Pelletier	*1820*	*CL*
J-B. Caventou	*1820*	*CL*
Christian Eherenberg	*1820–1875*	*B*
Jan Purkyne	*1823–1839*	*P*
Christian Pander	*1825–1865*	*EM*
Karl von Baer	*1826–1876*	*EM*
Henri Dutrochet	*1826–1839*	*BO, CY*
Robert Brown	*1827–1839*	*BO, CY*
John James Audubon	*1827–1839*	
Friedrich Wöhler	*1828–1832*	*CL*
Jean-Pierre Flourens	*1830–1865*	*B, P*
Marshall Hall	*1830–1833*	*M, P*
Augustino Bassi	*1835*	*MI*
Edward Blyth	*1835–1837*	*EB*
Theodor Schwann	*1835–1839*	*M, P, CY*
Jean Marie Poiseuffe	*1835–1846*	*M, P*
Robert Remak	*1836–1858*	*M, CY, B, EM*
Matthias Schleiden	*1838–1839*	*BO, CY*
Gerhardus Müller	*1838*	*CL*

Table 4.14: (Cont.)

Name	*fl.*	*Specialization*
Friedrich Henle	*1840*	*A, PA*
Karl Schimper	*1840–1865*	*B, BO*
Wilhelm Schimper	*1840–1890*	*BO*
Edward Forbes	*1841–1847*	
David Gruby	*1841–1852*	*MI, PA*
Julius Robert Mayer	*1842*	*M*
Crawford Long	*1842*	*S, M*
Gabril Gustav Valentin	*1844*	*M, P, CL*
Carl Friedrich Ludwig	*1844–1859*	*P*
Alexander von Humboldt	*1845–1859*	*BG*
Hugo von Mohl	*1846*	*BO, P, CY*
Ernest Heinrich Weber	*1846*	*A, P, CL*
Louis Pasteur	*1846–1885*	*MI, B, CL*
Ignaz Semmelweis	*1847*	*BA, M*
Herman von Helmholtz	*1847–1894*	*M, P*
Emil Du Bois-Reymond	*1849–1877*	*P*
Casimir Davaine	*1850–1882*	*M*
Julius Ferdinand Cohn	*1850–1881*	*BO, MI*
Rudolf Ludwig Virchow	*1856–1858*	*PA*
Nathanael Pringsheim	*1855–1868*	*BO*
Adolf Eugen Fick	*1856*	*P*
Claude Bernard	*1857*	*CY*
Max Schultze	*1858–1866*	*A, ZO, CY*
Charles Robert Darwin	*1858–1871*	*EB*
Alfred Russel Wallace	*1858*	*EB*
Jackson St. George Mivrat	*1860–1900*	*B*
Pierre-Paul Broca	*1861*	*M, A, AN*
Carl von Voit	*1861–1884*	*P*
Ernst Hoppe-Seyler	*1862–1871*	*P, CL*
Julius Sachs	*1862–1887*	*BO, P*
Ernst Haeckel	*1862–1899*	*B, M, EB*
Henri Baker Tristram	*1863–1895*	*ZO, OR*

Table 4.14: (Cont.)

Name	*fl.*	*Specialization*
Julius Friedrich Cohenheim	*1864–1884*	*A, PA*
Gregor Johann Mendel	*1865*	*BO, G, EB*
Heinrich Anton de Bary	*1865–1877*	*PA, MY*
Friedrich August Kekulé	*1865*	*P, CL*
John Hughlings	*1865–1911*	
Joseph Lister	*1867*	*M, S*
Johann Friedrich Miescher	*1868–1874*	*P, CY*
John Muir	*1868–1916*	
Jean Henri Fabre	*1870–1913*	*EN*
Carl Weigert	*1870–1904*	
Gustav Theodor Fritsch	*1870–1927*	*P, CL*
Eduard Hitzig	*1870–1907*	*P*
Charles Wyville Thomson	*1872–1876*	*B, A*
Camilo Golgi	*1873–1893*	*P, M, A*
Anton Schneider	*1873*	*CY, G*
Jacobus Van't Hoff	*1874*	*CL*
Joseph-Achille Le Bel	*1874*	*CL*
Santigo Ramon y Cajal	*1875–1928*	*A, P, M*
Luther Burbank	*1875–1920*	*BO*
David Ferrier	*1875–1925*	*P*
Robert Koch	*1876–1897*	*M, BA, P*
Oscar Hertwig	*1876*	*B, CY*
Herman Fol	*1876*	*B, CY*
Wilhelm Friedrich Pffefer	*1877–1881*	*BO*
Emile Roux	*1879–1891*	*BA, IM*
Walter Flemming	*1879–1898*	*CY, G*
Charles Louis Laveran	*1880*	*M, P*
Hugo de Vries	*1880–1935*	*H*
Walter Reed	*1881–1902*	*S, BA*
Charles Roy	*1881*	*P*
Paul Ehrlich	*1881–1912*	*M, MI, BA*
Edouard van Beneden	*1883–1887*	*CY, G, ZO*

Table 4.14: (Cont.)

Name	*fl.*	*Specialization*
Emil Fischer	*1884*	*CL*
Karl Martin Kossel	*1885–1896*	*BI*
Elie Metchnikov	*1886–1908*	*IM*
Ernest Thompson-Seton	*1886–1940*	*NA*
Eduard Strassburger	*1888*	*CY, G, BO*
Wilhelm von Waldeyer-Hartz	*1888–1891*	*CY, G, A*
Georges Fernand Widal	*1888–1906*	*M, P, PA*
Theobald Smith	*1889–1895*	*BA*
Oscar Hertwig	*1889*	*CY, G, ZO*
Oskar Minkowski	*1889*	*M, P*
Joseph von Mering	*1889*	*M, P*
Emil von Behring	*1890–1901*	*IM, P, MI*
Eugene Dubois	*1891–1921*	*M, A, PA*
August Weismann	*1892*	*CY, G, B*
Theodore Boveri	*1892–1903*	*CY, G, B*
Dimitri Ivanovski	*1892*	*MI, BO*
Richard Friedrich Pffifer	*1892–1894*	*MI, IM, BA*
Morde Wolfe Haffkine	*1893–1896*	*BA, IM, MI*
Zigmund Freud	*1894–1925*	*M*
Georg Oliver	*1894–1897*	*M, P*
Edward Sharpey-Schäfer	*1894–1897*	*M, P, BI*
Carl Correns	*1895–1933*	*BO, H*
David Bruce	*1895–1915*	*MI, PA*
Giovanni Batista Grassi	*1895–*	*BA*
John Jacob Abel	*1897–1926*	*P, BI*
Ronald Ross	*1897–1916*	*M, P*
Paul Frosch	*1898*	*MI, BA*
Friedrich Loeffler	*1898*	*MI, BA*
Sergei Winogradsky	*1888–1905*	*MI, E, BA*
Martinus Beijerinck	*1898*	*MI, BO*

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Indices

Name Index

(underlined page number indicates full biography)

A

B

C

D

F

G

H

I

J

K

L

M

P

Q

R

S

T

U

V

W

X

Y

Z

Subject Index

A

B

C

D

F

G

H

I

J

K

L

M

N

O

P

Q

R

S

T

U

V

W

X

Y

Z